Symbolic Analysis

Foundations of Computational Mathematics (FoCM) 2011

Symbolic Summation in Perturbative Quantum Field Theory

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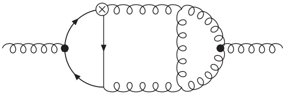
Evaluation of Feynman diagrams

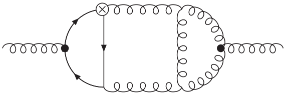
(joint project with J. Blümlein, Deutsches Elektronen–Synchrotron)



Behavior of particles

Evaluation of Feynman diagrams

(joint project with J. Blümlein, Deutsches Elektronen–Synchrotron)



Behavior of particles

$\longrightarrow$

$$\int \Phi(n,\epsilon,x)dx$$

Feynman integrals

Evaluation of Feynman diagrams

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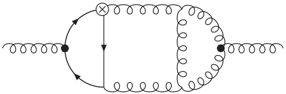


Behavior of particles

$$\longrightarrow \quad \int \Phi(n,\epsilon,x)dx$$

Feynman integrals

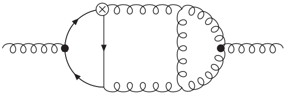
$\downarrow$ **DESY**

$$\sum f(n,\epsilon,k)$$

multi sums

Evaluation of Feynman diagrams

(joint project with J. Blümlein, Deutsches Elektronen–Synchrotron)



Behavior of particles

$$\int \Phi(n,\epsilon,x)dx$$

Feynman integrals

DESY

$$\sum f(n,\epsilon,k)$$

multi sums

symbolic summation

simple sum expressions

Evaluation of Feynman diagrams

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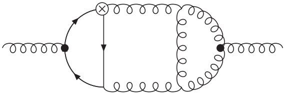
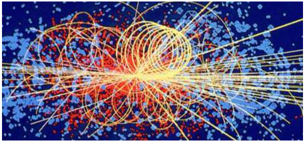


Behavior of particles

$$\int \Phi(n,\epsilon,x)dx$$

Feynman integrals



Evaluations required for the LHC experiment at CERN

DESY

$$\sum f(n,\epsilon,k)$$

multi sums

symbolic summation

simple sum expressions

processable by physicists

A warm up example: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i}(=H_n)$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

A warm up example: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!}+\frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j)=g(j+1)-g(j)}$$

$$f(j)=\frac{(j+1)!(j+k+n+1)!}{(j+k+1)!(j+n+1)!}\times$$
$$\times\ \frac{\frac{2j+k+n+2}{(j+k+1)(j+n+1)}-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)}{(j+1)(j+k+n+1)}$$

with shift in j:

A difference field for the **summand:** A **rational function field**

$$\mathbb{F}$$

and a **field automorphism** $\sigma:\mathbb{F}\to\mathbb{F}$ defined by

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, ,

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$$f(j) = \frac{(j+1)!(j+k+n+1)!}{(j+k+1)!(j+n+1)!} \times$$
$$\times \frac{\frac{2j+k+n+2}{(j+k+1)(j+n+1)} - S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n)}{(j+1)(j+k+n+1)}$$

with shift in j: Elements from $\mathbb{Q}(n,k)$ are constants

A difference field for the **summand:** A rational function field

$$\mathbb{F} := \mathbb{Q}(n,k)$$

and a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(\mathbf{c}) = \mathbf{c} \quad \forall \mathbf{c} \in \mathbb{Q}(\mathbf{n}, \mathbf{k}),$$

