FWF
Der Wissenschaftsfonds.

German-Japanese Workshop on

Modern Trends in Quantum Chromodynamics, DESY, Zeuthen

Multi-summation algorithms for particle physics

Carsten Schneider
Research Institute for Symbolic Computation (RISC)
J. Kepler University Linz, Austria

October 4, 2011

A warm up example: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!}+\frac{j!k!(j+k+N)!\,(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

where

$$S_1(N)=\sum_{i=1}^{N}\frac{1}{i}$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

A warm up example: Simplify

$$\sum_{k=0}^{\infty}\color{blue}{\sum_{j=0}^{\infty}}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!}+\frac{j!k!(j+k+N)!\,(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{j=0}^{a} f(N,k,j) =$$ ▸ Sigma

A warm up example: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\left(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)\right)}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{j=0}^{a} f(N,k,j) =$$ ▸ Sigma

FIND $g(j)$:

$$\boxed{f(N,k,j) = g(j+1) - g(j)}$$

A warm up example: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!}+\frac{j!k!(j+k+N)!\,(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{j=0}^{a} f(N,k,j) =$$ Sigma

FIND $g(j)$:

$$\boxed{f(N,k,j)=g(j+1)-g(j)}$$

Sigma (based on a refined version of M. Karr's difference fields (1981)) computes

$$g(j)=\frac{(j+k+1)(j+N+1)j!k!(j+k+N)!\big(S_1(j)-S_1(j+k)-S_1(j+N)+S_1(j+k+N)\big)}{kN(j+k+1)!(j+N+1)!(k+N+1)!}$$

A warm up example: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\,(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{j=0}^{a} f(N,k,j) =$$ ▸ Sigma

FIND $g(j)$:

$$\boxed{f(N,k,j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\boxed{\sum_{j=0}^{a} f(N,k,j) = g(a+1) - g(0)}$$

A warm up example: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\left(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!}+\frac{j!k!(j+k+N)!\left(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)\right)}{(j+k+1)!(j+N+1)!(k+N+1)!}\right)}^{f(N,k,j)}$$

$$\sum_{j=0}^{a} f(N,k,j) = \underbrace{\frac{(a+1)!(k-1)!(a+k+N+1)!(S_1(a)-S_1(a+k)-S_1(a+N)+S_1(a+k+N))}{N(a+k+1)!(a+N+1)!(k+N+1)!} + \frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!} + \frac{(2a+k+N+2)a!k!(a+k+N)!}{(a+k+1)(a+N+1)(a+k+1)!(a+N+1)!(k+N+1)!}}_{a\to\infty}$$

A warm up example: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!}+\frac{j!k!(j+k+N)!\,(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{j=0}^{\infty} f(N,k,j) = \frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

A warm up example: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\left(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)\right)}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{k=1}^{a}\sum_{j=0}^{\infty} f(N,k,j) = \sum_{k=1}^{a}\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

$$=$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence finder

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence solver

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} \in \left\{ c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)} \middle| c \in \mathbb{R} \right\}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1)-n\mathbf{A}(n)=\frac{(n+1)S_1(n)+1}{(n+1)^3}$$

$$\boxed{A(n)=\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}=\boxed{\frac{1}{2}\times\frac{1}{n(n+1)}+\frac{S_1(n)^2+S_2(n)}{2n(n+1)}}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i}\qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

Sigma

A warm up example: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\left(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)\right)}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(N,k,j) = \sum_{k=1}^{\infty}\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$
$$= \frac{S_1(N)^2+S_2(N)}{2N(N+1)!}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

More generally: Sigma's summation spiral

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n)=\sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

More generally: Sigma's summation spiral

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n) + \cdots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products and sums
(Abramov/Bronstein/Petkovšek/Schneider, in preparation)

More generally: Sigma's summation spiral

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \ldots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n) + \cdots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products and sums
(Abramov/Bronstein/Petkovšek/Schneider, in preparation)

3. Find a "closed form"

A(n)=combined solutions.

