

# Lattice Green's Functions of the Higher-Dimensional Face-Centered Cubic Lattices

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## Abstract

We study the lattice Green's functions of the face-centered cubic lattice (fcc) in up to six dimensions. We give computer algebra proofs of results that were conjectured by Guttmann and Broadhurst for the four- and five-dimensional fcc lattices. Additionally we derive a differential equation for the lattice Green's function of the six-dimensional fcc lattice, a result that was not believed to be achievable with current computer hardware.

## 1 Introduction

### 1.1 Bravais Lattices and their Applications

We consider lattices in  $\mathbb{R}^d$  that are given as infinite sets of points

$$\left\{ \sum_{i=1}^d n_i \mathbf{a}_i : n_1, \dots, n_d \in \mathbb{Z} \right\} \subseteq \mathbb{R}^d$$

for some linearly independent vectors  $\mathbf{a}_1, \dots, \mathbf{a}_d \in \mathbb{R}^d$  (throughout this paper, vectors are denoted by bold letters). In three dimensions such lattices are called *Bravais lattices*. The simplest instance of such a lattice is obtained by choosing  $\mathbf{a}_i = \mathbf{e}_i$ , the  $i$ -th unit vector; the result is the integer lattice  $\mathbb{Z}^d$  which is also called the *square lattice* (for  $d = 2$ ), or the *cubic lattice* (for  $d = 3$ ), or the *hypercubic lattice* (for  $d \geq 4$ ).

The *face-centered (hyper-) cubic lattice* (fcc lattice) is obtained from the (hyper-) cubic lattice by adding the center point of each (two-dimensional) face to the set of lattice points. In two dimensions this operation is trivial: the faces of the square lattice  $\mathbb{Z}^2$  are all unit squares with corners  $(m, n), (m + 1, n), (m + 1, n + 1), (m, n + 1)$  for integers  $m, n \in \mathbb{Z}$ . Their center points are  $\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$  which together with  $\mathbb{Z}^2$  again yields a square lattice, more precisely a copy of  $\mathbb{Z}^2$  which is rotated by 45 degrees and shrunk by a factor of  $\sqrt{2}$ . The situation becomes more interesting in higher dimensions. For example, in three dimensions there are 6 faces of the unit cube, and their center points together

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with all integral translates have to be included. It is not difficult to see that the three-dimensional fcc lattice consists of four copies of  $\mathbb{Z}^3$ , namely

$$\mathbb{Z}^3 \cup \left(\mathbb{Z}^3 + \left(\frac{1}{2}, \frac{1}{2}, 0\right)\right) \cup \left(\mathbb{Z}^3 + \left(\frac{1}{2}, 0, \frac{1}{2}\right)\right) \cup \left(\mathbb{Z}^3 + \left(0, \frac{1}{2}, \frac{1}{2}\right)\right).$$

Similarly the fcc lattice in four dimensions consists of 7 copies of  $\mathbb{Z}^4$ , and in general the  $d$ -dimensional fcc lattice is composed of  $1 + \binom{d}{2}$  translated copies of  $\mathbb{Z}^d$ .

The study of Bravais lattices was inspired by crystallography in as much as the atomic structure of crystals forms such regular lattices. While the cubic lattice is quite rarely found in nature (e.g., in polonium) due to its small *atomic packing factor* (the proportion of space that is filled when a sphere of maximal radius is put on each lattice point, in a way that these spheres do not overlap), the fcc lattice is more often encountered, for example, in aluminium, copper, silver, and gold. The atomic packing factor of the fcc lattice is  $\sqrt{2}\pi/6$ , the highest possible value as was shown by Hales in his famous proof of the Kepler conjecture [5]. Further applications, in particular in connection with random walks (see the next section) appear in physics, chemistry, ecology, economics, and computer science when lattice vibration problems (phonons), diffusion models, luminescence, Markov processes and other random processes are studied.

## 1.2 Random Walks

For sake of simplicity, the fcc lattice as introduced in the previous section, is stretched by a factor of 2 in all coordinate directions so that all lattice points have integral coordinates. This convention is kept throughout the paper as it does not change the relevant quantities that we are interested in (e.g., the return probability, see below).

The aim of this paper is to study random walks on the fcc lattice in several dimensions. We consider walks that allow only steps to the nearest neighbors of a point (with respect to the Euclidean metric). Furthermore it is assumed that all steps are taken with the same probability. For example, consider a point  $(k, m, n)$  in the three-dimensional cubic lattice  $(2\mathbb{Z})^3$ . It is the common corner point of 8 cubes. The nearest neighbors in the 3D fcc lattice are then the center points of some of those faces which have  $(k, m, n)$  as a corner point. Note that they all have distance  $\sqrt{2}$  whereas the other corner and face-center points are farther away (their distance is  $\geq 2$ ) and hence not reachable in a single step. Thus the number of possible steps is  $8 \cdot 3/2 = 12$  (number of adjacent cubes times the number of adjacent faces per cube, divided by two since each face belongs to two cubes). The same situation is encountered at the center point of some face and hence every point in the 3D fcc lattice has exactly 12 nearest neighbors; this number is called the *coordination number* of the lattice.

The above considerations can be generalized to arbitrary dimensions in a straight-forward manner; one finds that the set of permitted steps in the  $d$ -dimensional fcc lattice is given by

$$\{(s_1, \dots, s_d) \in \{0, -1, 1\}^d : |s_1| + \dots + |s_d| = 2\} \quad (1)$$

and thus its coordination number is  $4\binom{d}{2}$ .

