Advanced Computer Algebra for Determinants

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Abstract

We prove three conjectures concerning the evaluation of determinants, which are related to the counting of plane partitions and rhombus tilings. One of them has been posed by George Andrews in 1980, the other two are by Guoce Xin and Christian Krattenthaler. Our proofs employ computer algebra methods, namely the holonomic ansatz proposed by Doron Zeilberger and variations thereof. These variations make Zeilberger's original approach even more powerful and allow for addressing a wider variety of determinants. Finally we present, as a challenge problem, a conjecture about a closed form evaluation of Andrews's determinant.

1 Introduction

The concept of determinants evolved as early as 1545 when Girolamo Cardano tried to solve systems of linear equations. The mathematics community slowly realized the importance of determinants; we had to wait for more than 200 years before someone formally defined the term "determinant". It was first introduced by Carl Friedrich Gauß in his *disquisitiones arithmeticae* in 1801. There are many nice properties about determinants such as multiplicativity, invariance under row operations, Cramer's rule, etc. Every student nowadays learns how to compute the determinant of a specific given matrix, say with fixed dimension and containing numeric quantities as entries. On the other hand, there are lots of matrices with symbolic entries that have a nice closed-form formula for arbitrary dimension. The first example in the history of mathematics and still the most prominent one is the Vandermonde matrix.

Starting in the mid 1970's, the importance of evaluating determinants became even more significant when people related counting problems from combinatorics to the evaluation of certain determinants. Evaluating determinants

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of matrices whose dimension varies according to a parameter is a delicate problem but at the same time a very important one. Around the same time the field of computer algebra emerged. Doron Zeilberger was amongst the first mathematicians to realize the importance of computer algebra algorithms for combinatorial problems, special function identities, symbolic summation, and many more. In his paper [15] he has built the bridge between these two topics, namely symbolic determinant evaluation and computer algebra, and since then his *holonomic ansatz* has been successfully applied to many problems related to the evaluation of determinants. The most prominent one is probably a notorious conjecture from enumerative combinatorics, the so-called q-TSPP conjecture, which was the only remaining open problem from the famous list [13] by Richard Stanley (it also appeared in [11]) until it was recently proved [9] using Zeilberger's holonomic ansatz.

In this paper, we solve some of the problems that are listed in Christian Krattenthaler's complement [12] to his celebrated essay [11] (the attentive reader may already have observed that our title is an allusion to this reference). At the same time we show that the holonomic ansatz can be modified in various ways in order to apply it to particular problems that could not be addressed with the original method.

2 Zeilberger's Holonomic Ansatz

For sake of self-containedness we shortly describe the original holonomic ansatz for determinant evaluations as it was proposed by Zeilberger [15]. Its steps are completely automatic and produce a rigorous proof—provided that they can be successfully executed on the concrete example. In particular, the approach relies on the existence of a "nice" description for an auxiliary function (it appears as $c_{n,j}$ below); if such a description does not exist then the holonomic ansatz fails. That's why we call it an "approach" or an "ansatz", rather than an algorithm.

Generally speaking, Zeilberger's holonomic ansatz addresses determinant evaluations of the type

$$\det A_n = \det_{1\leqslant i,j\leqslant n}(a_{i,j}) = b_n \qquad (n \geqslant 1)$$

where the entries $a_{i,j}$ of the $n \times n$ matrix A_n and the (conjectured) evaluation b_n (where $b_n \neq 0$ is required for all $n \geqslant 1$) are explicitly given. The underlying principle is an induction argument on n. The base case $a_{1,1} = b_1$ is easily checked. Now assume that the determinant evaluation has been proven for $n-1$. In particular, it follows that $\det A_{n-1}$ is nonzero by the general assumption on b_n. Hence the rows of A_{n-1} are linearly independent and thus the linear system

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,n-1} & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{n,1} \\ \vdots \\ c_{n,n-1} \\ c_{n,n} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

has a unique solution. According to the well-known Laplace expansion formula

(here with respect to the last row)

$$\det_{1 \leqslant i,j \leqslant n} (a_{i,j}) = \sum_{j=1}^{n} (-1)^{n+j} M_{n,j} a_{n,j}$$

the expression $(-1)^{n+j} M_{n,j}$ is called the (n, j)-cofactor. The minor $M_{n,j}$ is the determinant of the matrix after removing the n-th row and the j-th column. The above linear system has been constructed in a way that the entry $c_{n,j}$ in its solution is precisely the (n, j)-cofactor of A_n divided by its (n, n)-cofactor (which is just the $(n-1)$-determinant). This fact can easily be seen by considering the matrix $A_n^{(i)}$ that we obtain from A_n by replacing the last row by the i-th row. For $1 \leqslant i < n$ the Laplace expansion of $A_n^{(i)}$ corresponds exactly to the i-th equation in our linear system. Now the determinant of A_n is given by

$$b_{n-1} \sum_{j=1}^{n} c_{n,j} a_{n,j}.$$

To complete the induction step it remains to show that this expression is equal to b_n.

