DESY 11–033
DO–TH 11–12
SFB/CPP-11-24
LPN 11/24
May 2011

arXiv:1105.6063 [math-ph]

Harmonic Sums and Polylogarithms Generated by Cyclotomic Polynomials

Jakob Ablinger[a], Johannes Blümlein[b], and Carsten Schneider[a]

[a] *Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Altenbergerstraße 69, A–4040, Linz, Austria*

[b] *Deutsches Elektronen–Synchrotron, DESY, Platanenallee 6, D-15738 Zeuthen, Germany*

Abstract

The computation of Feynman integrals in massive higher order perturbative calculations in renormalizable Quantum Field Theories requires extensions of multiply nested harmonic sums, which can be generated as real representations by Mellin transforms of Poincaré–iterated integrals including denominators of higher cyclotomic polynomials. We derive the cyclotomic harmonic polylogarithms and harmonic sums and study their algebraic and structural relations. The analytic continuation of cyclotomic harmonic sums to complex values of N is performed using analytic representations. We also consider special values of the cyclotomic harmonic polylogarithms at argument $x = 1$, resp., for the cyclotomic harmonic sums at $N \to \infty$, which are related to colored multiple zeta values, deriving various of their relations, based on the stuffle and shuffle algebras and three multiple argument relations. We also consider infinite generalized nested harmonic sums at roots of unity which are related to the infinite cyclotomic harmonic sums. Basis representations are derived for weight $\mathsf{w} = 1,2$ sums up to cyclotomy $\mathsf{l} = 20$.

Dedicated to Martinus Veltman on the occasion of his 80th birthday.

1 Introduction

The analytic calculation of Feynman integrals requires the complete understanding of the associated mathematical structures at a given loop level. By the pioneering work in Refs. [1, 2] this has been thoroughly achieved for one loop integrals occurring in renormalizable Quantum Field Theories. The corresponding complete framework in case of higher order calculations is, however, not yet available. Which mathematical functions are of relevance there is revealed stepwise in specific higher order calculations. Here higher transcendental functions, like the generalized hypergeometric functions and their generalizations [3], play a central role. Their series expansion in the dimensional parameter [1, 4] leads to nested infinite sums over products of digamma functions [5], cf. [6]. The nested harmonic sums [7, 8] are defined by

$$S_{b,\vec{a}}(N) \quad = \quad \sum_{k=1}^{N} \frac{(\mathsf{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k) \ , \qquad S_{\emptyset}(N) = 1 \ , \quad b, a_i \in \mathbb{Z}\backslash\{0\}. \tag{1.1}$$

They form a quasi-shuffle algebra [9]. Their values for $N \to \infty$ are the multiple zeta values $\zeta_{\vec{a}}$, resp. Euler-Zagier values [10] defined by

$$\zeta_{b,\vec{a}} = \lim_{N \to \infty} S_{b,\vec{a}}(N), \quad b \neq 1,$$

see [11]. In case of massless problems to 3–loop order the results for single scale quantities in Mellin space can be written by polynomial expressions in terms of $S_{\vec{a}}(N)$ and $\zeta_{\vec{a}}$ with coefficients being from the rational function field $\mathbb{Q}(N)$, cf. e.g. [12]. In this context we consider the Mellin transform

$$\mathbf{M}[f(x)](N) = \int_0^1 dx\ x^N\ f(x) \ . \tag{1.2}$$

In most of the applications below we assume $N \in \mathbb{N}_+ = \mathbb{N}\backslash\{0\}$.

In computations at even higher orders in the coupling constant and through finite mass effects, however, generalizations of the nested harmonic sums contribute, at least in intermediary results. One extension concerns the so-called generalized harmonic sums [13, 14] given by

$$S_{b,\vec{a}}(\zeta, \vec{\xi}; N) \quad = \quad \sum_{k=1}^{N} \frac{\zeta^k}{k^b} S_{\vec{\xi}}(\vec{r}; k) \ , \tag{1.3}$$

with $b, a_i \in \mathbb{N}_+; \zeta, \xi_i \in \mathbb{R}^* = \mathbb{R}\backslash\{0\}$. Known examples are related to the second index set $\xi_i \in \{1, -1, 1/2, -1/2, 2, -2\}$, cf. [12, 15, 16].

