

RISC, Hagenberg, Austria

LHCPHenoNet School: Integration, Summation and Special Functions in QFT

Difference field algorithms for Feynman integrals

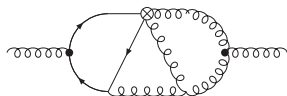
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July 12, 2012

Consider a massive 3-loop ladder graph (Ablinger, Blümlein, Hasselhuhn, Klein, CS, Wißbrock, 2012)



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$

Consider a massive 3-loop ladder graph (Ablinger, Blümlein, Hasselhuhn, Klein, CS, Wißbrock, 2012)



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$

||

Simplify

$$\sum_{j=0}^{n-3} \sum_{k=0}^j \sum_{l=0}^k \sum_{q=0}^{-j+n-3} \sum_{s=1}^{-l+n-q-3} \sum_{r=0}^{-l+n-q-s-3} (-1)^{-j+k-l+n-q-3} \times$$

$$\times \frac{\binom{j+1}{k+1} \binom{k}{l} \binom{n-1}{j+2} \binom{-j+n-3}{q} \binom{-l+n-q-3}{s} \binom{-l+n-q-s-3}{r} r! (-l+n-q-r-s-3)! (s-1)!}{(-l+n-q-2)! (-j+n-1) (n-q-r-s-2) (q+s+1)}$$

$$\left[\begin{aligned} &4S_1(-j+n-1) - 4S_1(-j+n-2) - 2S_1(k) \\ &- (S_1(-l+n-q-2) + S_1(-l+n-q-r-s-3) - 2S_1(r+s)) \\ &+ 2S_1(s-1) - 2S_1(r+s) \end{aligned} \right] + \mathbf{3 \text{ further 6-fold sums}}$$

$$\begin{aligned}
\boxed{F_0(n)} = & \\
& \frac{7}{12} S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{3n(n+1)} + \left(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2} \right) S_1(n)^2 \\
& + \left(-\frac{4(13n+5)}{n^2(n+1)^2} + \left(\frac{4(-1)^n(2n+1)}{n(n+1)} - \frac{13}{n} \right) S_2(n) + \left(\frac{29}{3} - (-1)^n \right) S_3(n) \right. \\
& + \left(2 + 2(-1)^n \right) S_{2,1}(n) - 28S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)} \Big) S_1(n) + \left(\frac{3}{4} + (-1)^n \right) S_2(n)^2 \\
& - 2(-1)^n S_{-2}(n)^2 + S_{-3}(n) \left(\frac{2(3n-5)}{n(n+1)} + (26 + 4(-1)^n) S_1(n) + \frac{4(-1)^n}{n+1} \right) \\
& + \left(\frac{(-1)^n(5-3n)}{2n^2(n+1)} - \frac{5}{2n^2} \right) S_2(n) + S_{-2}(n) (10S_1(n)^2 + \left(\frac{8(-1)^n(2n+1)}{n(n+1)} \right. \\
& + \left. \frac{4(3n-1)}{n(n+1)} \right) S_1(n) + \frac{8(-1)^n(3n+1)}{n(n+1)^2} + (-22 + 6(-1)^n) S_2(n) - \frac{16}{n(n+1)} \Big) \\
& + \left(\frac{(-1)^n(9n+5)}{n(n+1)} - \frac{29}{3n} \right) S_3(n) + \left(\frac{19}{2} - 2(-1)^n \right) S_4(n) + (-6 + 5(-1)^n) S_{-4}(n) \\
& + \left(-\frac{2(-1)^n(9n+5)}{n(n+1)} - \frac{2}{n} \right) S_{2,1}(n) + (20 + 2(-1)^n) S_{2,-2}(n) + (-17 + 13(-1)^n) S_{3,1}(n) \\
& - \frac{8(-1)^n(2n+1) + 4(9n+1)}{n(n+1)} S_{-2,1}(n) - (24 + 4(-1)^n) S_{-3,1}(n) + (3 - 5(-1)^n) S_{2,1,1}(n) \\
& + 32S_{-2,1,1}(n) + \left(\frac{3}{2} S_1(n)^2 - \frac{3S_1(n)}{n} + \frac{3}{2} (-1)^n S_{-2}(n) \right) \zeta(2)
\end{aligned}$$

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Simplify

$$\sum_{k=1}^n S_1(k)$$

where $S_1(k) = \sum_{i=1}^k \frac{1}{i}$

Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$f(k) = g(k+1) - g(k)$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Telescoping

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FIND $g(k)$:

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for all $1 \leq k \leq n$ and $n \geq 0$.