The problem is that we cannot expect to obtain a closed-form expression for the quantity $c_{n,j}$ (otherwise we certainly would be able to derive a closed form for the determinant and we were done). Instead, we will guess an implicit, recursive definition for a bivariate sequence $c_{n,j}$ and then prove that it satisfies

$$c_{n,n} = 1 \qquad (n \geqslant 1), \tag{1}$$

$$\sum_{j=1}^{n} c_{n,j} a_{i,j} = 0 \qquad (1 \leqslant i < n), \tag{2}$$

$$\sum_{j=1}^{n} c_{n,j} a_{n,j} = \frac{b_n}{b_{n-1}} \qquad (n \geqslant 1). \tag{3}$$

From the first two identities which correspond to the linear system given above, it follows that our guessed $c_{n,j}$ is indeed the normalized (n, j)-cofactor. Identity (3) then certifies that the determinant evaluates to b_n. Hence the sequence $c_{n,j}$ plays the rôle of a certificate for the determinant evaluation.

Now what kind of implicit definition for $c_{n,j}$ could we think of? Of course, there was a good reason for Zeilberger to name his approach the "holonomic ansatz". That is to say because he had the class of holonomic functions (or better: sequences) in mind when he formulated his approach. In short, this class consists of multi-dimensional sequences that satisfy linear recurrence relations with polynomial coefficients, such that the sequence is uniquely determined by specifying *finitely many* initial values (we omit some additional technical conditions here). What makes the use of this class of sequences convenient is the fact that it is not only closed under the basic arithmetic operations (zero test, addition, multiplication), but also under more advanced operations such as specialization, diagonalization, and definite summation; all these are needed to prove (1), (2), and (3). For more details on holonomic functions and related algorithms see, e.g., [14, 2, 7]). Thus if the matrix entries $a_{i,j}$ are holonomic and if luckily the auxiliary function $c_{n,j}$ turns out to be holonomic, then the approach will succeed to produce a recurrence for the quotient b_n/b_{n-1}.

3 An Old Problem by George Andrews

In the context of enumerating certain classes of plane partitions, namely cyclically symmetric ones and descending ones, George Andrews [1] encountered an intriguing determinant which he posed as a challenging problem. In Christian Krattenthaler's survey [12], it appears as *Problem 34*. This conjecture does not even deal with a closed-form evaluation, but it is "only" about the quotient of two consecutive determinants; a situation that strongly suggests to employ Zeilberger's holonomic ansatz!

Theorem 1. *Let the determinant $D_1(n)$ be defined by*

$$D_1(n) := \det_{1\leqslant i,j\leqslant n}\left(\delta_{i,j} + \binom{\mu+i+j-2}{j}\right)$$

where μ is an indeterminate and $\delta_{i,j}$ is the Kronecker delta function. Then the following relation holds:

$$\frac{D_1(2n)}{D_1(2n-1)} = (-1)^{(n-1)(n-2)/2}2^n\frac{\left(\frac{1}{2}(\mu+2n)\right)_{\lfloor(n+1)/2\rfloor}\left(\frac{1}{2}(\mu+4n+1)\right)_{n-1}}{(n)_n\left(\frac{1}{2}(-\mu-4n+3)\right)_{\lfloor(n-1)/2\rfloor}}.$$

Proof. By looking at the first few evaluations of $D_1(n)$ (for $1\leqslant n\leqslant 8$ they are explicitly displayed in [12]), we notice that only the quotient $D_1(2n)/D_1(2n-1)$ is nice, but not $D_1(2n+1)/D_1(2n)$. The reason is the occurrence of irreducible nonlinear factors that change every two steps, i.e., $D_1(2n)$ and $D_1(2n-1)$ share the same "ugly" factor (thus their quotient is nice), but in $D_1(2n+1)$ the "ugly" part will be different (and therefore the quotient $D_1(2n+1)/D_1(2n)$ does not factor nicely).

We first tried Zeilberger's original approach on the determinant $D_1(n)$. But we didn't even succeed to guess the recurrences for $c_{n,j}$ as they either are extraordinarily large or do not exist at all.

Therefore we have to come up with a variation of Zeilberger's approach, that pays attention to even n only. This means that we consider the normalized cofactors $c_{n,j}$ only for matrices of even size. What we basically do, is to replace the variable n in the equations (1)–(3) of the original approach by $2n$; the new identities, to be verified, will be

$$c_{n,2n} = 1 \qquad (n\geqslant 1), \tag{1a}$$

$$\sum_{j=1}^{2n} c_{n,j}a_{i,j} = 0 \qquad (1\leqslant i<2n), \tag{2a}$$

$$\sum_{j=1}^{2n} c_{n,j}a_{2n,j} = \frac{b_{2n}}{b_{2n-1}} \qquad (n\geqslant 1). \tag{3a}$$