“Background of our 3–loop computations”

- The following examples arise in the context of 2– and 3–loop massive single scale Feynman diagrams with operator insertion.
- These are related to the QCD anomalous dimensions and massive operator matrix elements.
- At 2-loop order all respective calculations are finished:

M. Buza, Y. Matiounine, J. Smith, R. Migneron, W.L. van Neerven, Nucl. Phys. **B472** (1996) 611;

I. Bierenbaum, J. Blümlein, S. Klein, Nucl. Phys. **B780** (2007) 40;

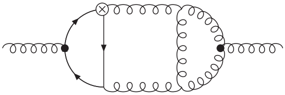
I. Bierenbaum, J. Blümlein, S. Klein, C. Schneider, Nucl.Phys. **B803** (2008)

and lead to representations in terms of harmonic sums.

Example 1: All n-Results for 3–Loop Ladder Graphs

Joint work with J. Ablinger (RISC), J. Blümlein (DESY), A. Hasselhuhn (DESY), S. Klein (RWTH)

In total around 50 diagrams (for this class) have been calculated, like e.g.



(containing three massive fermion propagators)

$$\Downarrow$$

Around 1000 sums have to be calculated for this diagram

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$||$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\boxed{\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\|$$

$$\sum_{j=0}^{n-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\|$$

$$\boxed{\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Bigg(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\displaystyle\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+$$
$$\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Bigg)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Bigg(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Bigg)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$||$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}},\ a\in\mathbb{Z}\setminus\{0\}.$

Multi-Summation

Mathematica Session:

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= $\mathbf{EvaluateMultiSum}\Big[\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!},$

$\{\{s,0,n-j+r-2\},\{r,0,j+1\},\{j,0,n-2\}\}\Big]$

Out[4]= $\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$

A typical sum

$$\sum_{j=0}^{n-2}\sum_{s=1}^{j+1}\sum_{r=0}^{n+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+n+s-2}{r}(n-j)!(s-1)!\sigma!S_1(r+2)}{(n-r)(r+1)(r+2)(-j+n+\sigma+1)(-j+n+\sigma+2)(-j+n+s+\sigma)!}$$

A typical sum

$$\sum_{j=0}^{n-2}\sum_{s=1}^{j+1}\sum_{r=0}^{n+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+n+s-2}{r}(n-j)!(s-1)!\sigma!S_1(r+2)}{(n-r)(r+1)(r+2)(-j+n+\sigma+1)(-j+n+\sigma+2)(-j+n+s+\sigma)!}$$

$$=\frac{\left(2n^2+6n+5\right)S_{-2}(n)^2}{2(n+1)(n+2)}+S_{-2,-1,2}(n)+S_{-2,1,-2}(n)$$
$$+\dots$$

where, e.g.,

$$S_{-2,1,-2}(n)=\sum_{i=1}^{n}\frac{(-1)^i\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{(-1)^k}{k^2}}{j}}{i^2}$$

Vermaseren 98/Blümlein/Kurth 99

A typical sum

$$\sum_{j=0}^{n-2}\sum_{s=1}^{j+1}\sum_{r=0}^{n+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+n+s-2}{r}(n-j)!(s-1)!\sigma!S_1(r+2)}{(n-r)(r+1)(r+2)(-j+n+\sigma+1)(-j+n+\sigma+2)(-j+n+s+\sigma)!}$$

$$=\frac{\left(2n^2+6n+5\right)S_{-2}(n)^2}{2(n+1)(n+2)}+S_{-2,-1,2}(n)+S_{-2,1,-2}(n)$$

$$+\dots-S_{2,1,1,1}(-1,2,\frac{1}{2},-1;n)+S_{2,1,1,1}(1,\frac{1}{2},1,2;n)$$

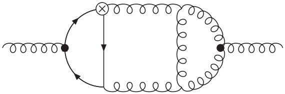
$$+\dots$$

where, e.g.,

145 S–sums occur

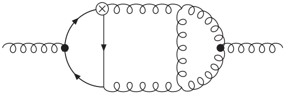
$$S_{2,1,1,1}(1,\frac{1}{2},1,2;n)=\sum_{i=1}^{n}\frac{\sum_{j=1}^{i}\frac{\left(\frac{1}{2}\right)^{j}\sum_{k=1}^{j}\frac{\sum_{l=1}^{k}\frac{2^l}{l}}{k}}{j}}{i^2}$$

