Discrete mathematics seminar
Faculty of Mathematics and Physics

University of Ljubljana

Symbolic Summation and its Application in Particle Physics

Carsten Schneider
RISC, J. Kepler University Linz, Austria

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Indefinite summation

Simplify

$$\sum_{k=0}^{a}\left(1+(n-2k)\,S_1(k)\right)\binom{n}{k} = \, ?,$$

where $S_1(k) := \sum_{i=1}^{k}\frac{1}{i} \;\; (= H_k)$.

GIVEN $f(k) = \big(1 + (n - 2k)\, S_1(k)\big) \binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

GIVEN $f(k) = \big(1 + (n - 2k)\, S_1(k)\big)\binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

Sigma computes

$$g(k) = (k\, S_1(k) - 1)\binom{n}{k}$$

GIVEN $f(k) = \big(1 + (n-2k)\,S_1(k)\big)\binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

Summing the telescoping equation over k from 0 to a gives

$$\boxed{\sum_{k=0}^{a}\big(1 + (n-2k)\,S_1(k)\big)\binom{n}{k}} = \boxed{g(a+1) - g(0)}$$

$$= 1 + (n-a)S_1(a)\binom{n}{a}.$$

GIVEN $f(k) = \big(1 + (\mathbf{n} - 2k)\, S_1(k)\big)\binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

A difference field for the **summand:**

Construct a rational function field

$$\mathbb{F} := \mathbb{Q}(\mathbf{n})$$

and a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(\mathbf{c}) = \mathbf{c} \quad \forall \mathbf{c} \in \mathbb{Q}(\mathbf{n}),$$

GIVEN $f(k) = \big(1 + (n - 2\mathbf{k})\, S_1(k)\big) \binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

A difference field for the **summand:**

Construct a rational function field

$$\mathbb{F} := \mathbb{Q}(n)(\mathbf{k})$$

and a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),$$

$$\sigma(\mathbf{k}) = \mathbf{k} + \mathbf{1}, \qquad\qquad \mathcal{S}\, k = k + 1,$$

GIVEN $f(k) = \left(1 + (n-2k)\,\mathbf{S_1(k)}\right)\binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

A difference field for the **summand:**

Construct a rational function field

$$\mathbb{F} := \mathbb{Q}(n)(k)(\mathbf{h})$$

and a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

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$$\sigma(\mathbf{h}) = \mathbf{h} + \frac{\mathbf{1}}{\mathbf{k+1}}, \qquad \mathcal{S}\,S_1(k) = S_1(k) + \frac{1}{k+1},$$

GIVEN $f(k) = \left(1 + (n-2k)\, S_1(k)\right) \binom{\mathbf{n}}{\mathbf{k}}$

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A difference field for the **summand:**

Construct a rational function field

$(\mathbb{F}, \sigma)$ is a $\Pi\Sigma$-field

$$\mathbb{F} := \mathbb{Q}(n)(k)(h)(\mathbf{b})$$

Karr 1981

and a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

$$\sigma(h) = h + \frac{1}{k+1}, \qquad \mathcal{S}\, S_1(k) = S_1(k) + \frac{1}{k+1},$$

$$\sigma(\mathbf{b}) = \frac{\mathbf{n} - \mathbf{k}}{\mathbf{k} + \mathbf{1}}\, \mathbf{b}, \qquad \mathcal{S} \binom{n}{k} = \frac{n-k}{k+1} \binom{n}{k}.$$

GIVEN $f(k) = \big(1 + (n-2k)\,S_1(k)\big)\binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

$$\downarrow$$

GIVEN $f := (1 + (n-2k)h)b \in \mathbb{F}$.
FIND $g \in \mathbb{F}$:

$$\boxed{f = \sigma(g) - g}$$

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$\downarrow$ Sigma

$$g = (k\,h - 1)b$$

GIVEN $f(k)=\left(1+(n-2k)\,S_1(k)\right)\binom{n}{k}$

FIND $g(k)$:

$$f(k)=g(k+1)-g(k)$$

$$\downarrow$$

GIVEN $f:=(1+(n-2k)h)b\in\mathbb{F}$.
FIND $g\in\mathbb{F}$:

$$f=\sigma(g)-g$$

$\downarrow$ Sigma

$$g=(k\,h-1)b$$

$h\equiv S_1(k)$
$b\equiv\binom{n}{k}$

A family of identities discovered by S. Ahlgren (see Paule/Schneider 03):

$$\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha}=?$$

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$\alpha=1$:

$$\sum_{k=0}^{a}\left(1+(n-2k)S_1(k)\right)\binom{n}{k}=1+(n-a)S_1(a)\binom{n}{a}$$