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$$f(j)=\frac{(j+1)!(j+k+n+1)!}{(j+k+1)!(j+n+1)!}\times$$
$$\times\ \frac{\frac{2j+k+n+2}{(j+k+1)(j+n+1)}-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)}{(j+1)(j+k+n+1)}$$

with shift in j: $\mathcal{S}\,j=j+1$

A difference field for the **summand:** A rational function field

$$\mathbb{F}{:=}\mathbb{Q}(n,k)\,(j)$$

and a field automorphism $\sigma:\mathbb{F}\to\mathbb{F}$ defined by

$$\sigma(c)=c\quad\forall c\in\mathbb{Q}(n,k),$$

$$\sigma(\mathbf{j})=\mathbf{j}+\mathbf{1},$$

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$$f(j)=\frac{(j+1)!(j+k+n+1)!}{(j+k+1)!(j+n+1)!}\times$$
$$\times\frac{\frac{2j+k+n+2}{(j+k+1)(j+n+1)}-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)}{(j+1)(j+k+n+1)}$$

with shift in j: $\mathcal{S}\,S_1(j)=S_1(j)+\frac{1}{j+1}$

A difference field for the **summand:** A rational function field

$$\mathbb{F}{:=}\mathbb{Q}(n,k)\ (j)\ (h_1)$$

and a field automorphism $\sigma:\mathbb{F}\to\mathbb{F}$ defined by

$$\sigma(c)=c\quad\forall c\in\mathbb{Q}(n,k),$$

$$\sigma(j)=j+1, \qquad ,$$

$$\sigma(\mathbf{h_1})=\mathbf{h_1}+\frac{\mathbf{1}}{\mathbf{j+1}}, \qquad ,$$

,

$$f(j) = \frac{(j+1)!(j+k+n+1)!}{(j+k+1)!(j+n+1)!} \times$$
$$\times \frac{\frac{2j+k+n+2}{(j+k+1)(j+n+1)} - S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n)}{(j+1)(j+k+n+1)}$$

with shift in j: $\mathcal{S}\, S_1(j+k) = S_1(j+k) + \frac{1}{j+k+1}$

A difference field for the **summand:** A rational function field

$$\mathbb{F} := \mathbb{Q}(n,k)\ (j)\ (h_1)\ (h_2)$$

and a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n,k),$$

$$\sigma(j) = j + 1, \qquad ,$$

$$\sigma(h_1) = h_1 + \frac{1}{j+1}, \qquad ,$$

$$\sigma(\mathbf{h_2}) = \mathbf{h_2} + \frac{\mathbf{1}}{\mathbf{j+k+1}},$$

$$f(j)=\frac{(j+1)!(j+k+n+1)!}{(j+k+1)!(j+n+1)!}\times$$
$$\times\frac{\frac{2j+k+n+2}{(j+k+1)(j+n+1)}-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)}{(j+1)(j+k+n+1)}$$

with shift in j: $\mathcal{S}\,S_1(j+n)=S_1(j+n)+\frac{1}{j+n+1}$

A difference field for the **summand:** A rational function field

$$\mathbb{F}:=\mathbb{Q}(n,k)\ (j)\ (h_1)\ (h_2)\ (h_3)$$

and a field automorphism $\sigma:\mathbb{F}\to\mathbb{F}$ defined by

$$\sigma(c)=c\quad\forall c\in\mathbb{Q}(n,k),$$

$$\sigma(j)=j+1,\qquad\qquad \sigma(\mathbf{h_3})=\mathbf{h_3}+\frac{\mathbf{1}}{\mathbf{j+n+1}},$$

$$\sigma(h_1)=h_1+\frac{1}{j+1},\qquad\qquad ,$$

$$\sigma(h_2)=h_2+\frac{1}{j+k+1},$$

$$f(j) = \frac{(j+1)!(j+k+n+1)!}{(j+k+1)!(j+n+1)!} \times$$
$$\times \frac{\frac{2j+k+n+2}{(j+k+1)(j+n+1)} - S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n)}{(j+1)(j+k+n+1)}$$

with shift in j: $\mathcal{S}\, S_1(j+n+k) = S_1(j+n+k) + \frac{1}{j+n+k+1}$

A difference field for the **summand:** A rational function field

$$\mathbb{F} := \mathbb{Q}(n,k)\ (j)\ (h_1)\ (h_2)\ (h_3)\ (h_4)$$

and a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n,k),$$

$$\sigma(j) = j+1,$$

$$\sigma(h_1) = h_1 + \frac{1}{j+1},$$

$$\sigma(h_2) = h_2 + \frac{1}{j+k+1},$$

$$\sigma(h_3) = h_3 + \frac{1}{j+n+1},$$

$$\boldsymbol{\sigma(h_4) = h_4 + \frac{1}{j+n+k+1}},$$

$$f(j)=\frac{(j+1)!(j+k+n+1)!}{(j+k+1)!(j+n+1)!}\times$$
$$\times\ \frac{\frac{2j+k+n+2}{(j+k+1)(j+n+1)}-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)}{(j+1)(j+k+n+1)}$$