S. Moch, P. Uwer, S. Weinzierl 02



$\Downarrow$ Sigma.m

Around 1000 sums are calculated containing in total 533 S–sums



$\Downarrow$ Sigma.m

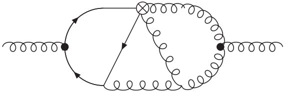
Around 1000 sums are calculated containing in total 533 S–sums

$\Downarrow$ J. Ablinger's HarmonicSum.m

After elimination the following sums remain:

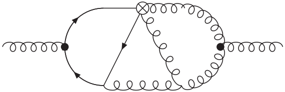
$$S_{-4}(n), S_{-3}(n), S_{-2}(n), S_{1}(n), S_{2}(n), S_{3}(n), S_{4}(n), S_{-3,1}(n),$$
$$S_{-2,1}(n), S_{2,-2}(n), S_{2,1}(n), S_{3,1}(n), S_{-2,1,1}(n), S_{2,1,1}(n)$$

So far, the most complicated 3–loop ladder graph:



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$

So far, the most complicated 3–loop ladder graph:



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$

$$\|$$

$$\boxed{\begin{aligned}&\sum_{j=0}^{n-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+n-3}\sum_{s=1}^{-l+n-q-3}\sum_{r=0}^{-l+n-q-s-3}(-1)^{-j+k-l+n-q-3}\times\\ &\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{n-1}{j+2}\binom{-j+n-3}{q}\binom{-l+n-q-3}{s}\binom{-l+n-q-s-3}{r}r!(-l+n-q-r-s-3)!(s-1)!}{(-l+n-q-2)!(-j+n-1)(n-q-r-s-2)(q+s+1)}\\ &\Big[4S_1(-j+n-1)-4S_1(-j+n-2)-2S_1(k)\\ &-(S_1(-l+n-q-2)+S_1(-l+n-q-r-s-3)-2S_1(r+s))\\ &+2S_1(s-1)-2S_1(r+s)\Big] + \textbf{3 further 6–fold sums}\end{aligned}}$$

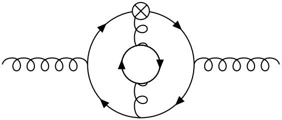
$$\boxed{F_0(n)} =$$

$$\frac{7}{12}S_1(n)^4+\frac{(17n+5)S_1(n)^3}{3n(n+1)}+\Big(\frac{35n^2-2n-5}{2n^2(n+1)^2}+\frac{13S_2(n)}{2}+\frac{5(-1)^n}{2n^2}\Big)S_1(n)^2$$
$$+\Big(-\frac{4(13n+5)}{n^2(n+1)^2}+\Big(\frac{4(-1)^n(2n+1)}{n(n+1)}-\frac{13}{n}\Big)S_2(n)+\Big(\frac{29}{3}-(-1)^n\Big)S_3(n)$$
$$+\big(2+2(-1)^n\big)S_{2,1}(n)-28S_{-2,1}(n)+\frac{20(-1)^n}{n^2(n+1)}\Big)S_1(n)+\Big(\frac{3}{4}+(-1)^n\Big)S_2(n)^2$$
$$-2(-1)^nS_{-2}(n)^2+S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)}+\big(26+4(-1)^n\big)S_1(n)+\frac{4(-1)^n}{n+1}\Big)$$
$$+\Big(\frac{(-1)^n(5-3n)}{2n^2(n+1)}-\frac{5}{2n^2}\Big)S_2(n)+S_{-2}(n)\Big(10S_1(n)^2+\Big(\frac{8(-1)^n(2n+1)}{n(n+1)}$$
$$+\frac{4(3n-1)}{n(n+1)}\Big)S_1(n)+\frac{8(-1)^n(3n+1)}{n(n+1)^2}+\big(-22+6(-1)^n\big)S_2(n)-\frac{16}{n(n+1)}\Big)$$
$$+\Big(\frac{(-1)^n(9n+5)}{n(n+1)}-\frac{29}{3n}\Big)S_3(n)+\Big(\frac{19}{2}-2(-1)^n\Big)S_4(n)+\big(-6+5(-1)^n\big)S_{-4}(n)$$
$$+\Big(-\frac{2(-1)^n(9n+5)}{n(n+1)}-\frac{2}{n}\Big)S_{2,1}(n)+\big(20+2(-1)^n\big)S_{2,-2}(n)+\big(-17+13(-1)^n\big)S_{3,1}(n)$$
$$-\frac{8(-1)^n(2n+1)+4(9n+1)}{n(n+1)}S_{-2,1}(n)-\big(24+4(-1)^n\big)S_{-3,1}(n)+\big(3-5(-1)^n\big)S_{2,1,1}(n)$$
$$+32S_{-2,1,1}(n)+\left(\frac{3}{2}S_1(n)^2-\frac{3S_1(n)}{n}+\frac{3}{2}(-1)^nS_{-2}(n)\right)\zeta(2)$$