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$$\sum_{k=0}^{n}\left(1+(n-2k)S_1(k)\right)\binom{n}{k} = 1$$

$\alpha = 2$:

$$\sum_{k=0}^{a}\left(1+2(n-2k)S_1(k)\right)\binom{n}{k}^{2} = \frac{(a-n)^2(1+2nS_1(a))}{n^2}\binom{n}{a}$$

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$\alpha=3$:
$$\sum_{k=0}^{n}\left(1+3(n-2k)S_1(k)\right)\binom{n}{k}^{3}=?$$

$\alpha=4$:
$$\sum_{k=0}^{n}\left(1+4(n-2k)S_1(k)\right)\binom{n}{k}^{4}=?$$

$\alpha=5$:
$$\sum_{k=0}^{n}\left(1+5(n-2k)S_1(k)\right)\binom{n}{k}^{5}=?$$

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Telescoping

GIVEN

$$\text{SUM}(n) := \sum_{k=0}^{n} \underbrace{(1+5(n-2k)S_1(k))\binom{n}{k}^5}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{\qquad f(n,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

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$$\text{SUM}(n) := \sum_{k=0}^{n} \underbrace{(1+5(n-2k)S_1(k))\binom{n}{k}^5}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)\,f(n+1,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

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$$\mathsf{SUM}(n) := \sum_{k=0}^{n} \underbrace{\big(1 + 5(n-2k)S_1(k)\big)\binom{n}{k}^5}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

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for all $0 \leq k \leq n$ and all $n \geq 0$.

☺ No solutions implies that the sequences

$$\langle S_1(a)\rangle_{a\geq 0}, \quad \langle\binom{n}{a}\rangle_{a\geq 0}, \quad \langle\sum_{k=0}^{a} f(n,k)\rangle_{a\geq 0}, \dots, \langle\sum_{k=0}^{a} f(n+3,k)\rangle_{a\geq 0} \in \mathbb{Q}(n)^{\mathbb{N}}$$

are algebraically independent over the field of rational sequences.

For more details see: Parameterized telescoping proves algebraic independence of sums, Annal. Comb. 2011

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$$\text{SUM}(n) := \sum_{k=0}^{n} \underbrace{(1+5(n-2k)S_1(k))\binom{n}{k}^5}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n), c_3(n), c_4(n)$

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+\dots+c_4(n)\,f(n+4,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Sigma *computes*: $c_0(n),\dots,c_4(n) \in \mathbb{Q}[n]$

$$g(n,k) := \binom{n}{k}^5 \frac{p_1(k,n,S_1(k))}{(k-n-4)^5(k-n-3)^5(k-n-2)^5(k-n-1)^5},$$

$$g(n,k+1) := \binom{n}{k}^5 \frac{p_2(k,n,S_1(k))}{(k-n-3)^5(k-n-2)^5(k-n-1)^5}.$$

Zeilberger's creative telescoping paradigm

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for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 0 to n gives:

$$\boxed{g(n,n+1)-g(n,0)} =$$

$$\boxed{\begin{array}{l} c_0(n)\,\mathsf{SUM}(n)+ \\ c_1(n)\left[\mathsf{SUM}(n+1)-f(n+1,n+1)\right] \\ \vdots \\ c_4(n)\left[\mathsf{SUM}(n+4)-f(n+4,n+1)-f(n+4,n+2)-\dots-f(n+4,n+4)\right]. \end{array}}$$