We compute

$$g(k) = (S_1(k) - 1)k.$$

Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$f(k) = g(k+1) - g(k)$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Summing this equation over k from 1 to n gives

$$\begin{aligned} \sum_{k=1}^n S_1(k) &= g(n+1) - g(1) \\ &= (S_1(n+1) - 1)(n+1). \end{aligned}$$

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^n S_1(k).$$

A difference field for the **summand**

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}$$

with the automorphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

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A difference field for the **summand**

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)$$

with the automorphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1,$$

$$S k = k + 1,$$

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^n S_1(k).$$

A difference field for the **summand**

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

with the automorphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1,$$

$$\sigma(h) = h + \frac{1}{k+1},$$

$$S k = k + 1,$$

$$S S_1(k) = S_1(k) + \frac{1}{k+1}.$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\sigma(g) - g = h.$$

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We compute

$$g = (h - 1)k \in \mathbb{F}.$$

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This gives

$$g(k + 1) - g(k) = S_1(k)$$

with

$$g(k) = (S_1(k) - 1)k.$$

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Hence,

$$(S_1(n + 1) - 1)(n + 1) = \sum_{k=1}^n S_1(k).$$

The basic summation algorithm

(a simplified version of Karr's algorithm, 1981)

CONSTRUCT a difference field (\mathbb{F}, σ) :

- ▶ a rational function field

$$\mathbb{F} := \mathbb{K}$$

- ▶ with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

CONSTRUCT a difference field (\mathbb{F}, σ) :

- ▶ a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)$$

- ▶ with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

$$\sigma(t_1) = a_1 t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \quad f_1 \in \mathbb{K}$$

CONSTRUCT a difference field (\mathbb{F}, σ) :

- ▶ a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)$$

- ▶ with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

$$\sigma(t_1) = a_1 t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \quad f_1 \in \mathbb{K}$$

$$\sigma(t_2) = a_2 t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, \quad f_2 \in \mathbb{K}(t_1)$$

CONSTRUCT a difference field (\mathbb{F}, σ) :

- ▶ a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2) \dots (t_e)$$

- ▶ with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

$$\sigma(t_1) = a_1 t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \quad f_1 \in \mathbb{K}$$

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$$\sigma(t_e) = a_e t_e + f_e, \quad a_e \in \mathbb{K}(t_1, \dots, t_{e-1})^*, \quad f_e \in \mathbb{K}(t_1, \dots, t_{e-1})$$

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GIVEN $f \in \mathbb{F}$;

FIND $g \in \mathbb{F}$ such that

$$\sigma(g) - g = f.$$

CONSTRUCT a $\Pi\Sigma$ -field (\mathbb{F}, σ) :

- ▶ a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2) \dots (t_e)$$

- ▶ with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

$$\sigma(t_1) = a_1 t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \quad f_1 \in \mathbb{K}$$

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such that

$$\text{const}_\sigma \mathbb{F} = \{c \in \mathbb{K}(t_1)(t_2) \dots (t_e) \mid \sigma(c) = c\} = \mathbb{K}.$$

GIVEN $f \in \mathbb{F}$;

FIND $g \in \mathbb{F}$ such that

$$\sigma(g) - g = f.$$

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^n S_1(k).$$

A $\Pi\Sigma^*$ -field for the summand

$$\text{const}_\sigma \mathbb{F} = \mathbb{Q}$$

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

with the automorphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1,$$

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FIND $g \in \mathbb{Q}(k)(h)$:

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Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]^*$:

$$d = 1$$

$$\forall g \in \mathbb{Q}(k)(h) : \sigma(g) - g = h \Rightarrow g d \in \mathbb{Q}(k)[h].$$

FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma\left(\frac{g'}{d}\right) - \frac{g'}{d} = h.$$

FIND $g \in \mathbb{Q}(k)(h)$:

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FIND $g' \in \mathbb{Q}(k)[h]$ with

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Degree bound: COMPUTE $b \geq 0$:

$$b = 2$$

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \Rightarrow \deg(g) \leq b.$$

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Polynomial Solution: FIND

$$g = hk - k$$

$$g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h].$$

ANSATZ $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]$

$$\sigma(g) - g = h$$



ANSATZ $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]$

$$\left[\sigma(g_2) \left(h + \frac{1}{k+1} \right)^2 + \sigma(g_1 h + g_0) \right] - [g_2 h^2 + g_1 h + g_0] = h$$