In case of single scale problems with two massive lines and $m_1 = m_2$ at 3-loop order summation is also required over terms

$$\frac{(\pm 1)^k}{(2k+1)^n} \ , \tag{1.4}$$

which is a special case of

$$\frac{(\pm 1)^k}{(l \cdot k + m)^n} \ , \tag{1.5}$$

with $l, m, n \in \mathbb{N}_+$. Note that we have deliberately chosen a real representation here, which is of practical importance in case of fast polynomial operations needed for solving nested summation problems [17] [1]. Sums containing fractional terms m/l were considered in the context of colored harmonic sums in [20–22]. It is expected that sums of this kind and their iterations will occur in a wide class of massive calculations in higher order loop calculations in Quantum Electrodynamics, Quantum Chromodynamics, and other renormalizable Quantum Field Theories, in particular studying single distributions, but also for more variable differential distributions. Usually objects of this kind emerge first at intermediary steps and, at even higher orders, they occur in the final results. Therefore, these quantities have to be understood and methods have to be provided to perform these sums. This does not apply to integer values of the Mellin variable N only, but also to the analytic continuation of these quantities to $N \in \mathbb{C}$, since the experimental applications require the Mellin inversion into momentum-fraction space.

We show that the single sums (1.5) and their nested iterations can be obtained from linear combinations of Mellin transforms of harmonic polylogarithms over an alphabet of letters containing $x^l/\Phi_k(x),\ \ 0 < l < \deg(\Phi_k(x))$, where $\Phi_k(x)$ denotes the kth cyclotomic polynomial [23]. One may form words by Poincaré-iterated integrals [25] over this alphabet, which leads to the cyclotomic harmonic polylogarithms $\mathfrak{H}$, forming a shuffle algebra. This class extends the harmonic polylogarithms [26]. The Mellin transform of elements of $\mathfrak{H}$ has support $x \in [0, 1]$, but requires an extension in the Mellin variable $N \in \mathbb{N}$,

$$N \to k \cdot N \ , \tag{1.6}$$

where k denotes the index of $\Phi_k(x)$. This assumption allows to associate nested harmonic sums of the cyclotomic type by this Mellin transform. Special values are obtained by either the cyclotomic harmonic polylogarithms $H_{\vec{a}}(x)$ at $x = 1$ or the associated nested harmonic sums for $N \to \infty$. They extend the multiple zeta values and Euler-Zagier values, cf. Section 5. In the present paper we investigate relations and representations of these three classes of quantities. Special emphasis has been put on the class of nested sums (1.4) where all derived algorithms have been incorporated within the computer algebra package HarmonicSums [27].

The paper is organized as follows. In Section 2 we establish the connection between the cyclotomic harmonic sums and the cyclotomic harmonic polylogarithms through the Mellin transform, at modified argument kN. The basic properties of the cyclotomic harmonic polylogarithms are investigated in Section 3. The cyclotomic harmonic polylogarithms obey a shuffle algebra. The nested sums at finite values of N are studied in Section 4, including their algebraic and structural relations, generalizing [28–31]. Here are also three multiple argument relations of interest. The analytic continuation of the new harmonic sums to complex values of N is presented based on recursion and asymptotic representations, similar to the case of the nested harmonic sum [29–32]. The representation of the cyclotomic harmonic sums requires to know their values at $N \to \infty$, which are equivalently given by the cyclotomic harmonic polylogarithms at $x = 1$. The set of special numbers spanning the Euler-Zagier and multiple zeta values [11] are extended. We study the cases of weight $\mathsf{w} = 1{,}2$ up to cyclotomy $\mathsf{l} = 20$ and derive the corresponding relations based on the stuffle and shuffle algebra and three multiple argument relations. Furthermore, we investigate the relations of the cyclotomic harmonic sums for $N \to \infty$ extending for words resulting from the alphabet $(\pm 1)^k/k, (\pm 1)^k/(2k+1)$, cf. Section 5. For the cyclotomic harmonic polylogarithms, harmonic sums and their values at $N \to \infty$ basis representations are derived. In Section 6 we study the relations of the infinite nested harmonic sums with numerators at lth root of unity, $\mathsf{l} \le 20$ for weight $\mathsf{w} = 1{,}2$. Here, also the distribution relation, cf. [19], is considered beyond the relations mentioned before. We also consider a relation valid for finite generalized