In order to come up with an appropriate guess for the yet unknown function $c_{n,j}$, we compute the normalized cofactors for all even-size matrices up to dimension 30. This gives a 15×30 array with values in $\mathbb{Q}(\mu)$ that is used for guessing linear recurrences for $c_{n,j}$. For this step, Manuel Kauers's Mathematica package `Guess.m` has been employed, see [6] for more details. To give the reader an impression of how the output looks like, we display the results here in truncated form: whenever the abbreviation $\langle k \text{ terms}\rangle$ appears, it indicates that this polynomial cannot be factored into smaller pieces, and that for better readability it is not displayed in full size here.

$$
\begin{aligned}
&2n(j+1)(2n-1)(2j+\mu)(j-2n)(j-2n+1)\\
&\qquad\times(\mu+4n-5)(\mu+4n-3)(j+\mu+2n-1)c_{n,j} =\\
&j(j+\mu-1)(2j+\mu-1)(j-2n+3)(\mu+4n-3)\\
&\qquad\times\left(j^4+2j^3\mu+\ldots\langle 24 \text{ terms}\rangle+12\right)c_{n-1,j+1} -\\
&(j+1)(j+\mu+2n-3)\left(2j^6\mu+8j^6n+\ldots\langle 92 \text{ terms}\rangle-210\mu n\right)c_{n-1,j}
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
&(j-1)(j+\mu-3)(j+\mu-2)(2j+\mu-4)(j-2n)(j+\mu+2n-1)c_{n,j} =\\
&j(j+\mu-3)\left(4j^4+8j^3\mu+\ldots\langle 26 \text{ terms}\rangle+16\right)c_{n,j-1} -\\
&j(j-1)(j+\mu-2)(2j+\mu-2)(j-2n-2)(j+\mu+2n-3)c_{n,j-2}
\end{aligned}
\tag{5}
$$

When translated into operators, these two recurrences constitute a Gröbner basis of the left ideal that they generate in the corresponding operator algebra. As a consequence it follows that they are compatible; this means that starting from some given initial values, these recurrences will always produce the same value for a particular c_{n_0,j_0}, independently from the order in which they are applied. The support of the recurrences suggests that fixing the initial values $c_{1,1} = -\mu/2$ and $c_{1,2} = 1$ is sufficient: (5) will produce $c_{1,j}$ for all $j > 2$, and then (4) can be used to obtain the full array of values.

Unfortunately it is not that easy, since there are two disturbing phenomena. The first is the factor $j - 2n$ that appears in both leading coefficients. Hence for computing $c_{n,2n}$ none of the recurrences is applicable and we are stuck. We overcome this problem by finding a recurrence of the form

$$
p_2 c_{n+2,j+4} + p_1 c_{n+1,j+2} + p_0 c_{n,j} = 0, \qquad p_0, p_1, p_2 \in \mathbb{Q}[n, j, \mu] \tag{6}
$$

in the ideal generated by (4) and (5). The coefficient p_2 does not vanish for $j = 2n$, and thus, together with the additional initial value $c_{2,4} = 1$, (6) allows to compute the values $c_{n,2n}$.

The second unpleasant phenomenon is the free parameter μ which can cause the same effect for certain choices. For example, if $\mu = -6$ then we cannot compute $c_{2,3}$ from the given initial values since both leading coefficients vanish simultaneously by virtue of the factors $j-2n+1$ and $j+\mu+2n-1$. We handle it by restricting the parameter μ to the real numbers $> \mu_0$ for a certain $\mu_0 \in \mathbb{R}$, and by showing that all our calculations are sound under this assumption (in most cases we use $\mu > 0$ or $\mu > 2$). But since the determinant for every $n \in \mathbb{N}$ is a polynomial in μ we can extend our result afterwards to all $\mu \in \mathbb{C}$.

We have now prepared the stage for executing the main part of Zeilberger's approach. Identity (1a) is easily shown with help of the recurrence (6): Substituting $j \to 2n$ produces a recurrence for the entries $c_{n,2n}$, which together with the initial values implies that $c_{n,2n} = 1$ for all n. The identities (2a) and (3a) are proven automatically as well, since they are standard applications of holonomic closure properties and summation techniques (creative telescoping). For these tasks we have used the first author's Mathematica package

`HolonomicFunctions` [8]. The interested reader is invited to study our computations in detail, by downloading the electronic supplementary material from our webpage [10]. □

4 Interlude: The Double-Step Variant

In this section we propose a variant of Zeilberger's holonomic ansatz, described in Section 2, that enlarges the class of determinants which can be treated by this kind of ansatz. The condition $b_n \neq 0$ for all $n \geqslant 1$ imposes already some restriction. For example, when studying a Pfaffian $\operatorname{Pf}(A)$ for some skew-symmetric matrix A, one could be tempted to apply Zeilberger's approach to the determinant of A; recall that $\operatorname{Pf}(A)^2 = \det(A)$. The problem then is that $\det(A)$ is zero whenever the dimension of A is odd. Hence one would like to study the quotient b_n/b_{n-2} instead of the forbidden b_n/b_{n-1}; as in the previous section, n has to be restricted to the even integers. This dilemma can be solved by modifying Zeilberger's ansatz subject to the Laplace expansion of Pfaffians, see [5].