with shift in j: $\mathcal{S}\,\frac{(j+1)!(j+k+n+1)!}{(j+k+1)!(j+n+1)!}=\frac{(j+2)(j+k+n+2)}{(j+k+2)(j+n+2)}\cdot\frac{(j+1)!(j+k+n+1)!}{(j+k+1)!(j+n+1)!}$

A difference field for the **summand:** A rational function field

$$\mathbb{F}{:=}\mathbb{Q}(n,k)\ (j)\ (h_1)\ (h_2)\ (h_3)\ (h_4)\ (p)$$

and a field automorphism $\sigma:\mathbb{F}\to\mathbb{F}$ defined by

Karr's $\Pi\Sigma$-fields (1981)

$$\sigma(c)=c\quad\forall c\in\mathbb{Q}(n,k),$$

$$\sigma(j)=j+1,\qquad \sigma(h_3)=h_3+\frac{1}{j+n+1},$$

$$\sigma(h_1)=h_1+\frac{1}{j+1},\qquad \sigma(h_4)=h_4+\frac{1}{j+n+k+1},$$

$$\sigma(h_2)=h_2+\frac{1}{j+k+1},\qquad \boldsymbol{\sigma(\mathbf{p})=\frac{(\mathbf{j+2})(\mathbf{j+k+n+2})}{(\mathbf{j+k+2})(\mathbf{j+n+2})}\,\mathbf{p}}.$$

$$f(j)=\frac{(j+1)!(j+k+n+1)!}{(j+k+1)!(j+n+1)!}\times$$
$$\times\ \frac{\frac{2j+k+n+2}{(j+k+1)(j+n+1)}-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)}{(j+1)(j+k+n+1)}$$

FIND $g(j)$:

$$f(j)=g(j+1)-g(j)$$

$$\downarrow$$

GIVEN $f:=p\times\frac{\frac{2j+k+n+2}{(j+k+1)(j+n+1)}-h_1+h_2+h_3-h_4}{(j+1)(j+k+n+1)}$

FIND $g\in\mathbb{F}$:

$$\boxed{f=\sigma(g)-g}$$

$$f(j) = \frac{(j+1)!(j+k+n+1)!}{(j+k+1)!(j+n+1)!} \times$$
$$\times \frac{\frac{2j+k+n+2}{(j+k+1)(j+n+1)} - S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n)}{(j+1)(j+k+n+1)}$$

FIND $g(j)$:

$$f(j) = g(j+1) - g(j)$$

$$\downarrow$$

GIVEN $f := p \times \frac{\frac{2j+k+n+2}{(j+k+1)(j+n+1)} - h_1 + h_2 + h_3 - h_4}{(j+1)(j+k+n+1)}$

FIND $g \in \mathbb{F}$:

$$\boxed{f = \sigma(g) - g}$$

$\downarrow$ recursive Ansatz

$$g = p \times \frac{(h_1 - h_2 - h_3 + h_4)(j+k+1)(j+n+1)}{(j+1)kn(j+k+n+1)}$$

$$f(j)=\frac{(j+1)!(j+k+n+1)!}{(j+k+1)!(j+n+1)!}\times$$
$$\times\frac{\frac{2j+k+n+2}{(j+k+1)(j+n+1)}-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)}{(j+1)(j+k+n+1)}$$

FIND $g(j)$:

$$f(j)=g(j+1)-g(j)$$

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GIVEN $f:=p\times\frac{\frac{2j+k+n+2}{(j+k+1)(j+n+1)}-h_1+h_2+h_3-h_4}{(j+1)(j+k+n+1)}$