[1] Complex representations are related to the so-called colored harmonic sums $S_{b,\vec{a}}(p, \vec{r}; N) = \sum_{k=1}^{N} \frac{p^k}{k^b} S_{\vec{a}}(\vec{r}; k)$ with $b, a_i \in \mathbb{N}_+, p, r_i \in \cup_{l=2}^{M}\{\exp[2\pi i(n/l)], n \in \{1, ..., l-1\}\}$, cf. [18, 19].

harmonic sums with root numerator weights. Section 7 contains the conclusions. Some technical details are given in the Appendices.

2 Basic Formalism

We consider cyclotomic harmonic sums defined by

$$S_{\{a_1,b_1,c_1\},...,\{a_l,b_l,c_l\}}(s_1,...,s_l;N) = \sum_{k_1=1}^{N} \frac{s_1^k}{(a_1k_1+b_1)^{c_1}} S_{\{a_2,b_2,c_2\};...;\{a_l,b_l,c_l\}}(s_2,...,s_l;k_1), \quad S_\emptyset = 1, \tag{2.1}$$

where $a_i, c_i \in \mathbb{N}_+, b_i \in \mathbb{N}, \quad s_i = \pm 1, a_i > b_i$; the weight of this sum is defined by $c_1 + \cdots + c_l$ and $\{a_i, b_i, c_i\}$ denote lists, not sets. One may generalize this case further allowing $s_i \in \mathbb{R}^*$, [14]. Of special interest will be the infinite cyclotomic harmonic polylogarithms defined by

$$\sigma_{\{a_1,b_1,c_1\},...,\{a_l,b_l,c_l\}}(s_1,...,s_l) = \lim_{N\to\infty} S_{\{a_1,b_1,c_1\},...,\{a_l,b_l,c_l\}}(s_1,...,s_l;N)$$

which diverge if $c_1 = 1, s_1 = 1$. Sometimes we will use the notation

$$S_{\{a_1,b_1,c_1\},...,\{a_l,b_l,c_l\}}(s_1,...,s_l;N) = S_{\{a_1,b_1,s_1c_1\},...,\{a_l,b_l,s_lc_l\}}(N) \tag{2.2}$$

$$\sigma_{\{a_1,b_1,c_1\},...,\{a_l,b_l,c_l\}}(s_1,...,s_l) = \sigma_{\{a_1,b_1,s_1c_1\},...,\{a_l,b_l,s_lc_l\}} \tag{2.3}$$

as shortcut below.

For further considerations, we rely on the following procedure which transforms the sums to integrals. The denominators have the following integral representation

$$\frac{(\pm 1)^k}{ak+b} = \int_0^1 dx\ x^{ak+b-1}(\pm 1)^k \tag{2.4}$$

$$\frac{(\pm 1)^k}{(ak+b)^c} = \int_0^1 \frac{dx_1}{x_1}\int_0^{x_1}\frac{dx_2}{x_2}...\int_0^{x_{c-2}}\frac{dx_{c-1}}{x_{c-1}}\int_0^{x_{c-1}} dx_c\ x_c^{ak+b-1}(\pm 1)^k, \tag{2.5}$$

and the sum over k yields

$$\sum_{k=1}^{l}(\pm 1)^k x^{ak+b-1} = x^{a+b-1}\frac{(\pm x^a)^{l+1}-1}{(\pm x^a)-1}\ . \tag{2.6}$$