### 1.3 Lattice Green's Function

Let  $p_n(\mathbf{x})$  denote the probability that a random walk which started at the origin  $\mathbf{0}$  ends at point  $\mathbf{x}$  after  $n$  steps. Note that in our setting of unrestricted walks,  $c^n p_n(\mathbf{x})$  is an integer and gives the total number of walks that end at location  $\mathbf{x}$  after  $n$  steps, where  $c$  is the coordination number of the lattice.

In order to achieve information about random walks on the fcc lattice, the following multivariate generating function is introduced:

$$P(\mathbf{x}; z) = \sum_{n=0}^{\infty} p_n(\mathbf{x}) z^n. \quad (2)$$

This function is called the *lattice Green's function* (LGF). By defining the *structure function*  $\lambda(\mathbf{k}) = \lambda(k_1, \dots, k_d)$  of a lattice to be the discrete Fourier transform

$$\lambda(\mathbf{k}) = \sum_{\mathbf{x} \in \mathbb{R}^d} p_1(\mathbf{x}) e^{i\mathbf{x} \cdot \mathbf{k}}$$

of the single-step probability function  $p_1(\mathbf{x})$ , the generating function (2) can be expressed as the  $d$ -dimensional integral

$$P(\mathbf{x}; z) = \frac{1}{\pi^d} \int_0^\pi \dots \int_0^\pi \frac{e^{i\mathbf{x} \cdot \mathbf{k}}}{1 - z\lambda(\mathbf{k})} dk_1 \dots dk_d.$$

We shall be interested in walks which return to the origin and which we therefore call *excursions*. The LGF for excursions is given by

$$P(\mathbf{0}; z) = \sum_{n=0}^{\infty} p_n(\mathbf{0}) z^n = \frac{1}{\pi^d} \int_0^\pi \dots \int_0^\pi \frac{dk_1 \dots dk_d}{1 - z\lambda(\mathbf{k})}. \quad (3)$$

In the following, we will only refer to this special instance when talking about lattice Green's functions. This function allows one to calculate the *return probability*  $R$ , sometimes also referred to as the *Pólya number*, of the lattice. It signifies the probability that a random walk that started at the origin will eventually return to the origin. It can be computed via the formula

$$R = 1 - \frac{1}{P(\mathbf{0}; 1)} = 1 - \frac{1}{\sum_{n=0}^{\infty} p_n(\mathbf{0})}. \quad (4)$$

**Example 1.** Consider the square lattice  $\mathbb{Z}^2$  which admits the steps  $(-1, 0)$ ,  $(1, 0)$ ,  $(0, -1)$ , and  $(0, 1)$ . Its structure function is

$$\lambda(k_1, k_2) = \frac{1}{4} (e^{-ik_1} + e^{ik_1} + e^{-ik_2} + e^{ik_2}) = \frac{1}{2} (\cos k_1 + \cos k_2).$$

and therefore its LGF is (see, e.g., [4])

$$P(0, 0; z) = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \frac{dk_1 dk_2}{1 - \frac{z}{2} (\cos k_1 + \cos k_2)} = \frac{2}{\pi} \mathbf{K}(z^2)$$

where  $\mathbf{K}(z)$  is the complete elliptic integral of the first kind. The fact the the above integral diverges for  $z = 1$  immediately implies that the return probability  $R = 1$ ; in other words that every random walk will eventually return to the origin, a result that was already proven in 1921 by Pólya [12].  $\square$

**Example 2.** We have already remarked that the two-dimensional fcc lattice is nothing else but a rotated and stretched version of the square lattice. Nevertheless let's have a look at the LGF when the step set  $\{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$  is taken. Its structure function is

$$\begin{aligned}\lambda(k_1, k_2) &= \frac{1}{4} \left( e^{-i(k_1+k_2)} + e^{-i(k_1-k_2)} + e^{i(k_1-k_2)} + e^{i(k_1+k_2)} \right) \\ &= \frac{1}{2} \left( \cos(k_1 + k_2) + \cos(k_1 - k_2) \right) = \cos k_1 \cos k_2,\end{aligned}$$

using the well-known angle-sum identity  $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$ .

Although the structure function differs quite a lot from that in Example 1 the LGF is the same:

$$P(0, 0, z) = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \frac{dk_1 dk_2}{1 - z \cos k_1 \cos k_2} = \frac{2}{\pi} \mathbf{K}(z^2) \quad (5)$$

as shown in Equation (6) of [4]. Note also that the different distances between nearest-neighboring lattice points—1 in Example 1 and  $\sqrt{2}$  in Example 2—carry no weight since only excursions (and not walks with arbitrary end points) are investigated.  $\square$

It is now an easy exercise to compute the structure function  $\lambda(\mathbf{k})$  for the  $d$ -dimensional fcc lattice:

$$\lambda(\mathbf{k}) = \binom{d}{2}^{-1} \sum_{m=1}^d \sum_{n=m+1}^d \cos k_m \cos k_n.$$

The LGF is then given as the  $d$ -fold integral (3) and the return probability can be computed by integrating over  $1/(1 - \lambda(\mathbf{k}))$

## 1.4 The Differential Equation Detour

The return probability in the fcc lattice in three dimensions was first computed by Watson [13] as one of the three integrals which were later named after him, and which give the return probabilities in different three-dimensional lattices. These probabilities can be expressed in terms of algebraic numbers,  $\pi$ , and values of the Gamma function at rational numbers. For example, the probability of returning to the origin in the 3D fcc lattice is given by

$$1 - \frac{16\sqrt[3]{4}\pi^4}{9(\Gamma(\frac{1}{3}))^6}.$$

For the three-dimensional fcc lattice, Joyce [6] expressed the lattice Green's function in terms of complete elliptic integrals, too, but the expression is fairly complicated and for the higher-dimensional fcc lattices no such evaluation is known at all. Similarly we don't know of any closed-form representation of the return probabilities in higher dimensions.