On the other hand there are determinants which do factor nicely for even dimensions but which do not for odd ones. Also here, we expect the quotient b_n/b_{n-2} to be nice, whereas the expression b_n/b_{n-1} might not even satisfy a linear recurrence and hence could not be handled by Zeilberger's holonomic ansatz at all. Similarly when the closed form b_n is different for even and odd n: while here the original approach could probably work in principle, one may not succeed because of the computational complexity that is caused by studying the quotient b_n/b_{n-1}, which is expected to be more complicated than b_n/b_{n-2}. See Theorems 2 and 5 below for such examples, which actually have motivated us to propose the following variant.

As we announced we now generalize Zeilberger's method in order to produce a recurrence for the quotient of determinants whose dimensions differ by two. As before, let $M = (a_{i,j})_{1\leqslant i,j\leqslant n}$ and let b_n denote the conjectured evaluation of $\det(M)$, which for all n in question has to be nonzero. We pull out of the hat two discrete functions $c'_{n,j}$ and $c''_{n,j}$ and verify the following identities:

$$c'_{n,n-1} = c''_{n,n} = 1, \qquad c'_{n,n} = c''_{n,n-1} = 0, \tag{1b}$$

$$\sum_{j=1}^{n} a_{i,j} c'_{n,j} = \sum_{j=1}^{n} a_{i,j} c''_{n,j} = 0 \qquad (1 \leqslant i \leqslant n-2), \tag{2b}$$

$$\left(\sum_{j=1}^{n} a_{n-1,j}\, c'_{n,j}\right)\left(\sum_{j=1}^{n} a_{n,j}\, c''_{n,j}\right) - \left(\sum_{j=1}^{n} a_{n-1,j}\, c''_{n,j}\right)\left(\sum_{j=1}^{n} a_{n,j}\, c'_{n,j}\right) = \frac{b_n}{b_{n-2}}. \tag{3b}$$

Then the determinant evaluation follows as a consequence, using a similar induction argument as in Section 2.

Let's try to give the motivation for this approach which also explains why it works. The idea is based on the formula for the determinant of a block matrix:

$$\det(M) = \det\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \det(M_1)\det(M_4 - M_3 M_1^{-1} M_2).$$

We want to divide the matrix M into blocks such that M_4 is a 2×2 matrix. Let $C = (C', C'')$ denote the $(n-2) \times 2$ matrix whose first column is $C' =$

$(c'_{n,j})_{1\leqslant j\leqslant n-2}$ and whose second column is $C'' = (c''_{n,j})_{1\leqslant j\leqslant n-2}$. With this notation, the two equations (2b) can be written as

$$(M_1, M_2)\begin{pmatrix} C' & C'' \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = (M_1, M_2)\begin{pmatrix} C \\ I_2 \end{pmatrix} = M_1 C + M_2 = 0,$$

where the conditions of Equation (1b) have been employed to constitute the identity matrix I_2. Now by the induction hypothesis we may assume that $\det_{1\leqslant i,j\leqslant n-2}(a_{i,j}) = \det(M_1)$ equals b_{n-2}, which by our general assumption is nonzero. Thus the above system determines C uniquely and we can write $C = -M_1^{-1}M_2$.

Finally, we obtain the missing part from the block matrix formula that gives us the quotient $\det(M)/\det(M_1)$, as the determinant of the 2×2 matrix

$$(M_3, M_4)\begin{pmatrix} C \\ I_2 \end{pmatrix} = M_3 C + M_4 = M_4 - M_3 M_1^{-1} M_2.$$

This is exactly what is expressed in Equation (3b), with all matrix multiplications explicitly written out.

Even though it turned out during our research that there is no need to apply the double-step method in the present context, we decided to keep it, as we are sure that it will be useful for future use. Let us also remark that it is straightforward to generalize this idea in order to produce a triple-step method, etc. But since the identities to be verified become more and more complicated, we don't believe that these large-step methods are relevant in practice.

5 Small Change, Big Impact

The next determinants we want to study appear as *Conjecture 35* and *Conjecture 36* in [12]. Christian Krattenthaler (private communication) describes the story of how they were raised:

> I wrote this article during a stay at the Mittag-Leffler Institut. Guoce Xin was also there, as well as Alain Lascoux. They followed the progress on the article with interest. So, it was Guoce Xin, who had been looking at similar determinants at the time (in the course of his work with Ira Gessel on his big article on determinants and path counting [4]), who told me what became Conjecture 35. I made some experiments and discovered Conjecture 36. Alain Lascoux saw all this, and he came up with Conjecture 37.

Guoce Xin's observation was that a certain matrix, very similar to the one of Section 3, has a determinant that factors completely; the only change is the sign of the Kronecker delta $\delta_{i,j}$. But still, the evaluation is given as a case distinction between even and odd dimensions of the matrix.