FIND $g\in\mathbb{F}$:

$$\boxed{f=\sigma(g)-g}$$

$h_1\equiv S_1(j)$

$h_2\equiv S_1(j+k)$

$\vdots$

$\downarrow$ recursive Ansatz

$$g=p\times\frac{(h_1-h_2-h_3+h_4)(j+k+1)(j+n+1)}{(j+1)kn(j+k+n+1)}$$

A warm up example: Simplify

$$\underbrace{\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

$$\uparrow$$

$$g(j) = \frac{(j+k+1)(j+n+1)j!k!(j+k+n)!\Big(S_1(j)-S_1(j+k)-S_1(j+n)+S_1(j+k+n)\Big)}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

A warm up example: Simplify

$$\underbrace{\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j)=g(j+1)-g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^{a} f(n,k,j) = g(a+1)-g(0)$$

$$=\underbrace{\frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a)-S_1(a+k)-S_1(a+n)+S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}$$

$a \to \infty$

A warm up example: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{j=0}^{\infty} f(n,k,j) = \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!}$$

A warm up example: Simplify

$$\sum_{k=0}^{\infty}\underbrace{\sum_{j=0}^{\infty}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}f(n,k,j)=\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

summation paradigms

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

summation paradigms

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} \in \left\{ c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)} \middle| c \in \mathbb{R} \right\}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

summation paradigms

$$\boxed{A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}} = \boxed{\frac{1}{2} \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)}}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

A warm up example: Simplify

$$\sum_{k=0}^{\infty}\underbrace{\sum_{j=0}^{\infty}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(n,k,j) = \sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!}$$

$$= \frac{S_1(n)^2+S_2(n)}{2n(n+1)!}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n)=\sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

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FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n),\dots,a_d(n),h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n)+\dots+a_d(n)A(n+d)=h(n);$$

FIND all solutions expressible by indefinite nested products and sums
(Abramov/Bronstein/Petkovšek/Schneider, in preparation)

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FIND all solutions expressible by indefinite nested products and sums
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NOTE: By construction, the solutions are highly nested.

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

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2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n) + \dots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products and sums
(Abramov/Bronstein/Petkovšek/Schneider, in preparation)

3. Indefinite summation for simplification

A difference field approach (M. Karr, 1981)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with $f \in \mathbb{F}$.

FIND $g \in \mathbb{F}$:

$$\sigma(g) - g = f.$$

A symbolic summation approach

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F},\sigma)$ with $f\in\mathbb{F}$.

2. FIND $g\in\mathbb{F}$:

$$\sigma(g)-g=f.$$

A symbolic summation approach

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with $f \in \mathbb{F}$.

2. FIND an appropriate extension $\mathbb{E} > \mathbb{F}$ with $g \in \mathbb{E}$:

$$\sigma(g) - g = f.$$

A symbolic summation approach

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F},\sigma)$ with $f\in\mathbb{F}$.

2. FIND an appropriate extension $\mathbb{E}>\mathbb{F}$ with $g\in\mathbb{E}$:

$$\sigma(g)-g=f.$$

appropriate = sum representations with optimal nesting depth

Example:

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{i}}{j}}{k}=\frac{1}{6}\left(\sum_{i=1}^{n}\frac{1}{i}\right)^3+\frac{1}{2}\left(\sum_{i=1}^{n}\frac{1}{i^2}\right)\left(\sum_{i=1}^{n}\frac{1}{i}\right)+\frac{1}{3}\sum_{i=1}^{n}\frac{1}{i^3}$$

depth 3 | depth 1

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
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FIND a recurrence for $A(n)$

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GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n) + \dots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products and sums
(Abramov/Bronstein/Petkovšek/Schneider, in preparation)

4. Find a "closed form"

A(n)=combined solutions.