Instead we will derive differential equations for the corresponding LGFs. Although less explicit than the previously mentioned closed-form results, such an implicit representation of the LGF provides considerable insight. It allows one to efficiently compute the number of excursions for any fixed number of

steps, as well as the return probability with very high precision (see Section 3). But also the differential equations themselves reveal very interesting properties that are worth investigation.

To motivate our approach and to illuminate the origin of Equation (3), consider an arbitrary lattice in  $\mathbb{Z}^d$  with some finite set  $S \subset \mathbb{Z}^d$  of permitted steps. Then clearly the probability function  $p_n(\mathbf{x})$  satisfies the constant-coefficient recurrence

$$p_{n+1}(\mathbf{x}) = \frac{1}{|S|} \sum_{\mathbf{s} \in S} p_n(\mathbf{x} - \mathbf{s}). \quad (6)$$

Let  $F(\mathbf{y}; z)$  denote the multivariate generating function

$$F(\mathbf{y}; z) = \sum_{n=0}^{\infty} \sum_{\mathbf{x} \in \mathbb{Z}^d} p_n(\mathbf{x}) \mathbf{y}^{\mathbf{x}} z^n.$$

Multiplying both sides of (6) by  $\mathbf{y}^{\mathbf{x}} z^n$  and summing with respect to  $n$  and  $\mathbf{x}$  gives

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{\mathbf{x} \in \mathbb{Z}^d} p_{n+1}(\mathbf{x}) \mathbf{y}^{\mathbf{x}} z^n &= \frac{1}{|S|} \sum_{n=0}^{\infty} \sum_{\mathbf{x} \in \mathbb{Z}^d} \sum_{\mathbf{s} \in S} p_n(\mathbf{x} - \mathbf{s}) \mathbf{y}^{\mathbf{x}} z^n \\ \frac{1}{z} \sum_{n=1}^{\infty} \sum_{\mathbf{x} \in \mathbb{Z}^d} p_n(\mathbf{x}) \mathbf{y}^{\mathbf{x}} z^n &= \frac{1}{|S|} \sum_{\mathbf{s} \in S} \sum_{n=0}^{\infty} \sum_{\mathbf{x} \in \mathbb{Z}^d} p_n(\mathbf{x}) \mathbf{y}^{\mathbf{x}+\mathbf{s}} z^n \\ \frac{1}{z} (F(\mathbf{y}; z) - 1) &= \frac{1}{|S|} \sum_{\mathbf{s} \in S} \mathbf{y}^{\mathbf{s}} F(\mathbf{y}; z) \end{aligned}$$

Thus we obtain

$$F(\mathbf{y}; z) = \frac{1}{1 - \frac{z}{|S|} \sum_{\mathbf{s} \in S} \mathbf{y}^{\mathbf{s}}}$$

and the lattice Green's function  $P(\mathbf{0}; z)$  is nothing else but the constant term  $\langle \mathbf{y}^{\mathbf{0}} \rangle F(\mathbf{y}; z)$ . A differential equation for this expression can be derived from an operator

$$A(z, D_z) + D_{y_1} B_1 + \cdots + D_{y_d} B_d \quad (7)$$

that annihilates the expression  $F(\mathbf{y}; z)/(y_1 \cdots y_d)$ . Here the symbol  $D_x$  denotes the partial derivative w.r.t.  $x$  and the  $B_j$  are differential operators that may involve  $y_1, \dots, y_d, z$  as well as  $D_{y_1}, \dots, D_{y_d}, D_z$ . The fact that  $A$  may only depend on  $z$  and  $D_z$  is crucial and therefore explicitly indicated. In Section 3 we will discuss how to find such an operator. From

$$\langle y_1^{-1} \cdots y_d^{-1} \rangle A(z, D_z) \frac{F(\mathbf{y}; z)}{y_1 \cdots y_d} + \sum_{j=1}^d \langle y_1^{-1} \cdots y_d^{-1} \rangle D_{y_j} B_j \frac{F(\mathbf{y}; z)}{y_1 \cdots y_d} = 0$$

and the fact that the coefficient of  $y^{-1}$  in an expression of the form  $D_{y_j} \sum_{n=-\infty}^{\infty} a_n y^n$  is always zero, it follows that  $A(\langle \mathbf{y}^{\mathbf{0}} \rangle F(\mathbf{y}; z)) = A(P(\mathbf{0}; z)) = 0$ . Also in Section 3 we will see that finding the operator (7) is part of computing the  $d$ -fold integral w.r.t.  $y_1, \dots, y_d$ . The integral

$$\int \cdots \int \frac{d\mathbf{y}}{(y_1 \cdots y_d) (1 - \frac{z}{|S|} \sum_{\mathbf{s} \in S} \mathbf{y}^{\mathbf{s}})} = \int \cdots \int \frac{d\mathbf{k}}{1 - z\lambda(\mathbf{k})}$$

is transformed to (3) by means of the substitution  $y_j \rightarrow e^{ik_j}$ .