Theorem 2. *Let μ be an indeterminate and n be a nonnegative integer. Then the determinant*

$$\det_{1\leqslant i,j\leqslant n}\left(-\delta_{i,j} + \binom{\mu+i+j-2}{j}\right) \tag{7}$$

equals

$$(-1)^{n/2}\, 2^{n(n+2)/4} \frac{\left(\frac{\mu}{2}\right)_{n/2}}{\left(\frac{n}{2}\right)!} \left(\prod_{i=0}^{(n-2)/2} \frac{i!^2}{(2i)!^2} \right) \times \left(\prod_{i=1}^{\lfloor n/4 \rfloor} \left(\tfrac{1}{2}(\mu+6i-1)\right)^2_{(n-4i+2)/2} \left(\tfrac{1}{2}(-\mu-3n+6i)\right)^2_{(n-4i)/2} \right)$$

if n is even, and it equals

$$(-1)^{(n-1)/2}\, 2^{(n+3)(n+1)/4} \left(\tfrac{1}{2}(\mu-1)\right)_{(n+1)/2} \left(\prod_{i=0}^{(n-1)/2} \frac{i!(i+1)!}{(2i)!(2i+2)!} \right) \times \left(\prod_{i=1}^{\lfloor (n+1)/4 \rfloor} \left(\tfrac{1}{2}(\mu+6i-1)\right)^2_{(n-4i+1)/2} \left(\tfrac{1}{2}(-\mu-3n+6i-3)\right)^2_{(n-4i+3)/2} \right)$$

if n is odd.

Proof. We could try to solve this problem directly, either by Zeilberger's original ansatz or in the way we did in Section 3. However, this does not work in practice as the computations become too large (in the second case we were even able to guess the recurrences for $c_{n,j}$, but their size destroyed any hope to prove (2a) and (3a)). Next, we could give the double-step method a try, which we described in Section 4. We succeeded to make the proof for some concrete integer μ, but again it seems intractable for symbolic μ. Instead we make a small detour and break this determinant into pieces in order to make the calculations smaller. At the end, we put these pieces together by the Desnanot-Jacobi adjoint matrix theorem to obtain the desired result. Let us introduce the following notation

$$b_n(I,J) := b_n(I,J,\mu) := \det_{\substack{I \leqslant i \leqslant n-1+I \\ J \leqslant j \leqslant n-1+J}} \left(-\delta_{i,j} + \binom{\mu+i+j-2}{j} \right)$$

so that our determinant (7) is denoted by $b_n(1,1)$. In this notation the Desnanot-Jacobi identity is stated as follows:

$$b_n(0,0)b_{n-2}(1,1) = b_{n-1}(0,0)b_{n-1}(1,1) - b_{n-1}(0,1)b_{n-1}(1,0). \tag{8}$$

Substituting $n \to 2n+2$ and $n \to 2n+1$ in (8) and using the fact $b_{2n}(0,0) = -b_{2n-1}(1,1)$ (to be shown later) gives the equations

$$b_{2n+1}(1,1) = \frac{b_{2n+1}(0,1)b_{2n+1}(1,0)}{b_{2n}(1,1)+b_{2n+1}(0,0)}$$

$$b_{2n-1}(1,1) = \frac{-b_{2n}(0,1)b_{2n}(1,0)}{b_{2n}(1,1)+b_{2n+1}(0,0)}$$

from which the desired quotient can be obtained:

$$\frac{b_{2n+1}(1,1)}{b_{2n-1}(1,1)} = -\frac{b_{2n+1}(0,1)}{b_{2n}(0,1)} \cdot \frac{b_{2n+1}(1,0)}{b_{2n}(1,0)}. \tag{9}$$

Similarly, we substitute $n \to 2n+1$ and $n \to 2n$ into (8) and use the fact $b_{2n-1}(0,0) = 0$ (again, to be shown later), to derive the quotient of even determinants

$$\frac{b_{2n}(1,1)}{b_{2n-2}(1,1)} = -\frac{b_{2n}(0,1)}{b_{2n-1}(0,1)} \cdot \frac{b_{2n}(1,0)}{b_{2n-1}(1,0)}. \tag{10}$$

We now use the first variation of Zeilberger's ansatz (see Section 3) to derive recurrences for the quotients $b_{2n+1}(0,1)/b_{2n}(0,1)$, etc. which appear on the right hand sides of (9) and (10). Since the arguments closely follow the lines of the proof of Theorem 1, we don't detail further this part and refer to the electronic material [10]. Although for our purposes it would suffice to work with these recurrences, we succeed in solving them in closed form:

$$Q_1(n) := \frac{b_{2n+1}(0,1)}{b_{2n}(0,1)} = \frac{2\left(\frac{\mu}{2}+2n\right)_{n+1}\left(\mu+2n-1\right)_{n+1}}{\left(n+2\right)_{n+1}\left(\frac{\mu}{2}+n\right)_{n+1}}, \tag{11}$$

$$Q_2(n) := \frac{b_{2n}(0,1)}{b_{2n-1}(0,1)} = \frac{(\mu+2n-2)\left(\frac{\mu}{2}+2n-1\right)_{n-1}\left(\mu+2n+1\right)_{n-1}}{n\left(n\right)_{n-1}\left(\frac{\mu}{2}+n+1\right)_{n-1}}, \tag{12}$$

$$Q_3(n) := \frac{b_{2n+1}(1,0)}{b_{2n}(1,0)} = \frac{2\left(\frac{\mu}{2}+2n\right)_{n+1}\left(\mu+2n+1\right)_{n-1}}{\left(n+1\right)_{n}\left(\frac{\mu}{2}+n+1\right)_{n}}, \tag{13}$$

$$Q_4(n) := \frac{b_{2n}(1,0)}{b_{2n-1}(1,0)} = \frac{2\left(\frac{\mu}{2}+2n-1\right)_{n-1}\left(\mu+2n+1\right)_{n-1}}{\left(n\right)_{n-1}\left(\frac{\mu}{2}+n+1\right)_{n-1}}. \tag{14}$$