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A family of identities discovered by S. Ahlgren (see Paule/Schneider 03):

$$\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha}=?$$

extended → Krattenthaler/Rivoal 07

$\alpha=1$:
$$\sum_{k=0}^{n}\left(1+(n-2k)S_1(k)\right)\binom{n}{k}=1$$

$\alpha=2$:
$$\sum_{k=0}^{n}\left(1+2(n-2k)S_1(k)\right)\binom{n}{k}^2=0$$

$\alpha=3$:
$$\sum_{k=0}^{n}\left(1+3(n-2k)S_1(k)\right)\binom{n}{k}^3=(-1)^n$$

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$$\sum_{k=0}^{n}\left(1+4(n-2k)S_1(k)\right)\binom{n}{k}^4=(-1)^n\binom{2\,n}{n}$$

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$$\sum_{k=0}^{n}\left(1+5(n-2k)S_1(k)\right)\binom{n}{k}^5=(-1)^n\sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{j}$$

The other direction:

$$\boxed{\sum_{k=0}^{n}\big(1+\alpha(n-2k)S_1(k)\big)\binom{n}{k}^{\alpha}=?}$$

$\alpha=-1$:
$$\sum_{k=0}^{a}\big(1-(n-2k)S_1(k)\big)\binom{n}{k}^{-1}=\frac{(a+1)S_1(a)+1}{\binom{n}{a}}$$

The other direction:

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The other direction:

$$\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha} =?$$

$\alpha=-1$:

$$\sum_{k=0}^{n}\left(1-(n-2k)S_1(k)\right)\binom{n}{k}^{-1}=(n+1)S_1(n)+1$$

$\alpha=-2$:

$$\sum_{k=0}^{a}(1-2(n-2k)S_1(k))\binom{n}{k}^{-2}$$

$$=\frac{(n+1)^2}{(n+2)^2}+\frac{(a+1)(-a+2n+2(a+1)(n+2)S_1(a)+3)}{(n+2)^2\binom{n}{a}^{-2}}$$

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$$\sum_{k=0}^{n}\left(1-2(n-2k)S_1(k)\right)\binom{n}{k}^{-2}$$
$$=\frac{(n+1)^2}{(n+2)^2}+\frac{\left(n+2\left(n^2+3n+2\right)S_1(n)+3\right)(n+1)}{(n+2)^2}$$

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$\alpha=-3$:

$$\sum_{k=0}^{n}(1-3(n-2k)S_1(k))\binom{n}{k}^{-3}=\ ?$$

The other direction:

$$\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha}=?$$

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$\alpha=-3$:

$$\sum_{k=0}^{n}\left(1-3(n-2k)S_1(k)\right)\binom{n}{k}^{-3}=$$

$$=5(-1)^nS_{-3}(n)(n+1)^3$$

$$-6(-1)^nS_{-2,1}(n)(n+1)^3+6S_1(n)(n+1)+1$$

The other direction:

$$\boxed{\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha}=?}$$

$\alpha=-4$:

$$\sum_{k=0}^{n}\left(1-4(n-2k)S_1(k)\right)\binom{n}{k}^{-4}=\frac{(10(n+1)S_1(n)+3)(n+1)}{2n+3}$$
$$+\frac{(-1)^n\binom{2n}{n}^{-1}(n+1)^5}{(4n(n+2)+3)}\left(\frac{7}{2}\sum_{i=1}^{n}\frac{(-1)^i\binom{2i}{i}}{i^3}-5\sum_{i=1}^{n}\frac{(-1)^i\binom{2i}{i}S_1(i)}{i^2}\right)$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$S(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $S(n)$

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FIND a recurrence for $S(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)S(n) + \cdots + a_d(n)S(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products and sums
(Abramov/Bronstein/Petkovšek/Schneider, in preparation right now)

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NOTE: By construction, the solutions are highly nested.

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3. Indefinite summation (by Sigma's refined summation theory of $\Pi\Sigma^*$-fields)

Simplify the solutions:

- The sums have minimal nested depth.
- No algebraic relations occur among the sums.

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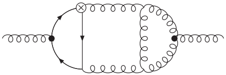
4. Find a "closed form"

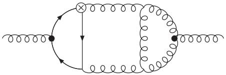
S(n)=combined solutions.