## 2 An Experimental Mathematics Approach

This section presents some results that were obtained in a non-rigorous way using the method of guessing. That is, for finding a linear differential equation

$$c_m(x)f^{(m)}(x) + \cdots + c_1(x)f'(x) + c_0(x)f(x) = 0$$

for a certain function  $f(x)$ , one computes the first terms of the Taylor expansion of  $f(x)$  and then makes an ansatz with undetermined polynomial coefficients  $c_j(x)$ . If the resulting linear system is overdetermined (i.e., if sufficiently many Taylor coefficients were used) but still admits a nontrivial solution, then the detected ODE is very likely to be correct. Another strategy to gain confidence in the result, is to test it with further Taylor coefficients, that were not used in the computation. However, this method can never produce a rigorous proof of the result and there is always a (very small) probability that the guess is indeed wrong.

### 2.1 Starting from the Integral Representation

In this section we briefly recapitulate some previous work done by Broadhurst and Guttman, who used the integral representation (3) of the LGF as a starting point.

A differential equation for the LGF of the four-dimensional fcc lattice was computed by Guttman [2] (see also Theorem 1): for this purpose, the four-fold integral was rewritten as a double integral whose integrand was expanded as a power series. Term-by-term integration yielded a truncated Taylor expansion of the LGF which allowed him to apply the method of guessing.

Very recently, Guttman [3] reported that David Broadhurst had obtained an ODE for the LGF of the five-dimensional fcc lattice (see also Theorem 3), a result that required several days of PARI calculations. Broadhurst's strategy consisted in expanding the integrand in (3) as a geometric series  $\sum_{n=0}^{\infty} \lambda(\mathbf{k})^n z^n$  and in expanding  $\lambda(\mathbf{k})^n$  using the multinomial theorem (which gives a  $(m-1)$ -fold sum if  $m$  is the number of summands in  $\lambda(\mathbf{k})$ ). The inner terms can now be integrated using Wallis' formula

$$\int_0^\pi \cos(x)^{2n} dx = \frac{\pi}{4^n} \binom{2n}{n}.$$

The structure function of the 5D fcc lattice consists of 10 summands. Thus the computation of the  $n$ -th Taylor coefficient of  $P(\mathbf{0}; z)$  requires the evaluation of a 9-fold sum, or in other words, has complexity  $O(n^9)$ .

### 2.2 Counting the walks

A different way to crank out as many Taylor coefficients of the LGF as necessary is to explicitly count all possible excursions with a certain number of steps. Let  $a_n(\mathbf{x})$  be the number of walks from  $\mathbf{0}$  to  $\mathbf{x}$  with  $n$  steps and let  $c$  denote the coordination number of the lattice, then the lattice Green's function  $P(\mathbf{0}; z) = \sum_{n=0}^{\infty} a_n(\mathbf{0})(z/c)^n$ , as we have already remarked earlier. The values of the  $(d+1)$ -dimensional sequence ( $d$  again denotes the dimensionality of the lattice) can be computed with the recurrence (6). To obtain the first  $n$  Taylor coefficients

hence requires one to fill the  $(d + 1)$ -dimensional array  $(a_m(\mathbf{x}))_{0 \leq m, x_1, \dots, x_d < n}$  with values (by symmetry it suffices to consider the first octant only, which again by symmetry can be restricted to the wedge  $x_1 \geq x_2 \geq \dots \geq x_d$ ). Still, this has complexity  $O(n^{d+1})$ . Further optimizations consist in cutting off the regions where the sequence can be predicted to be zero (e.g. when  $x_j > n$ ), and to truncate the  $x_j$ -coordinates at  $n/2$  (since we are interested in excursions, values too far from the origin  $\mathbf{0}$  are not relevant). Although the complexity is better than before, computing the full array of values can be quite an effort. Again, in the example of the 5D fcc lattice, about 115 Taylor coefficients are necessary to recover the recurrence for  $a_n(\mathbf{0})$  (which then gives rise to the differential equation of  $P(\mathbf{0}; z)$ ), and hence the full array contains about  $115 \cdot 58^5 / 5! \approx 6.3 \cdot 10^8$  values! Fortunately we can do better.

### 2.3 Multi-Step guessing

How can the recurrence for  $a_n(\mathbf{0})$  be computed without calculating all the values of the multivariate sequence  $a_n(\mathbf{x})$  in the box  $[0, n]^{d+1}$  (or some slightly optimized version of it)? In the previous section, we first computed lots of data, then threw away most of it, and did a single guessing step. But the guessing can be done in several steps which we call *multi-step guessing*. The method is illustrated on the 5D fcc example. As before, we start with the recurrence (6) to crank out a moderate number of values for the six-dimensional sequence  $a_n(x_1, \dots, x_5)$ , namely in the box  $[0, 15]^6$ , which takes about 30 seconds only. From this array we pick the values of  $a_n(x_1, x_2, x_3, 0, 0)$  which constitute a four-dimensional sequence that we denote with  $b_n(x_1, x_2, x_3)$ . The data is now used to guess recurrences for this new function  $b$ . One of these recurrences is

$$\begin{aligned} & (n+1)b_n(x_1, x_2+3, x_3+1) - (n+1)b_n(x_1, x_2+1, x_3+3) + \\ & (n+1)b_n(x_1+1, x_2, x_3+3) - (n+1)b_n(x_1+1, x_2+3, x_3) + \\ & (n+1)b_n(x_1+1, x_2+3, x_3+4) - (n+1)b_n(x_1+1, x_2+4, x_3+3) - \\ & (n+1)b_n(x_1+3, x_2, x_3+1) + (n+1)b_n(x_1+3, x_2+1, x_3) - \\ & (n+1)b_n(x_1+3, x_2+1, x_3+4) + (n+1)b_n(x_1+3, x_2+4, x_3+1) + \\ & (n+1)b_n(x_1+4, x_2+1, x_3+3) - (n+1)b_n(x_1+4, x_2+3, x_3+1) + \\ & (x_2+2)b_{n+1}(x_1+1, x_2+2, x_3+3) - (x_3+2)b_{n+1}(x_1+1, x_2+3, x_3+2) - \\ & (x_1+2)b_{n+1}(x_1+2, x_2+1, x_3+3) + (x_1+2)b_{n+1}(x_1+2, x_2+3, x_3+1) + \\ & (x_3+2)b_{n+1}(x_1+3, x_2+1, x_3+2) - (x_2+2)b_{n+1}(x_1+3, x_2+2, x_3+1) = 0 \end{aligned}$$