Example 2: 3-Loop All N-Results for the N_f Contributions

Joint work with J. Ablinger (RISC), J. Blümlein (DESY), F. Wißbrock (DESY), S. Klein (RWTH)

E.g., for the diagram



768 sums are simplified.

Simple example:

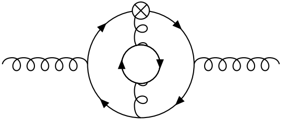
$$\sum_{j=1}^{n-2}\frac{j(j+1)(j+2)(n-j)(j-1)!^2(-j+n-1)!^2}{-j+n-1}$$
$$=\frac{\big(-n^3-5n^2-4n+6\big)(n!)^2}{(n-1)^2n^2}$$
$$+\frac{3}{2}\frac{(n!)^2\big(n^3+6n^2+11n+6\big)}{(n-1)n(2n+1)\binom{2n}{n}}\sum_{i=1}^{n}\frac{\binom{2i}{i}}{i}.$$

Not expressible in terms of harmonic sums or S-sums!

The final expression is given in terms of 703 indefinite nested sums and products. Typical examples are:

$$\sum_{i=1}^{n} \frac{\displaystyle\sum_{j=1}^{i} \frac{\binom{2j}{j}}{j}}{\binom{2\,i}{i}}, \quad \sum_{i=1}^{n} \frac{S_1(i) \displaystyle\sum_{j=1}^{i} \frac{\binom{2j}{j}}{j}}{\binom{2i}{i}};$$

Sigma finds all algebraic relations among them. We get:



$$\Downarrow$$

$$
\begin{aligned}
&-\frac{20S(1,n)^4}{27(n+1)(n+2)}+\frac{32\left(6n^3+61n^2-21n+24\right)S_1(n)^3}{81n^2(n+1)(n+2)}-\frac{16\left(48n^5+746n^4+2697n^3+2746n^2+1104n+240\right)S_1(n)^2}{81n^2(n+1)^2(n+2)^2}\\
&+\frac{32\left(264n^7+4046n^6+21591n^5+52844n^4+74856n^3+66812n^2+30576n+2640\right)S_1(n)}{243n^2(n+1)^3(n+2)^3}\\
&-\frac{4\left(48n^2+101n+96\right)S_2(n)^2}{9n(n+1)(n+2)}-\frac{32\left(363n^7+6758n^6+41285n^5+121235n^4+190235n^3+150758n^2+46964n+2904\right)}{243n(n+1)^4(n+2)^3}\\
&+\Bigg(-\frac{40S_1(n)^2}{9(n+1)(n+2)}+\frac{32\left(6n^3+61n^2-21n+24\right)S_1(n)}{27n^2(n+1)(n+2)}\\
&-\frac{16\left(124n^5+198n^4-2387n^3-6162n^2-3632n-480\right)}{81n^2(n+1)^2(n+2)^2}\Bigg)S_2(n)+\\
&+\left(-\frac{32\left(9n^3-623n^2+894n+276\right)}{81n^2(n+1)(n+2)}-\frac{160S_1(n)}{27(n+1)(n+2)}\right)S_3(n)-\frac{8\left(56n^2+169n+112\right)S_4(n)}{9n(n+1)(n+2)}\\
&+\left(\frac{64S_1(n)}{3(n+1)(n+2)}-\frac{128\left(n^3+9n^2-10n-6\right)}{9n^2(n+1)(n+2)}\right)S_{2,1}(n)+\frac{64S_{3,1}(n)}{3(n+1)(n+2)}+\frac{64\left(3n^2+7n+6\right)}{3n(n+1)(n+2)}S_{2,1,1}(n)\\
&+\zeta(2)\left(\frac{8S_1(n)^2}{3(n+1)(n+2)}+\frac{16\left(3n^3-n^2+30n+12\right)S_1(n)}{9n^2(n+1)(n+2)}-\frac{16\left(3n^3+2n^2+17n+6\right)}{9n(n+1)^2(n+2)}-\frac{8\left(4n^2+9n+8\right)S_2(n)}{3n(n+1)(n+2)}\right)+\\
&+\zeta(3)\left(\frac{448}{9(n+1)(n+2)}-\frac{448S_1(n)}{9(n+1)(n+2)}\right)
\end{aligned}
$$

Note: for our computations we used properties of

Harmonic sums (J. Vermaseren, J. Blümlein)

$$S_{2,1}(n) = \boxed{\sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}}$$

Note: for our computations we used properties of
Harmonic sums (J. Vermaseren, J. Blümlein)

$$S_{2,1}(n) = \boxed{\sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}}$$

Integral representation:

$$= \int_0^1 \frac{x^n - 1}{1 - x} \left(\int_0^x \frac{\int_0^y \frac{1}{1-z} dz}{y} dy - \zeta(2) \right) dx, \qquad \zeta(z) := \sum_{i=1}^{\infty} 1/i^z$$

Note: for our computations we used properties of

Harmonic sums (J. Vermaseren, J. Blümlein)

$$S_{2,1}(n) = \boxed{\sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}}$$

Integral representation:

$$= \int_0^1 \frac{x^n - 1}{1-x} \left(\int_0^x \frac{\int_0^y \frac{1}{1-z} dz}{y} dy - \zeta(2) \right) dx, \qquad \zeta(z) := \sum_{i=1}^{\infty} 1/i^z$$

Asymptotic expansion:

$$= \left(\frac{1}{30n^5} - \frac{1}{6n^3} + \frac{1}{2n^2} - \frac{1}{n} \right) \ln(n)$$

$$- \frac{1}{100n^5} - \frac{1}{6n^4} + \frac{13}{36n^3} - \frac{1}{4n^2} - \frac{1}{n} + 2\zeta(3) + O\left(\frac{\ln(n)}{n^6}\right).$$

limit computations **numerical evaluation**

Note: for our computations we used properties of
Generalization to cyclotomic harmonic sums (J. Ablinger, J. Blümlein, CS)

$$\boxed{\sum_{k=1}^{n}\frac{(-1)^k}{2k+1}} =$$

Integral representation:

$$=-(-1)^n\int_0^1\frac{x^{2n}}{x^2+1}dx+\frac{(-1)^n}{2n+1}-1+\frac{\pi}{4},$$

Asymptotic expansion:

$$=(-1)^n\left(-\tfrac{3}{64n^5}-\tfrac{1}{16n^4}+\tfrac{3}{16n^3}-\tfrac{1}{4n^2}+\tfrac{1}{4n}\right)+\tfrac{\pi}{4}-1+O(\tfrac{1}{n^6})$$

limit computations **numerical evaluation**

Discovering algebraic relations (J. Ablinger, J. Blümlein, CS)

multiple ζ–values

$$\sum_{i=1}^{\infty}\frac{1}{i^3}\sum_{j=1}^{i}\frac{(-1)^j}{j^2}\sum_{k=1}^{j}\frac{1}{k}$$