Evaluation of Feynman Integrals



Feynman diagrams

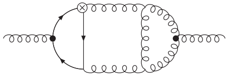
Evaluation of Feynman Integrals



Feynman diagrams $\longrightarrow$ $$\int \Phi(N, \epsilon, x) dx$$

Feynman integrals

Evaluation of Feynman Integrals



Feynman diagrams

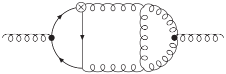
$$\int \Phi(N, \epsilon, x) dx$$

Feynman integrals

Reduction

multi-sums with
upper bound N

Evaluation of Feynman Integrals



Feynman diagrams

$$\int \Phi(N,\epsilon,x)dx$$

Feynman integrals

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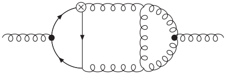
multi-sums with upper bound N

Sigma

sum expressions being processable by physicists

Evaluation of Feynman Integrals

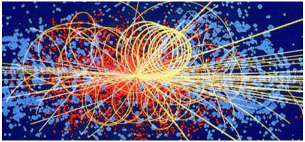


Feynman diagrams

$$\int \Phi(N, \epsilon, x)dx$$

Feynman integrals

Reduction



multi-sums with upper bound N

Sigma

sum expressions
being processable by physicists

"Background of our 3–loop computations"

- The following examples arise in the context of 2– and 3–loop massive single scale Feynman diagrams with operator insertion.
- These are related to the QCD anomalous dimensions and massive operator matrix elements.
- At 2-loop order all respective calculations are finished:

M. Buza, Y. Matiounine, J. Smith, R. Migneron, W.L. van Neerven, Nucl. Phys. **B472** (1996) 611;

I. Bierenbaum, J. Blümlein, S. Klein, Nucl. Phys. **B780** (2007) 40;

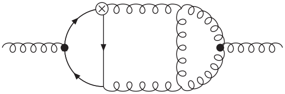
I. Bierenbaum, J. Blümlein, S. Klein, C. Schneider, Nucl.Phys. **B803** (2008)

and lead to representations in terms of harmonic sums.

Example 1: All N-Results for 3–Loop Ladder Graphs

Joint work with J. Ablinger (RISC), J. Blümlein (DESY),
A. Hasselhuhn (DESY), S. Klein (RWTH)

Consider, e.g., the diagram



(containing three massive fermion propagators)

$$\Downarrow$$

Around 1000 sums have to be calculated

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

Simple sum

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

$$||$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{N-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}}$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

$$||$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{N-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}}$$

$$||$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^r(-j+N-2)!r!}{(N+1)(-j+N+r-1)(-j+N+r)!}+\frac{(-1)^{N+r}(j+1)!(-j+N-2)!(-j+N-1)_r r!}{(N-1)N(N+1)(-j+N+r)!(-j-1)_r(2-N)_j}\Big)}$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

$$||$$

$$\boxed{\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+N-2)!r!}{(N+1)(-j+N+r-1)(-j+N+r)!}+\frac{(-1)^{N+r}(j+1)!(-j+N-2)!(-j+N-1)_r r!}{(N-1)N(N+1)(-j+N+r)!(-j-1)_r(2-N)_j}\Big)}$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

$$||$$

$$\sum_{j=0}^{N-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+N-2)!r!}{(N+1)(-j+N+r-1)(-j+N+r)!}+\frac{(-1)^{N+r}(j+1)!(-j+N-2)!(-j+N-1)_r r!}{(N-1)N(N+1)(-j+N+r)!(-j-1)_r(2-N)_j}\Big)}$$

$$||$$

$$\boxed{\Big(\frac{N^2-N+1}{(N-1)^2N^2(N+1)(2-N)_j}+\frac{\sum_{i=1}^{j}\frac{(2-N)_i}{(-i+N-1)^2(i+1)!}}{(N+1)(2-N)_j}+\frac{(-1)^{j+N}(-j-2)(-j+N-2)!}{(j-N+1)(N+1)^2N!}\Big)(j+1)!-\frac{1}{(N+1)^2(-j+N-1)}}$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