"Background of our 3–loop computations"

- The following examples arise in the context of 2– and 3–loop massive single scale Feynman diagrams with operator insertion.
- These are related to the QCD anomalous dimensions and massive operator matrix elements.
- At 2-loop order all respective calculations are finished:

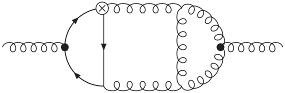
M. Buza, Y. Matiounine, J. Smith, R. Migneron, W.L. van Neerven, Nucl. Phys. **B472** (1996) 611;
I. Bierenbaum, J. Blümlein, S. Klein, Nucl. Phys. **B780** (2007) 40;
I. Bierenbaum, J. Blümlein, S. Klein, C. Schneider, Nucl.Phys. **B803** (2008)

and lead to representations in terms of harmonic sums.

Main–Example: All n-Results for 3–Loop Ladder Graphs

Joint work with J. Ablinger (RISC), J. Blümlein (DESY),
A. Hasselhuhn (DESY), S. Klein (RWTH)

In total around 50 diagrams (for this class) have been calculated, like e.g.



(containing three massive fermion propagators)

$$\Downarrow$$

Around 1000 sums have to be calculated for this diagram

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$||$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\boxed{\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^{r}(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\|$$

$$\boxed{\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\|$$

$$\boxed{\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Bigg(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Bigg)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\|$$

$$\sum_{j=0}^{n-2}\Bigg(\Bigg(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Bigg)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$\|$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}},\ a\in\mathbb{Z}\setminus\{0\}.$ Sigma

A typical sum

$$\sum_{j=0}^{n-2}\sum_{s=1}^{j+1}\sum_{r=0}^{n+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+n+s-2}{r}(n-j)!(s-1)!\sigma!S_1(r+2)}{(n-r)(r+1)(r+2)(-j+n+\sigma+1)(-j+n+\sigma+2)(-j+n+s+\sigma)!}$$

A typical sum

$$\sum_{j=0}^{n-2}\sum_{s=1}^{j+1}\sum_{r=0}^{n+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+n+s-2}{r}(n-j)!(s-1)!\sigma!S_1(r+2)}{(n-r)(r+1)(r+2)(-j+n+\sigma+1)(-j+n+\sigma+2)(-j+n+s+\sigma)!}$$

$$=\frac{\left(2n^2+6n+5\right)S_{-2}(n)^2}{2(n+1)(n+2)}+S_{-2,-1,2}(n)+S_{-2,1,-2}(n)$$
$$+\dots$$

where, e.g.,

$$S_{-2,1,-2}(n)=\sum_{i=1}^{n}\frac{(-1)^i\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{(-1)^k}{k^2}}{j}}{i^2}$$

Vermaseren 98/Blümlein/Kurth 99

A typical sum

$$\sum_{j=0}^{n-2}\sum_{s=1}^{j+1}\sum_{r=0}^{n+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+n+s-2}{r}(n-j)!(s-1)!\sigma!S_1(r+2)}{(n-r)(r+1)(r+2)(-j+n+\sigma+1)(-j+n+\sigma+2)(-j+n+s+\sigma)!}$$

$$=\frac{(2n^2+6n+5)\,S_{-2}(n)^2}{2(n+1)(n+2)}+S_{-2,-1,2}(n)+S_{-2,1,-2}(n)$$

$$+\dots-S_{2,1,1,1}(-1,2,\frac{1}{2},-1;n)+S_{2,1,1,1}(1,\frac{1}{2},1,2;n)$$

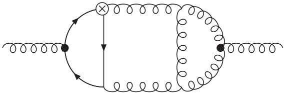
$$+\dots$$

where, e.g.,

145 S–sums occur

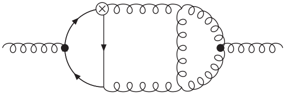
$$S_{2,1,1,1}(1,\frac{1}{2},1,2;n)=\sum_{i=1}^{n}\frac{\sum_{j=1}^{i}\frac{\left(\frac{1}{2}\right)^j\sum_{k=1}^{j}\frac{\sum_{l=1}^{k}\frac{2^l}{l}}{k}}{j}}{i^2}$$