These quotients immediately give closed form evaluations of the corresponding determinants (see also Theorems 3 and 4. It remains to justify the assumptions $b_{2n}(0,0) = -b_{2n-1}(1,1)$ and $b_{2n-1}(0,0) = 0$ that were used to derive (9) and (10).

In order to evaluate the quotient $b_{2n}(0,0)/b_{2n-1}(1,1)$ we need to modify the method presented in Section 3: we apply Laplace expansion with respect to the first row of the matrix, instead of the n-th row, and we normalize the auxiliary function $c_{n,j}$ such that $c_{n,0} = 1$. If we come up with a recursive description of some function $c_{n,j}$ and are able to verify the following identities, then we are done:

$$c_{n,0} = 1 \qquad (n \geqslant 1), \tag{1c}$$

$$\sum_{j=0}^{2n-1} c_{n,j} a_{i,j} = 0 \qquad (0 < i \leqslant 2n-1), \tag{2c}$$

$$\sum_{j=0}^{2n-1} c_{n,j} a_{0,j} = \frac{b_{2n}(0,0)}{b_{2n-1}(1,1)} = -1 \qquad (n \geqslant 1). \tag{3c}$$

As before, the details can be found in [10].

Last but not least we have to argue that $b_{2n-1}(0,0)$ vanishes. Christian Krattenthaler kindly pointed us to Theorem 11 in [3] (see also Theorem 35 in [11]) which contains this statement. Anyway we have produced an alternative, computerized proof: actually it is very simple, since we just have to come up with a guess for the coefficients of a linear combination of the columns (or rows) that gives 0, and then prove that our guess does the job. Hence we find a recursive description of a function $c_{n,j}$, $n \geqslant 1$, $0 \leqslant j \leqslant 2n-2$, such that

$$c_{n,0}\begin{pmatrix} a_{0,0} \\ \vdots \\ a_{2n-2,0} \end{pmatrix} + \cdots + c_{n,2n-2}\begin{pmatrix} a_{0,2n-2} \\ \vdots \\ a_{2n-2,2n-2} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

and such that there is an index j for which $c_{n,j} \neq 0$. For guessing, we compute $c_{n,j}$ for all n up to some bound and normalize them. Luckily the nullspace of the above system has always dimension 1, otherwise it would be trickier to find a suitable linear combination (however, in our problem this is no surprise since we are finally aiming at proving that the minor $M_{0,0}$ of this matrix evaluates to some nonzero expression). So we are successful in guessing the recurrences of $c_{n,j}$ and use them to prove

$$c_{n,2n-2} = 1 \qquad (n \geqslant 1)$$
$$\sum_{j=0}^{2n-2} c_{n,j} a_{i,j} = 0 \qquad (0 \leqslant i \leqslant 2n-2).$$

This concludes the proof of Theorem 2. □

Since the evaluations of the determinants $b_n(0,1)$ and $b_n(1,0)$ are interesting results in their own right, but are somehow hidden in the proof of Theorem 2, we are going to state them explicitly here.

Theorem 3. *Let μ be an indeterminate and n be a nonnegative integer. Let $Q_1(n)$ and $Q_2(n)$ be defined as in (11) and (12), respectively. Then the determinant*

$$b_n(0,1) = \det_{1\leqslant i,j\leqslant n}\left(-\delta_{i-1,j} + \binom{\mu+i+j-3}{j}\right)$$

equals

$$(\mu-1)\prod_{k=1}^{(n-1)/2}\Big(Q_1(k)Q_2(k)\Big)$$

if n is even, and it equals

$$\left(\prod_{k=0}^{n/2-1} Q_1(k)\right)\left(\prod_{k=1}^{n/2} Q_2(k)\right)$$

if n is odd.

Proof. It is analogous to the proof of Theorem 1 and can be found in [10]. □

Theorem 4. *Let μ be an indeterminate and n be a nonnegative integer. Let $Q_3(n)$ and $Q_4(n)$ be defined as in (13) and (14), respectively. Then the determinant*

$$b_n(1,0) = \det_{1\leqslant i,j\leqslant n}\left(-\delta_{i,j-1} + \binom{\mu+i+j-3}{j-1}\right)$$

equals

$$\prod_{k=1}^{(n-1)/2}\Big(Q_3(k)Q_4(k)\Big)$$

if n is even, and it equals

$$\left(\prod_{k=0}^{n/2-1} Q_3(k)\right)\left(\prod_{k=1}^{n/2} Q_4(k)\right)$$

if n is odd.