This representation is applied to the innermost sum ($a = a_l, b = b_l, c = c_l$). One now may perform the next sum in the same way, provided $a_{l-1}|a_l$. If this is not the case, one transforms the integration variables in (2.4, 2.5) such that the next denominator can be generated, etc. In this way the sums (2.1) can be represented in terms of linear combinations of Poincaré-iterated integrals. Evidently, the representation of the cyclotomic harmonic sum (2.1) in terms of a (properly regularized) Mellin transform will be related to the Mellin variable kN, with k the least common multiple of $a_1, ..., a_l$.

Let us illustrate the principle steps in case of the following example :

$$S_{\{3,2,2\},\{2,1,1\}}(1,-1;N) = \sum_{k=1}^{N}\frac{1}{(3k+2)^2}\sum_{l=1}^{k}\frac{(-1)^l}{(2l+1)}\ . \tag{2.7}$$

The first sum yields

$$S_{\{3,2,2\},\{2,1,1\}}(1,-1;N) = \sum_{k=1}^{N} \int_0^1 dx \frac{x^2}{x^2+1} \frac{(-x^2)^k+1}{(3k+2)^2} \ . \tag{2.8}$$

Setting $x = y^3$ one obtains

$$\begin{aligned} S_{\{3,2,2\},\{2,1,1\}}(1,-1;N) &= 12 \int_0^1 dy \ \frac{y^8}{y^6+1} \sum_{k=1}^{N} \frac{(-y^6)^k - 1}{(6k+4)^2} \\ &= 12 \int_0^1 dy \ \frac{y^4}{y^6+1} \left\{ \int_0^y \frac{dz}{z} \int_0^z dt \ t^9 \ \frac{(-t^6)^N - 1}{t^6+1} \right. \\ &\qquad \left. -y^4 \int_0^1 \frac{dz}{z} \int_0^z dt \ t^9 \ \frac{t^{6N}-1}{t^6-1} \right\} \qquad (2.9) \\ &= 12 \int_0^1 dy \ \frac{y^4}{y^6+1} \int_0^y \frac{dz}{z} \int_0^z dt \ t^9 \ \frac{(-t^6)^N - 1}{t^6+1} \\ &\qquad - (4-\pi) \int_0^1 \frac{dz}{z} \int_0^z dt \ t^9 \ \frac{t^{6N}-1}{t^6-1} \Bigg\} \ . \qquad (2.10) \end{aligned}$$

In general, the polynomials

$$x^a - 1 \tag{2.11}$$

in (2.6) decompose in a product of cyclotomic polynomials, except for $a = 1$ for which the expression is $\Phi_1(x)$, see Section 3. Moreover, the polynomials

$$x^a + 1 = \frac{x^{2a} - 1}{x^a - 1} \tag{2.12}$$

are either cyclotomic for $a = 2^n, n \in \mathbb{N}$ or decompose into products of cyclotomic polynomials in other cases, see Appendix A. All factors divide $(x^a)^l - 1$, resp. $(-x^a)^l - 1$. We remark that Eq. (2.10) is not yet written in terms of a Mellin transform (1.2). To achieve this in an automatic fashion, the cyclotomic harmonic polylogarithms are introduced in Section 3. Furthermore, their special values at $x = 1$ contribute to which we turn in Section 5.