S. Moch, P. Uwer, S. Weinzierl 02



$\Downarrow$ Sigma.m

Around 1000 sums are calculated containing in total 533 S–sums



$\Downarrow$ Sigma.m

Around 1000 sums are calculated containing in total 533 S–sums

$\Downarrow$ J. Ablinger's HarmonicSum.m

After elimination the following sums remain:

$$S_{-4}(n), S_{-3}(n), S_{-2}(n), S_{1}(n), S_{2}(n), S_{3}(n), S_{4}(n), S_{-3,1}(n),$$
$$S_{-2,1}(n), S_{2,-2}(n), S_{2,1}(n), S_{3,1}(n), S_{-2,1,1}(n), S_{2,1,1}(n)$$

For 3–loop ladder graphs we dealt (so far) with up to 6–fold sums. E.g.,

$$\sum_{l=2}^{N}\sum_{j=2}^{l}\sum_{k=1}^{j}\sum_{r=0}^{l-k}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{2(-1)^{j+k+l+r}(k-1)!\binom{j}{k}\binom{l}{j}\binom{l-k}{r}\binom{N}{l}}{(N-1)N(k+m+n+r+2)(k+m)!}$$
$$\frac{(k+m-1)!(N-j)!(l+r-2)!(n+r+1)!\,(k+m+n+r-1)!}{(-j+N+2)!(k+r-1)!(l+n+r-1)!\,(k+m+n+r+1)!}$$

$$\begin{aligned}=\frac{1}{N(N+1)(N+2)}\Big(&2\big((3-2^{N+3})-(-1)^N\big)\zeta_3\\
&+\tfrac{1}{6}S_1(N)^3+\tfrac{4(2N+3)}{(N+1)^2(N+2)}S_1(N)+\tfrac{8(2N+3)}{(N+1)^3(N+2)}\\
&-\frac{-56-40N-3N^2+2S_1(N)+3NS_1(N)+N^2S_1(N)}{2(1+N)(2+N)}S_2(N)\\
&+\tfrac{\left(16+12N+N^2\right)}{2(1+N)\,(2+N)}S_1(N)^2+\tfrac{1}{3}(-3N-17)S_3(N)\\
&-(-1)^NS_{-3}(N)+(-N-3)S_{2,1}(N)-2(-1)^NS_{-2,1}(N)\\
&+2^{N+4}S_{1,2}\big(\tfrac{1}{2},1,N\big)+2^{N+3}S_{1,1,1}\big(\tfrac{1}{2},1,1,N\big)\Big)\end{aligned}$$

and ...

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+N-3}\sum_{s=1}^{-l+N-q-3}\sum_{r=0}^{-l+N-q-s-3}$$
$$\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{N-1}{j+2}\binom{-j+N-3}{q}\binom{-l+N-q-3}{s}\binom{-l+N-q-s-3}{r}r!(-l+N-q-r-s-3)!(s-1)!}{(-l+N-q-2)!}$$
$$\Bigg(\frac{\frac{2\ (-1)^{-j+k-l+N-q-3}(2S_1(-j+N-1)-S_1(-j+N-2)}{-j+N-1}-\frac{(-1)^{-j+k-l+N-q-3}S_1(k)}{-j+N-1}}{(N-q-r-s-2)(q+s+1)}$$
$$-\frac{(-1)^{-j+k-l+N-q-3}(S_1(-l+N-q-2)+S_1(-l+N-q-r-s-3)-2S_1(r+s))}{(-j+N-1)(N-q-r-s-2)(q+s+1)}$$
$$+\frac{2(-1)^{-j+k-l+N-q-3}(S_1(s-1)-S_1(r+s))}{(-j+N-1)(N-q-r-s-2)(q+s+1)}\Bigg)$$

$=$ polynomial expression in terms of 49 harmonic sums and S–sums

Concluding remarks

- Symbolic summation is ready for many classes of 3–loop problems
- Relies on complex analysis assisted by computer algebra: e.g., asymptotic expansions of nested sums and products
  (see, e.g., J. Ablinger, J. Blümlein, C.S. 2011)
- Alternative summation methods are available
  (see, e.g., J. Blümlein, S. Klein, C.S., F. Stan; 2010)