Proof. It is analogous to the proof of Theorem 1 and can be found in [10]. □

By replacing j by $j+1$ at the bottom of the binomial coefficient in the entries of (7), we arrive at our last determinant; it has been discovered by Christian Krattenthaler and appears as *Conjecture 36* in his paper [12] (note that the formula there is erroneous, as one of the product quantors is missing such that the corresponding factor slided into the previous product). Also this problem has its own combinatorial interpretation in terms of counting certain rhombus tilings.

Theorem 5. *Let μ be an indeterminate. For any odd nonnegative integer n there holds*

$$\det_{1\leqslant i,j\leqslant n}\left(-\delta_{i,j}+\binom{\mu+i+j-2}{j+1}\right)=$$
$$(-1)^{(n-1)/2}\,2^{(n-1)(n+5)/4}\,(\mu+1)\,\frac{\left(\frac{1}{2}(\mu-2)\right)_{(n+1)/2}}{\left(\frac{1}{2}(n+1)\right)!}\left(\prod_{i=0}^{(n-1)/2}\frac{i!^2}{(2i)!^2}\right)$$
$$\times\left(\prod_{i=1}^{\lfloor (n+3)/4\rfloor}\left(\tfrac{1}{2}(\mu+6i-3)\right)^2_{(n-4i+3)/2}\right)$$
$$\times\left(\prod_{i=1}^{\lfloor (n+1)/4\rfloor}\left(\tfrac{1}{2}(-\mu-3n+6i-1)\right)^2_{(n-4i+1)/2}\right).$$

Proof. Because of the similarity of this determinant with (7), we are able to relate these two problems via shifting the starting points:

$$\det_{1\leqslant i,j\leqslant 2n-1}\left(-\delta_{i,j}+\binom{\mu+i+j-2}{j+1}\right)=\det_{2\leqslant i,j\leqslant 2n}\left(-\delta_{i,j}+\binom{(\mu-2)+i+j-2}{j}\right).$$

By using the notation from above, the determinant of Theorem 5 is denoted by $b_{2n-1}(2,2,\mu-2)$. Analogously to (1c)–(3c), we apply a variation of Zeilberger's approach to derive a recurrence for

$$q_n(\mu)=\frac{b_{2n}(1,1,\mu)}{b_{2n-1}(2,2,\mu)}.$$

The result is $q_{n+1}(\mu)-q_n(\mu)=0$ which reveals that the quotient $q_n(\mu)$ is constant. Together with the initial value $q_1(\mu)=-4/(\mu+3)$ and the fact that $b_{2n}(1,1,\mu)$ is already known from Theorem 2, we get the desired result. Once again, we refer to [10] for the details of the computations. □

As mentioned above, Alain Lascoux found that the more general determinant

$$\det_{1\leqslant i,j\leqslant n}\left(-\delta_{i,j+r-1}+\binom{\mu+i+j-2}{j+r-1}\right)$$

factors completely for odd natural numbers n and r, and its complicated evaluation, which was figured out by Christian Krattenthaler, appears as *Conjecture 37* in [12]. We remark that the formula given there holds only for $r\leqslant n$; otherwise the Kronecker delta does not show up in the matrix and the evaluation is much simpler. We cannot attack this determinant directly with Zeilberger's ansatz since the matrix entries do not evaluate to polynomials in μ for concrete integers i and j, but keeping r symbolically. Therefore the guessing for $c_{n,j}$ will not work. A different strategy would consist in finding some connection between the cases r and $r+2$; then induction on r would provide a proof, using Theorem 2 as the base case $r=1$. Unfortunately we were not able to achieve this goal.

6 A Challenge Problem

We want to conclude our article with a challenge problem for the next generation of computer algebra tools. In Section 3 we have only proven a statement about the quotient of two consecutive determinants (Theorem 1). But so far nobody has come up with a closed form for the determinant $D_1(n)$. We now present a conjectured closed form, which, however, we are unable to prove with the methods described in the present paper.

Conjecture 6. *Let μ be an indeterminate and let the sequences $C(n)$, $F(n)$, and $G(n)$ be defined as follows:*

$$C(n) = \frac{(-1)^n+3}{2}\prod_{i=1}^{n}\frac{\lfloor\frac{i}{2}\rfloor!}{i!}$$

$$E(n) = (\mu+1)_n\left(\prod_{i=1}^{\lfloor\frac{3}{2}\lfloor\frac{1}{2}(n-1)\rfloor-2\rfloor}\Big(\mu+2i+6\Big)^{2\lfloor\frac{1}{3}(i+2)\rfloor}\right) \times\left(\prod_{i=1}^{\lfloor\frac{3}{2}\lfloor\frac{n}{2}\rfloor-2\rfloor}\Big(\mu+2i+2\left\lfloor\tfrac{3}{2}\left\lfloor\tfrac{n}{2}+1\right\rfloor\right\rfloor-1\Big)^{2\lfloor\frac{1}{2}\lfloor\frac{n}{2}\rfloor-\frac{1}{3}(i-1)\rfloor-1}\right)$$