ANSATZ $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]$

$$\begin{aligned} & [\sigma(g_2)(h + \frac{1}{k+1})^2 + \sigma(g_1 h + g_0)] \\ & - [g_2 h^2 + g_1 h + g_0] = h \end{aligned}$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

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$$\sigma(g_2) - g_2 = 0$$

$$g_2 = c \in \mathbb{Q}$$

$$\sigma(g_1 h + g_0) - (g_1 h + g_0) = h - c \left[\frac{2h(k+1)+1}{(k+1)^2} \right]$$

$$\text{ANSATZ } g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]$$

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$$\sigma(g_1) - g_1 = 1 - c \frac{2}{k+1}$$

$$\sigma(g_0) - g_0 = -1 - d \frac{1}{k+1}$$

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$$\left[\sigma(g_2) \left(h + \frac{1}{k+1} \right)^2 + \sigma(g_1 h + g_0) \right] - [g_2 h^2 + g_1 h + g_0] = h$$

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$$g = hk - k$$

$$\sigma(g_2) - g_2 = 0$$

$$g_2 = c \in \mathbb{Q}$$

$$\sigma(g_1 h + g_0) - (g_1 h + g_0) = h - c \left[\frac{2h(k+1)+1}{(k+1)^2} \right]$$

coeff. comp.

$$\sigma(g_1) - g_1 = 1 - c \frac{2}{k+1}$$

$$\begin{aligned} g_0 &= -k \\ d &= 0 \end{aligned}$$

$$\sigma(g_0) - g_0 = -1 - d \frac{1}{k+1}$$

$$c = 0, \quad \begin{aligned} g_1 &= k + d \\ d &\in \mathbb{Q} \end{aligned}$$

Difference equations in difference fields

Let (\mathbb{F}, σ) be a $\Pi\Sigma$ -field with constant field \mathbb{K}

Telescoping

- ▶ Given $f \in \mathbb{F}$.
- ▶ Find $g \in \mathbb{F}$:

$$\sigma(g) - g = f.$$

Difference equations in difference fields

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↓

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Parameterized Telescoping

- ▶ Given $f_0, \dots, f_d \in \mathbb{F}$.
- ▶ Find all $c_0, \dots, c_d \in \mathbb{K}, g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = c_0 f_0 + \dots + c_d f_d.}$$

Difference equations in difference fields

Let (\mathbb{F}, σ) be a $\Pi\Sigma$ -field with constant field \mathbb{K}

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- ▶ Given $f_0, \dots, f_d \in \mathbb{F}$.
- ▶ Find all $c_0, \dots, c_d \in \mathbb{K}$, $g \in \mathbb{F}$:

$$\sigma(g) - g = c_0 f_0 + \dots + c_d f_d.$$

↓

↑

Parameterized first order difference equation

- ▶ Given $f_0, \dots, f_d \in \mathbb{F}$, $a_0, a_1 \in \mathbb{F}$.
- ▶ Find all $c_0, \dots, c_d \in \mathbb{K}$, $g \in \mathbb{F}$:

$$a_1 \sigma(g) + a_0 g = c_0 f_0 + \dots + c_d f_d.$$

Constructing $\Pi\Sigma$ -fields

CONSTRUCT a $\Pi\Sigma$ -field (\mathbb{F}, σ) :

- ▶ a rational function field

$$\mathbb{F} := \mathbb{K}$$

- ▶ with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

such that

$$\text{const}_\sigma \mathbb{F} = \{c \in \mathbb{K}$$

$$| \sigma(c) = c \} = \mathbb{K}.$$

CONSTRUCT a $\Pi\Sigma$ -field (\mathbb{F}, σ) :

- ▶ a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)$$

- ▶ with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

$$\sigma(t_1) = a_1 t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \quad f_1 \in \mathbb{K}$$

such that

$$\text{const}_\sigma \mathbb{F} = \{c \in \mathbb{K}(t_1) \mid \sigma(c) = c\} = \mathbb{K}.$$

CONSTRUCT a $\Pi\Sigma$ -field (\mathbb{F}, σ) :

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- ▶ with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

$$\sigma(t_1) = a_1 t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \quad f_1 \in \mathbb{K}$$

$$\sigma(t_2) = a_2 t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, \quad f_2 \in \mathbb{K}(t_1)$$

$$\vdots$$

$$\sigma(t_e) = a_e t_e + f_e, \quad a_e \in \mathbb{K}(t_1, \dots, t_{e-1})^*, \quad f_e \in \mathbb{K}(t_1, \dots, t_{e-1})$$

such that

$$\text{const}_\sigma \mathbb{F} = \{c \in \mathbb{K}(t_1)(t_2) \dots (t_e) \mid \sigma(c) = c\} = \mathbb{K}.$$