$$||$$

$$\sum_{j=0}^{N-2}\Bigg(\Big(\frac{N^2-N+1}{(N-1)^2N^2(N+1)(2-N)_j}+\frac{\sum_{i=1}^{j}\frac{(2-N)_i}{(-i+N-1)^2(i+1)!}}{(N+1)(2-N)_j}+\frac{(-1)^{j+N}(-j-2)(-j+N-2)!}{(j-N+1)(N+1)^2N!}\Big)(j+1)!-\frac{1}{(N+1)^2(-j+N-1)}\Bigg)$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

$$||$$

$$\sum_{j=0}^{N-2}\Bigg(\Bigg(\frac{N^2-N+1}{(N-1)^2N^2(N+1)(2-N)_j}+\frac{\sum_{i=1}^{j}\frac{(2-N)_i}{(-i+N-1)^2(i+1)!}}{(N+1)(2-N)_j}+\frac{(-1)^{j+N}(-j-2)(-j+N-2)!}{(j-N+1)(N+1)^2N!}\Bigg)(j+1)!-\frac{1}{(N+1)^2(-j+N-1)}\Bigg)$$

$$||$$

$$\frac{-N^2-N-1}{N^2(N+1)^3}+\frac{(-1)^N\left(N^2+N+1\right)}{N^2(N+1)^3}-\frac{2S_{-2}(N)}{N+1}+\frac{S_1(N)}{(N+1)^2}+\frac{S_2(N)}{-N-1}$$

Note: $S_a(N)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}}$, $a\in\mathbb{Z}\setminus\{0\}$.

Sigma

A typical sum

$$\sum_{j=0}^{N-2}\sum_{s=1}^{j+1}\sum_{r=0}^{N+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+N+s-2}{r}(N-j)!(s-1)!\sigma!S_1(r+2)}{(N-r)(r+1)(r+2)(-j+N+\sigma+1)(-j+N+\sigma+2)(-j+N+s+\sigma)!}$$

A typical sum

$$\sum_{j=0}^{N-2}\sum_{s=1}^{j+1}\sum_{r=0}^{N+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+N+s-2}{r}(N-j)!(s-1)!\sigma!S_1(r+2)}{(N-r)(r+1)(r+2)(-j+N+\sigma+1)(-j+N+\sigma+2)(-j+N+s+\sigma)!}$$

$$=\frac{\left(2N^2+6N+5\right)S_{-2}(N)^2}{2(N+1)(N+2)}+S_{-2,-1,2}(N)+S_{-2,1,-2}(N)$$
$$+\dots$$

where, e.g.,

$$S_{-2,1,-2}(N)=\sum_{i=1}^{N}\frac{(-1)^i\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{(-1)^k}{k^2}}{j}}{i^2}$$

Vermaseren 98/Blümlein/Kurth 99

A typical sum

$$\sum_{j=0}^{N-2}\sum_{s=1}^{j+1}\sum_{r=0}^{N+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+N+s-2}{r}(N-j)!(s-1)!\sigma!S_1(r+2)}{(N-r)(r+1)(r+2)(-j+N+\sigma+1)(-j+N+\sigma+2)(-j+N+s+\sigma)!}$$

$$=\frac{(2N^2+6N+5)\,S_{-2}(N)^2}{2(N+1)(N+2)}+S_{-2,-1,2}(N)+S_{-2,1,-2}(N)$$

$$+\dots-S_{2,1,1,1}(-1,2,\frac{1}{2},-1;N)+S_{2,1,1,1}(1,\frac{1}{2},1,2;N)$$

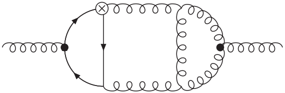
$$+\dots$$

where, e.g.,

145 S–sums occur

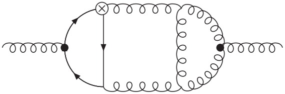
$$S_{2,1,1,1}(1,\frac{1}{2},1,2;N)=\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{\left(\frac{1}{2}\right)^j\sum_{k=1}^{j}\frac{\sum_{l=1}^{k}\frac{2^l}{l}}{k}}{j}}{i^2}$$