(Tremendous literature: M.E. Hoffman, D. Zagier,P. Cartier,
D.J. Broadhurst, D. Kreimer, M. Waldschmidt,
D.M. Bradley, J. Vermaseren, J. Bümlein, etc.)

computer algebra

cyclotomic ζ-values

$$\sum_{i=1}^{\infty}\frac{1}{i^3}\sum_{j=1}^{i}\frac{(-1)^j}{(2j+1)^2}\sum_{k=1}^{j}\frac{1}{k}$$

(up to weight 6)

$\longleftrightarrow$

ζ-values with roots of unity

$$\sum_{i=1}^{\infty}\frac{1}{i^3}\sum_{j=1}^{i}\frac{(-1)^{\frac{j}{3}}}{j^2}\sum_{k=1}^{j}\frac{1}{k}$$

(up weight 2, up to 20th root of unity)
(siehe auch P. Deligne/A. Goncharov, D.J. Broadhurst)

Concluding remarks

- Symbolic summation is ready for many classes of 3–loop problems

- Relies on complex analysis assisted by computer algebra: e.g., asymptotic expansions of nested sums and products

  Algorithms are implemented in J. Ablinger's HarmonicSums package

- Alternative summation methods are available
  (see, e.g., J. Blümlein, S. Klein, C.S., F. Stan; 2010)

General Tactic

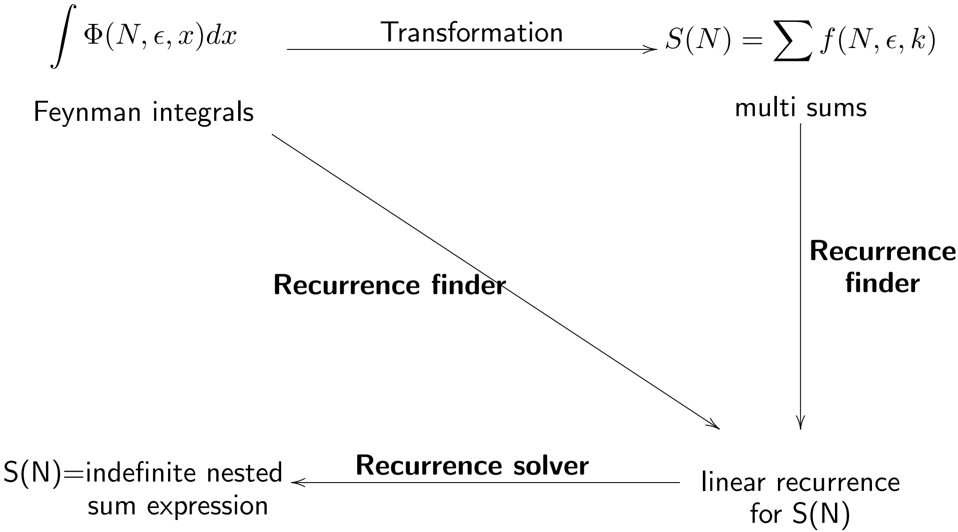
$$\int \Phi(N,\epsilon,x)dx \quad \xrightarrow{\text{Transformation}} \quad S(N)=\sum f(N,\epsilon,k)$$

Feynman integrals → multi sums

multi sums → **Recurrence finder** → linear recurrence for S(N)

linear recurrence for S(N) → **Recurrence solver** → S(N)=indefinite nested sum expression

General Tactic



Concluding remarks

- Symbolic summation is ready for many classes of 3–loop problems

- Relies on complex analysis assisted by computer algebra: e.g., asymptotic expansions of nested sums and products

  Algorithms are implemented in J. Ablinger's HarmonicSums package

- Alternative summation methods are available
  (see, e.g., J. Blümlein, S. Klein, C.S., F. Stan; 2010)

- Currently, alternative integration methods are developed
  (J. Blümlein, J. Ablinver, C.S.)