GIVEN $f \in \mathbb{F}$;

FIND $g \in \mathbb{F}$ such that

$$\sigma(g) - g = f.$$

Construction of Σ^* -extensions

- ▶ Let (\mathbb{F}, σ) be a difference field with constant field

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Symbolic summation in $\Pi\Sigma$ -fields

A difference field approach (M. Karr, 1981)

GIVEN a $\Pi\Sigma$ -field (\mathbb{F}, σ) with $f \in \mathbb{F}$.

FIND $g \in \mathbb{F}$:

$$\sigma(g) - g = f.$$

A symbolic summation approach (see, e.g., J. Symb. Comput 2008; arXiv:0808.2543)

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appropriate = sum representations with optimal nesting depth

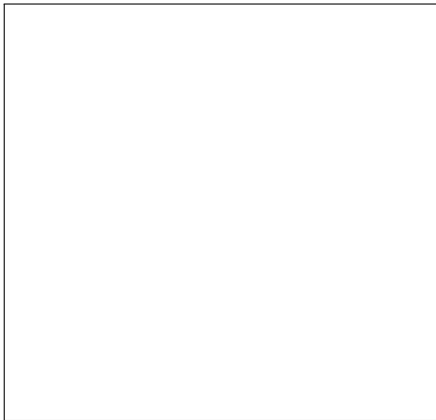
Example:

$$\sum_{k=1}^n \frac{\sum_{j=1}^k \frac{\sum_{i=1}^j \frac{1}{i}}{j}}{k} = \frac{1}{6} \left(\sum_{i=1}^n \frac{1}{i} \right)^3 + \frac{1}{2} \left(\sum_{i=1}^n \frac{1}{i^2} \right) \left(\sum_{i=1}^n \frac{1}{i} \right) + \frac{1}{3} \sum_{i=1}^n \frac{1}{i^3}$$

depth 3

depth 1

$$\sum_{k=1}^n \frac{\sum_{j=1}^k \frac{\sum_{i=1}^j \frac{1}{i}}{j}}{k}$$



$$\sum_{k=1}^n \frac{\sum_{j=1}^k \frac{1}{i}}{j} \quad h$$

$\Pi\Sigma$ -field $(\mathbb{Q}(k)(h), \sigma)$ with

$$\sigma(k) = k + 1$$

$$\sigma(h) = h + \frac{1}{k + 1}$$

$$\sum_{k=1}^n \frac{\sum_{j=1}^k \frac{1}{i}}{k} \quad \text{S}$$

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No simplification



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$$\sum_{k=1}^n \frac{\sum_{j=1}^k \frac{\sum_{i=1}^j 1}{i}}{j} = \frac{1}{6} \left(\sum_{i=1}^n \frac{1}{i} \right)^3 + \frac{1}{2} \left(\sum_{i=1}^n \frac{1}{i^2} \right) \left(\sum_{i=1}^n \frac{1}{i} \right) + \frac{1}{3} \sum_{i=1}^n \frac{1}{i^3}$$

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Apéry's proof (1979) of the irrationality of $\zeta(3)$ relies on the following fact:

$$a(n) = \sum_{k=0}^n \binom{n+k}{k}^2 \binom{n}{k}^2$$

and

$$b(n) = \sum_{k=0}^n \binom{n+k}{k}^2 \binom{n}{k}^2 \left(S_3(n) + \sum_{m=1}^k \frac{(-1)^{m-1}}{2m^3 \binom{n+m}{m} \binom{n}{m}} \right)$$

satisfy both the recurrence relation

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Van der Poorten (1979) points out that Henri Cohen and Don Zagier showed this fact by

“some rather complicated but ingenious explanations”

based on the creative telescoping method.

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$a(n)$ -case: trivial exercise by Zeilberger's algorithm (1991)

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b(n)-case: skilful application of computer algebra

1. Generalization of the Cohen/Zagier method in the WZ-setting (Zeilberger, 1993)
2. Multi-summation + holonomic closure properties (Chyzak/Salvy, 1998)

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b(n)-case: plain sailing (and not plane sailing) by [Sigma](#)

Appendix

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3. $\boxed{\exists g \in \mathbb{F}^* : \sigma(g) = a^n g \text{ for } n > 1, \text{ but not for } n = 1}$ 