S. Moch, P. Uwer, S. Weinzierl 02



$\Downarrow$ Sigma.m

Around 1000 sums are calculated containing in total 533 S–sums



$\Downarrow$ Sigma.m

Around 1000 sums are calculated containing in total 533 S–sums

$\Downarrow$ J. Ablinger's HarmonicSum.m

After elimination the following sums remain:

$$S_{-4}(N), S_{-3}(N), S_{-2}(N), S_1(N), S_2(N), S_3(N), S_4(N), S_{-3,1}(N),$$
$$S_{-2,1}(N), S_{2,-2}(N), S_{2,1}(N), S_{3,1}(N), S_{-2,1,1}(N), S_{2,1,1}(N)$$

For 3–loop ladder graphs we dealt (so far) with up to 6–fold sums. E.g.,

$$\sum_{l=2}^{N}\sum_{j=2}^{l}\sum_{k=1}^{j}\sum_{r=0}^{l-k}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{2(-1)^{j+k+l+r}(k-1)!\binom{j}{k}\binom{l}{j}\binom{l-k}{r}\binom{N}{l}}{(N-1)N(k+m+n+r+2)(k+m)!}$$
$$\frac{(k+m-1)!(N-j)!(l+r-2)!(n+r+1)!\,(k+m+n+r-1)!}{(-j+N+2)!(k+r-1)!(l+n+r-1)!\,(k+m+n+r+1)!}$$

$$\begin{aligned}
=\frac{1}{N(N+1)(N+2)}\Big(2\big((3-2^{N+3}\big)-(-1)^N\Big)\zeta_3\\
+\tfrac{1}{6}S_1(N)^3+\tfrac{4(2N+3)}{(N+1)^2(N+2)}S_1(N)+\tfrac{8(2N+3)}{(N+1)^3(N+2)}\\
-\frac{-56-40N-3N^2+2S_1(N)+3NS_1(N)+N^2S_1(N)}{2(1+N)(2+N)}S_2(N)\\
+\tfrac{\left(16+12N+N^2\right)}{2(1+N)\ (2+N)}S_1(N)^2+\tfrac{1}{3}(-3N-17)S_3(N)\\
-(-1)^NS_{-3}(N)+(-N-3)S_{2,1}(N)-2(-1)^NS_{-2,1}(N)\\
+2^{N+4}S_{1,2}\left(\tfrac{1}{2},1,N\right)+2^{N+3}S_{1,1,1}\left(\tfrac{1}{2},1,1,N\right)\Big)
\end{aligned}$$

and ...

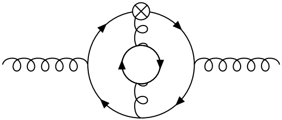
$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+N-3}\sum_{s=1}^{-l+N-q-3}\sum_{r=0}^{-l+N-q-s-3}$$
$$\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{N-1}{j+2}\binom{-j+N-3}{q}\binom{-l+N-q-3}{s}\binom{-l+N-q-s-3}{r}r!(-l+N-q-r-s-3)!(s-1)!}{(-l+N-q-2)!}$$
$$\Bigg(\frac{\frac{2\,(-1)^{-j+k-l+N-q-3}(2S_1(-j+N-1)-S_1(-j+N-2)}{-j+N-1}-\frac{(-1)^{-j+k-l+N-q-3}S_1(k)}{-j+N-1}}{(N-q-r-s-2)(q+s+1)}$$
$$-\frac{(-1)^{-j+k-l+N-q-3}(S_1(-l+N-q-2)+S_1(-l+N-q-r-s-3)-2S_1(r+s))}{(-j+N-1)(N-q-r-s-2)(q+s+1)}$$
$$+\frac{2(-1)^{-j+k-l+N-q-3}(S_1(s-1)-S_1(r+s))}{(-j+N-1)(N-q-r-s-2)(q+s+1)}\Bigg)$$

$=$ polynomial expression in terms of 49 harmonic sums and S–sums

Example 2: 3-Loop All N-Results for the N_f Contributions

Joint work with J. Ablinger (RISC), J. Blümlein (DESY), F. Wißbrock (DESY), S. Klein (RWTH)

E.g., for the diagram



768 sums are simplified.