which has the disadvantage that it does not allow us to compute the values  $b_n(0, 0, 0)$  since the leading coefficient vanishes; unfortunately this phenomenon occurs frequently in this context. An additional recurrence that does not suffer from this handicap is much larger and therefore not reproduced here. Anyway, guessing these recurrences can be done in less than a minute.

Now these recurrences can be used to compute more values for the sequence  $b_n(x_1, x_2, x_3)$  (in 30 seconds one can now go up to  $n = 30$ ) which in turn are used to guess recurrences for  $b_n(x_1, x_2, 0)$ . These latter recurrences allow to compute  $a_n(\mathbf{0}) = b_n(0, 0, 0)$  for  $0 \leq n \leq 115$  in about 2.5 minutes. Voilà! The whole computation takes less than 5 minutes on a modest laptop.

### 3 A Computer Algebra Approach

Again, we want to emphasize that the results presented in the previous section are certainly nice, but lack mathematical rigor. To achieve ultimate confidence in their correctness we have to apply a different method. One possible such method is called *creative telescoping*, a short introduction of which is given in the following section. After that we are able to state our results in the form of theorems.

#### 3.1 Creative Telescoping

This method has been popularized by Zeilberger in his seminal paper [15]. Since then it has been applied to innumerable identities involving hypergeometric summations, multisums, integrals of special functions, and various other kinds of problems. The basic idea is very simple and we illustrate it on the example of a definite integral  $F(z) = \int_a^b f(x, z) dx$ . The main step in the algorithm consists in finding a partial differential equation for  $f(x, z)$  that can be written in the form

$$(A(z, D_z) + D_x B(x, z, D_x, D_z))(f(x, z)) = 0 \quad (8)$$

where the *principal part*  $A \in \mathbb{C}(z)\langle D_z \rangle$  and the *delta part*  $B \in \mathbb{C}(x, z)\langle D_x, D_z \rangle$  are differential operators, with the previously introduced notation of  $D_x$  being the partial derivative w.r.t.  $x$ . By  $\mathbb{C}(z)\langle D_z \rangle$  we denote the non-commutative Ore algebra that can be viewed as a polynomial ring in the “variable”  $D_z$  with rational function coefficients. The result of the algorithm is a (possibly) inhomogeneous linear ODE for the integral  $F(z)$  that is obtained by integrating Equation (8):

$$A(F(z)) + \left[ B(f(x, z)) \right]_{x=a}^{x=b}.$$

In applications one frequently encounters the situation that the second part vanishes, yielding a homogeneous ODE. This is because many integrals that occur in practice, have *natural boundaries*. With our study of LGFs, we are in a similar situation: in Section 1.4 it was shown that the principal part automatically annihilates the LGF. Nevertheless, in some of the present cases we did do the additional (but superfluous) check that the differential equation is indeed homogeneous by plugging in the boundaries of the integral, and got confirmation.

If this method is applied to a one-dimensional integral with hyperexponential integrand (i.e., its logarithmic derivative is a rational function), then it is called the Almkvist-Zeilberger algorithm. Its summation counterpart is the celebrated Zeilberger algorithm for hypergeometric summation. For our purposes we have to generalize the input class for the integrand to the so-called  $\partial$ -finite holonomic functions: a function  $f(x_1, \dots, x_d)$  is called  $\partial$ -finite if for each  $x_i$  there exists a linear ODE for  $f$  with respect to  $x_i$ . If in addition  $f$  is holonomic (the definition of this notion is somewhat technical and is omitted here) then the existence of creative telescoping operators like (7) or (8) is guaranteed.

The first algorithm to compute (8) for general  $\partial$ -finite functions (our examples fall into this class, too) was proposed by Chyzak [1]. It can deal with single integrations only and thus has to be applied iteratively for multiple integrals. Its main drawback is its complexity that makes it impossible to apply it to the problems discussed in this paper. In [9] we have developed a different approach



to compute (8) which is much better suited for large examples involving multidimensional integrals. In addition, it can directly deal with multiple integrals by computing operators of the form (7), but in the present context it turned out to be more efficient to do the integrations step by step. Both algorithms are implemented in our Mathematica package `HolonomicFunctions` [10], and a detailed introduction into the topic is given in [8]. The following example demonstrates how this method is applied to the previously discussed two-dimensional lattice.