$$F_m(n) = \left(\prod_{i=1}^{\lfloor\frac{1}{4}(n-1)\rfloor}(\mu+2i+n+m)^{1-2i-m}\right)\left(\prod_{i=1}^{\lfloor\frac{n}{4}-1\rfloor}(\mu-2i+2n-2m+1)^{1-2i-m}\right)$$

$$F(n) = \begin{cases} E(n)F_0(n) & \text{if } n \text{ is even} \\ E(n)F_1(n)\displaystyle\prod_{i=1}^{\frac{1}{2}(n-5)}(\mu+2i+2n-1) & \text{if } n \text{ is odd}\end{cases}$$

$$T(k) = 55296k^6+41472(\mu-1)k^5+384(30\mu^2-66\mu+53)k^4+ 96(\mu-1)(15\mu^2-42\mu+61)k^3+4(19\mu^4-122\mu^3+419\mu^2-544\mu+72)k^2+ (\mu-1)(\mu^4-14\mu^3+101\mu^2-160\mu-84)k+2(\mu-3)(\mu-2)(\mu-1)(\mu+1)$$

$$S_1(n) = \sum_{k=1}^{n-1}\frac{2^{6k}(\mu+8k-1)\left(\frac{1}{2}\right)_{2k-1}^2\left(\frac{1}{2}(\mu+5)\right)_{2k-3}\left(\frac{1}{2}(\mu+4k+2)\right)_{k-2}\left(\frac{1}{2}(\mu+4k+2)\right)_{2n-2k-2}T(k)}{(2k)!\left(\frac{1}{2}(\mu+6k-3)\right)_{3k+4}}$$

$$S_2(n) = \sum_{k=1}^{n-1}\frac{2^{6k}(\mu+8k+3)\left(\frac{1}{2}\right)_{2k}^2\left(\frac{1}{2}(\mu+5)\right)_{2k-2}\left(\frac{1}{2}(\mu+4k+4)\right)_{k-2}\left(\frac{1}{2}(\mu+4k+4)\right)_{2n-2k-2}T\big(k+\frac{1}{2}\big)}{(2k+1)!\left(\frac{1}{2}(\mu+6k+1)\right)_{3k+5}}$$

$$P_1(n) = 2^{3n-1}\frac{\left(\frac{1}{2}(\mu+6n-3)\right)_{3n-2}}{\left(\frac{1}{2}(\mu+5)\right)_{2n-3}}\left(\frac{\left(\frac{1}{2}(\mu+2)\right)_{2n-2}}{(\mu+3)^2}+\frac{\mu(\mu-1)S_1(n)}{2^{13}}\right)$$

$$P_2(n) = 2^{3n-1}\frac{\left(\frac{1}{2}(\mu+6n+1)\right)_{3n-1}}{\left(\frac{1}{2}(\mu+5)\right)_{2n-2}}\left(\frac{(\mu+14)\left(\frac{1}{2}(\mu+4)\right)_{2n-2}}{(\mu+7)(\mu+9)}+\frac{\mu(\mu-1)S_2(n)}{2^{9}}\right)$$

$$G(n) = \begin{cases} P_1\left(\frac{1}{2}(n+1)\right) & \textit{if } n \textit{ is odd} \\ P_2\left(\frac{n}{2}\right) & \textit{if } n \textit{ is even} \end{cases}$$

Then for every positive integer n we have

$$\det_{1\leqslant i,j\leqslant n}\left(\delta_{i,j}+\binom{\mu+i+j-2}{j}\right) = C(n)F(n)G\left(\left\lfloor\tfrac{1}{2}(n+1)\right\rfloor\right).$$

Let us add a few remarks on our conjectured closed form. The elements of the sequence $C(n)$ are rational numbers of the form $1/k$ where k is an integer. The sequences $F(n)$ and $G(n)$ consist of monic polynomials in μ with integer coefficients. The $F(n)$ factor completely into linear factors of the form $(\mu+k)$ where $k \in \mathbb{N}$, and thus have positive coefficients. The $G(n)$ have positive coefficients as well, but turn out to be mostly irreducible; the only counterexample that we found is $G(4) = (\mu+34)(\mu^3+47\mu^2+954\mu+5928)$. They correspond to the "ugly factors" mentioned in Section 3. For the convenience of the reader, we provide the Mathematica code for all quantities introduced in Conjecture 6 in the supplementary material [10].

In order to come up with this complicated conjecture, we computed the determinants $D_1(n)$ for $1 \leqslant n \leqslant 295$ which gave us the first 148 polynomials of the sequence $G(n)$. These data enabled us to guess recurrences for the subsequences $P_1(n)$ and $P_2(n)$; the recurrences by the way are used in [10] to provide a fast procedure for computing $D_1(n)$. The Maple package `LREtools` was able to find "closed form" solutions which, after lots of automatic and manual simplifications, became the formulae for $T(k)$, $S_i(n)$, and $P_i(n)$.

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