Simple example:

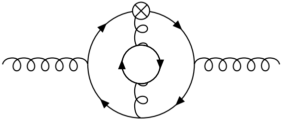
$$\sum_{j=1}^{N-2} \frac{j(j+1)(j+2)(N-j)(j-1)!^2(-j+N-1)!^2}{-j+N-1}$$
$$= \frac{\big(-N^3-5N^2-4N+6\big)(N!)^2}{(N-1)^2N^2}$$
$$+\frac{3}{2}\frac{(N!)^2\big(N^3+6N^2+11N+6\big)}{(N-1)N(2N+1)\binom{2N}{N}}\sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}.$$

Not expressible in terms of harmonic sums or S-sums!

The final expression is given in terms of 703 indefinite nested sums and products. Typical examples are:

$$\sum_{i=1}^{N} \frac{\displaystyle\sum_{j=1}^{i} \frac{\binom{2j}{j}}{j}}{\binom{2\,i}{i}}, \quad \sum_{i=1}^{N} \frac{S_1(i) \displaystyle\sum_{j=1}^{i} \frac{\binom{2j}{j}}{j}}{\binom{2i}{i}};$$

Sigma finds all algebraic relations among them. We get:



$$\Downarrow$$

$$
\begin{aligned}
&-\frac{20S(1,N)^4}{27(N+1)(N+2)}+\frac{32\left(6N^3+61N^2-21N+24\right)S_1(N)^3}{81N^2(N+1)(N+2)}-\frac{16\left(48N^5+746N^4+2697N^3+2746N^2+1104N+240\right)S_1(N)^2}{81N^2(N+1)^2(N+2)^2}\\
&+\frac{32\left(264N^7+4046N^6+21591N^5+52844N^4+74856N^3+66812N^2+30576N+2640\right)S_1(N)}{243N^2(N+1)^3(N+2)^3}\\
&-\frac{4\left(48N^2+101N+96\right)S_2(N)^2}{9N(N+1)(N+2)}-\frac{32\left(363N^7+6758N^6+41285N^5+121235N^4+190235N^3+150758N^2+46964N+2904\right)}{243N(N+1)^4(N+2)^3}\\
&+\Bigg(-\frac{40S_1(N)^2}{9(N+1)(N+2)}+\frac{32\left(6N^3+61N^2-21N+24\right)S_1(N)}{27N^2(N+1)(N+2)}\\
&-\frac{16\left(124N^5+198N^4-2387N^3-6162N^2-3632N-480\right)}{81N^2(N+1)^2(N+2)^2}\Bigg)S_2(N)+\\
&+\left(-\frac{32\left(9N^3-623N^2+894N+276\right)}{81N^2(N+1)(N+2)}-\frac{160S_1(N)}{27(N+1)(N+2)}\right)S_3(N)-\frac{8\left(56N^2+169N+112\right)S_4(N)}{9N(N+1)(N+2)}\\
&+\left(\frac{64S_1(N)}{3(N+1)(N+2)}-\frac{128\left(N^3+9N^2-10N-6\right)}{9N^2(N+1)(N+2)}\right)S_{2,1}(N)+\frac{64S_{3,1}(N)}{3(N+1)(N+2)}+\frac{64\left(3N^2+7N+6\right)}{3N(N+1)(N+2)}S_{2,1,1}(N)\\
&+\zeta_2\left(\frac{8S_1(N)^2}{3(N+1)(N+2)}+\frac{16\left(3N^3-N^2+30N+12\right)S_1(N)}{9N^2(N+1)(N+2)}-\frac{16\left(3N^3+2N^2+17N+6\right)}{9N(N+1)^2(N+2)}-\frac{8\left(4N^2+9N+8\right)S_2(N)}{3N(N+1)(N+2)}\right)+\\
&+\zeta_3\left(\frac{448}{9(N+1)(N+2)}-\frac{448S_1(N)}{9(N+1)(N+2)}\right)
\end{aligned}
$$