**Example 3.** Looking at the integrand of Equation (5) one realizes that it is not  $\partial$ -finite since no linear ODE with respect to  $k_1$  can be found (and analogously for  $k_2$ ). But by means of the simple substitutions  $\cos k_1 \rightarrow x_1$  and  $\cos k_2 \rightarrow x_2$  we can overcome this trouble: the integral now reads

$$P(z) = \frac{1}{\pi^2} \int_{-1}^1 \int_{-1}^1 \frac{dx_1 dx_2}{(1 - zx_1x_2)\sqrt{1-x_1^2}\sqrt{1-x_2^2}}. \quad (9)$$

Let  $f(x_1, x_2, z)$  denote the above integrand; it is easily verified that it is a  $\partial$ -finite function. The three ODEs w.r.t.  $x_1$ ,  $x_2$ , and  $z$  are given by the operators

$$\begin{aligned} G_1 &= (x_1x_2z - 1)D_z + x_1x_2, \\ G_2 &= (x_2^2 - 1)(x_1x_2z - 1)D_{x_2} + (2x_1x_2^2z - x_1z - x_2), \\ G_3 &= (x_1^2 - 1)(x_1x_2z - 1)D_{x_1} + (2x_1^2x_2z - x_1 - x_2z), \end{aligned}$$

so that  $G_i(f(x_1, x_2, z)) = 0$  for  $i = 1, 2, 3$ . In this example it is an easy exercise to check that the creative telescoping operator

$$z(z^2 - 1)D_z^2 + (3z^2 - 1)D_z + z + D_{x_1} \frac{x_2 - x_1^2x_2}{x_1x_2z - 1} + D_{x_2} \frac{x_2z - x_2^3z}{x_1x_2z - 1} \quad (10)$$

annihilates the integrand  $f$ . Indeed, it can be written as a linear combination

$$\left( \frac{z(z^2 - 1)}{x_1x_2z - 1} D_z + \frac{x_1x_2z(z^2 + 1) - 3z^2 + 1}{(x_1x_2z - 1)^2} \right) G_1 - \frac{x_2}{(x_1x_2z - 1)^2} (zG_2 + G_3)$$

of the previously computed operators. It follows that the double integral (9) satisfies the ODE

$$z(z^2 - 1)P''(z) + (3z^2 - 1)P'(z) + zP(z) = 0$$

whose solution is the elliptic integral  $\mathbf{K}(z^2)$ .

Alternatively, the two integrations can be performed in two steps (the strategy that will be applied to the higher-dimensional fcc lattices). In the first step (integration w.r.t.  $x_1$ ) the following two creative telescoping operators are found:

$$\begin{aligned} &(x_2^2z^2 - 1)D_z + x_2^2z + D_{x_1}(x_1^2 - 1)x_2 \\ &(x_2^2 - 1)(x_2^2z^2 - 1)D_{x_2} + x_2(2x_2^2z^2 - z^2 - 1) + D_{x_1}(x_1^2 - 1)(x_2^2 - 1)z. \end{aligned}$$

They certify that the integral  $\int_{-1}^1 f(x_1, x_2, z) dx_1$  is annihilated by the operators  $(x_2^2z^2 - 1)D_z + x_2^2z$  and  $(x_2^2 - 1)(x_2^2z^2 - 1)D_{x_2} + x_2(2x_2^2z^2 - z^2 - 1)$ . Next the operator

$$z(z^2 - 1)D_z^2 + (3z^2 - 1)D_z + z - D_{x_2} \frac{x_2z(x_2^2 - 1)}{(x_2^2z^2 - 1)}$$

which is a linear combination of the previous ones, again reveals the same ODE for the double integral.  $\square$

## 3.2 Results

Using the above methodology and software, we have computed differential equations for the LGFs of the fcc lattices in four, five, and six dimensions, and rigorously proved their correctness. Additionally, this allows the computation of the return probabilities in the respective lattices up to very high precision.

**Theorem 1.** *The lattice Green's function of the four-dimensional face-centered cubic lattice*

$$P(z) = \frac{1}{\pi^4} \int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi \frac{dk_1 dk_2 dk_3 dk_4}{1 - \frac{z}{6} (\cos k_1 \cos k_2 + \cos k_1 \cos k_3 + \dots + \cos k_3 \cos k_4)}$$

satisfies the differential equation

$$\begin{aligned} & (z-1)(z+2)(z+3)(z+6)(z+8)(3z+4)^2 z^3 P^{(4)}(z) + \\ & 2(3z+4)(21z^6 + 356z^5 + 2079z^4 + 4920z^3 + 3676z^2 - 2304z - 3456)z^2 P^{(3)}(z) + \\ & 6(81z^7 + 1286z^6 + 7432z^5 + 19898z^4 + 25286z^3 + 11080z^2 - 5248z - 5376)z P''(z) + \\ & 12(45z^7 + 604z^6 + 2939z^5 + 6734z^4 + 7633z^3 + 3716z^2 + 224z - 384)P'(z) + \\ & 12(9z^5 + 98z^4 + 382z^3 + 702z^2 + 632z + 256)z P(z) = 0. \end{aligned}$$

*Proof.* Here we give only an outline of the proof. The calculations in extenso are provided as a Mathematica notebook in the electronic supplementary material [11] (to be downloaded from <http://www.risc.jku.at/people/ckoutsch/fcc/>).

The substitutions  $\cos k_j \rightarrow x_j$  transform the integrand of the four-fold integral to

$$f(x_1, \dots, x_4, z) = \frac{1}{(1 - \frac{z}{6}(x_1 x_2 + x_1 x_3 + \dots + x_3 x_4)) \cdot \prod_{j=1}^4 \sqrt{1 - x_j^2}}. \quad (11)$$

This expression is  $\partial$ -finite and thus a Gröbner basis of the zero-dimensional annihilating left ideal can be computed (`ann0` in the notebook). Next, some operators  $A_j(x_2, x_3, x_4, z, D_{x_2}, D_{x_3}, D_{x_4}, D_z)$  and  $B_j(x_1, \dots, x_4, z, D_{x_1}, \dots, D_{x_4}, D_z)$  for  $1 \leq j \leq 4$  are computed, such that  $A_j + D_{x_1} B_j$  is an element in the left ideal generated by `ann0`. This fact can be easily tested by reducing it with the Gröbner basis: the remainder being 0 answers the membership question in an affirmative way. In the notebook, the  $A_j$ 's are collected in the variable `ann0`, and the  $B_j$ 's in the variable `delta1`. We conclude that  $A_1, A_2, A_3$ , and  $A_4$  generate an annihilating left ideal for the integral  $\int_0^\pi f(x_1, x_2, x_3, x_4, z) dx_1$ . In a similar fashion, the integrations with respect to  $x_2, x_3$ , and  $x_4$  are performed, yielding a single ODE in  $z$  that annihilates  $P(z)$ .  $\square$

Note that this theorem confirms the result given in [2]. Guttmann also observed that the differential equation given in Theorem 1 has *maximal unipotent monodromy* (MUM), i.e., its indicial equation is of the form  $\lambda^n$  and hence has only 0 as a root, and additionally satisfies the *Calabi-Yau condition*. Many LGFs of other lattices fall into this class, too, and therefore this fact may not seem too surprising.

In order to receive the return probability in the four-dimensional fcc lattice, holonomic closure properties are applied to compute a differential equation for

$$\frac{P(z)}{1-z} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n p_k(\mathbf{0}) \right) z^n,$$

which in turn gives a recurrence for  $f(n) = \sum_{k=0}^n p_k(\mathbf{0})$ :

$$\begin{aligned}
& (n+2)(n+3)^2(n+4)(35n^2+420n+1252)f(n)+ \\
& (n+3)(n+4)(595n^4+11375n^3+79874n^2+244384n+276024)f(n+1)+ \\
& 3(n+4)(1015n^5+24780n^4+240253n^3+1156976n^2+ \\
& \quad 2769392n+2638272)f(n+2)+ \\
& (3325n^6+107100n^5+1427695n^4+10080600n^3+39767416n^2+ \\
& \quad 83134488n+71984160)f(n+3)- \\
& 4(2065n^6+62580n^5+788848n^4+5295615n^3+19973086n^2+ \\
& \quad 40139838n+33590844)f(n+4)- \\
& 12(735n^6+25200n^5+359282n^4+2725632n^3+11601091n^2+ \\
& \quad 26259960n+24690708)f(n+5)+ \\
& 288(35n^2+350n+867)(n+6)^4f(n+6) = 0.
\end{aligned}$$

The initial values

$$f(0) = 1, \quad f(1) = 1, \quad f(2) = \frac{25}{24}, \quad f(3) = \frac{19}{18}, \quad f(4) = \frac{1637}{1536}, \quad f(5) = \frac{549}{512}$$

are easily (of course, not by hand!) computed by counting the number of excursions of length up to 5. For the return probability

$$R = 1 - \left( \lim_{n \rightarrow \infty} f(n) \right)^{-1}$$

we need to evaluate the limit of the sequence  $f(n)$ . This can be done very accurately when knowing the asymptotics of the sequence. We apply the method of Wimp and Zeilberger [14], which has been implemented in Mathematica by Kauers [7], and obtain the following basis of asymptotic solutions:

$$\begin{aligned}
s_1(n) &= \frac{1}{n^2} \left( -\frac{1}{2} \right)^n \left( 1 - \frac{5}{6n} + \frac{67}{24n^2} + \frac{1459}{144n^3} + O\left(\frac{1}{n^4}\right) \right), \\
s_2(n) &= \frac{1}{n^2} \left( -\frac{1}{3} \right)^n \left( 1 - \frac{5}{2n} + \frac{51}{8n^2} - \frac{143}{8n^3} + O\left(\frac{1}{n^4}\right) \right), \\
s_3(n) &= \frac{1}{n^2} \left( -\frac{1}{6} \right)^n \left( 1 - \frac{45}{14n} + \frac{4633}{392n^2} - \frac{112407}{5488n^3} + O\left(\frac{1}{n^4}\right) \right), \\
s_4(n) &= \frac{1}{n^2} \left( -\frac{1}{8} \right)^n \left( 1 - \frac{52}{9n} + \frac{812}{27n^2} - \frac{45820}{243n^3} + O\left(\frac{1}{n^4}\right) \right), \\
s_5(n) &= \frac{1}{n} \left( 1 - \frac{1}{n} + \frac{7}{9n^2} - \frac{7}{18n^3} + O\left(\frac{1}{n^4}\right) \right), \\
s_6(n) &= 1.
\end{aligned}$$

Obviously, the first four solutions do not significantly contribute as they tend to 0 very rapidly. Thus by taking into account  $s_5(n)$  and  $s_6(n)$  only, and computing the asymptotic expansion to a higher order (e.g., 30), allows us to obtain (at least) 100 correct digits of the limit.

**Corollary 2.** *The LGF of the four-dimensional fcc lattice, evaluated at  $z = 1$  is*

$$P(1) \approx 1.10584379792120476018299547088585107443954623663875285836499,$$

and therefore the return probability is

$$R \approx 0.09571315417256289673531676490121018570070881963801735768774.$$

**Theorem 3.** *The lattice Green's function of the five-dimensional face-centered cubic lattice*

$$P(z) = \frac{1}{\pi^5} \int_0^\pi \cdots \int_0^\pi \frac{dk_1 dk_2 dk_3 dk_4 dk_5}{1 - \frac{z}{10} (\cos k_1 \cos k_2 + \cos k_1 \cos k_3 + \cdots + \cos k_4 \cos k_5)}$$

*satisfies the differential equation*

$$\begin{aligned} & 16(z-5)(z-1)(z+5)^2(z+10)(z+15)(3z+5)(15678z^6 + 144776z^5 + \\ & 449735z^4 + 933650z^3 - 1053375z^2 + 3465000z - 675000)z^4 P^{(6)}(z) + \\ & 8(z+5)(3057210z^{12} + 97471734z^{11} + 1048560285z^{10} + 3939663705z^9 - \\ & 4878146975z^8 - 87265479875z^7 - 304623830625z^6 - 266627903125z^5 + \\ & 254876515625z^4 - 1289447109375z^3 - 503550000000z^2 + \\ & 1774828125000z - 354375000000)z^3 P^{(5)}(z) + \\ & 10(27279720z^{13} + 923795772z^{12} + 11725276842z^{11} + 68439921540z^{10} + \\ & 148313757125z^9 - 382134335775z^8 - 3351125770500z^7 - 7801785421250z^6 - \\ & 3779011321875z^5 - 7716298734375z^4 - 39702348750000z^3 + \\ & 3393646875000z^2 + 23905125000000z - 5568750000000)z^2 P^{(4)}(z) + \\ & 5(255864960z^{13} + 7892060544z^{12} + 92744995638z^{11} + 524857986060z^{10} + \\ & 1350059072325z^9 - 465440555100z^8 - 13545524756500z^7 - 26918293320000z^6 - \\ & 3649915059375z^5 - 77498059625000z^4 - 190176960000000z^3 + \\ & 40530375000000z^2 + 45343125000000z - 13162500000000)z P^{(3)}(z) + \\ & 5(496679040z^{13} + 13819981248z^{12} + 149186684934z^{11} + 810956145330z^{10} + \\ & 2287368823475z^9 + 1646226060075z^8 - 8282515456375z^7 - 6199228765625z^6 + \\ & 13367806743750z^5 - 110925736437500z^4 - 133825053750000z^3 + \\ & 44457862500000z^2 + 5055750000000z - 3240000000000)P''(z) + \\ & 10(167064768z^{12} + 4143853440z^{11} + 40678130502z^{10} + 209673119160z^9 + \\ & 607021304825z^8 + 689643286650z^7 - 135661728250z^6 + 3711617481250z^5 + \\ & 2664478321875z^4 - 21210430812500z^3 - 7268326875000z^2 + \\ & 4816462500000z - 189000000000)P'(z) + \\ & 30(7525440z^{11} + 163913184z^{10} + 1443544710z^9 + 6925739310z^8 + \\ & 19123388575z^7 + 21336230625z^6 + 36477006875z^5 + 187923165625z^4 - \\ & 55567000000z^3 - 346865625000z^2 + 84037500000z + 27000000000)P(z) \end{aligned}$$

*Proof.* The proof is very analogous to that of Theorem 1 and is given in detail in the supplementary material [11].  $\square$

Remarkably enough, the indicial equation of the differential equation presented in Theorem 3 is  $\lambda^5(\lambda-1)$  and hence the ODE lacks MUM. For the same reason it is not a Calabi-Yau differential equation.

**Corollary 4.** *The LGF of the five-dimensional fcc lattice, evaluated at  $z = 1$  is*

$$P(1) \approx 1.04885235135491485162956376369999275945402550465206640313845,$$

*and therefore the return probability is*

$$R \approx 0.04657695746384802419337442059480329107640239774632112930532.$$

**Theorem 5.** *The lattice Green's function of the six-dimensional face-centered cubic lattice*

$$P(z) = \frac{1}{\pi^6} \int_0^\pi \cdots \int_0^\pi \frac{dk_1 dk_2 dk_3 dk_4 dk_5 dk_6}{1 - \frac{z}{15} (\cos k_1 \cos k_2 + \cos k_1 \cos k_3 + \cdots + \cos k_5 \cos k_6)}$$

satisfies a differential equation of order 8 and with polynomial coefficients of degree 43. Its leading coefficient is

$$z^6(z-3)(z-1)(z+4)(z+5)(z+9)(z+15)^2(z+24)(2z+3)(2z+15) \\ \times (4z+15)(7z+60)\text{poly}(z)$$

where  $\text{poly}(z)$  stands for a certain irreducible polynomial of degree 25, and its indicial equation is  $\lambda^6(\lambda-1)^2$ . The full equation is too long to be printed here, but can be found in [11].

*Proof.* The proof is very analogous to that of Theorem 1 and is given in detail in the supplementary material [11].  $\square$

As in the five-dimensional fcc lattice, the differential equation of Theorem 5 lacks MUM and therefore is not Calabi-Yau.

**Corollary 6.** *The LGF of the six-dimensional fcc lattice, evaluated at  $z = 1$  is*

$$P(1) \approx 1.02774910062749883985936367927396850209243990900114872425172,$$

and therefore the return probability is

$$R \approx 0.02699987828795612426936417542619638021612262676239501413843.$$

## 4 Outlook and Acknowledgements

While the calculations for Theorem 1 and Theorem 3 are done in a few minutes respectively hours, it was a major effort of several days to compute the certificates that prove Theorem 5; they are several hundred MegaBytes in size. With the methods described in this paper and with the current hardware, it is completely out of the question to attack the fcc lattice in seven dimensions. An interesting question is whether the pattern that showed up in dimensions four to six continues. This would suggest a differential equation of order 10 with indicial equation  $\lambda^7(\lambda-1)^3$ . But who knows?

For the three corollaries we computed the approximations for the return probabilities with more than one hundred valid digits. But we have no clue what their exact values are. Cyril Banderier evaluated these numbers up to several thousand digits, but also he was unable to identify the closed forms. So we leave these questions open, as a challenge for future research.

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