3 Cyclotomic Harmonic Polylogarithms

To account for the newly emerging sums (2.1) in perturbative calculations in Quantum Field Theory we introduce Poincaré-iterated integrals over the alphabet $\mathfrak{A}$

$$\mathfrak{A} = \left\{ \frac{1}{x} \right\} \cup \left\{ \frac{x^l}{\Phi_k(x)} \middle| k \in \mathbb{N}_+, 0 \le l < \varphi(k) \right\}, \tag{3.1}$$

where $\Phi_k(x)$ denotes the kth cyclotomic polynomial [23], and $\varphi(k)$ denotes Euler's totient function [24].

$$\Phi_n(x) = \frac{x^n - 1}{\prod_{d|n, d<n} \Phi_d(x)} \ , \qquad d, n \in \mathbb{N}_+, \tag{3.2}$$

and the first cyclotomic polynomials are given by

$$\Phi_1(x) = x - 1 \tag{3.3}$$
$$\Phi_2(x) = x + 1 \tag{3.4}$$
$$\Phi_3(x) = x^2 + x + 1 \tag{3.5}$$
$$\Phi_4(x) = x^2 + 1 \tag{3.6}$$
$$\Phi_5(x) = x^4 + x^3 + x^2 + x + 1 \tag{3.7}$$
$$\Phi_6(x) = x^2 - x + 1 \tag{3.8}$$
$$\Phi_7(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \tag{3.9}$$
$$\Phi_8(x) = x^4 + 1 \tag{3.10}$$
$$\Phi_9(x) = x^6 + x^3 + 1 \tag{3.11}$$
$$\Phi_{10}(x) = x^4 - x^3 + x^2 - x + 1 \tag{3.12}$$
$$\Phi_{11}(x) = x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \tag{3.13}$$
$$\Phi_{12}(x) = x^4 - x^2 + 1, \quad \text{etc.} \tag{3.14}$$

The alphabet $\mathfrak{A}$ is an extension of the alphabet

$$\mathfrak{A}_{\rm H} = \left\{\frac{1}{x}, \frac{1}{\Phi_1(x)}, \frac{1}{\Phi_2(x)}\right\}, \tag{3.15}$$

generating the usual harmonic polylogarithms [26] [2]. As a shorthand notation we define the letters of $\mathfrak{A}$ by

$$f_0^0(x) = \frac{1}{x} \tag{3.16}$$
$$f_k^l(x) = \frac{x^l}{\Phi_k(x)}, \qquad k \in \mathbb{N}_+, l \in \mathbb{N}, l \le \varphi(k)\ . \tag{3.17}$$

Here the labels k and l form a double index, which always appears at a common position. Mellin transformations associated to $1/\Phi_4(x)$ and $1/\Phi_6(x)$ were discussed long ago in [36],

$$\frac{1}{2}\beta\left(\frac{x}{2}\right) = \int_0^1 dt \frac{t^{x-1}}{t^2+1} \tag{3.18}$$
$$\beta\left(\frac{x}{3}\right) = \beta(x) - \int_0^1 dt\ t^{x-1}\frac{t-2}{t^2-t+1}\ , \tag{3.19}$$

with Stirling's β-function [37]

$$\beta(x) = \frac{1}{2}\left[\Psi\left(\frac{x+1}{2}\right) - \Psi\left(\frac{x}{2}\right)\right]\ . \tag{3.20}$$

Integrals of this kind emerge also in particle physics problems in [20, 38–40].

[2] Note that we defined here the second letter by $1/(x-1)$ which differs in sign from the corresponding letter in [26]. Numerical implementations were given in [33,34]. A few extensions of iterated integrals introduced in [26] based on linear denominator functions of different kind, which are used in quantum-field theoretic calculations, were made in [34, 35].

We form the Poincaré iterated integrals

$$
\begin{aligned}
C_{k_1,\ldots,k_m}^{l_1,\ldots,l_m}(z) &= \frac{1}{m!}\ln(x)^m && \text{if } (l_1,\ldots,l_m)=(0,\ldots,0),(k_1,\ldots,k_m)=(0,\ldots,0),\\
C_{k_m}^{l_m}(z) &= \int_0^z dx\; f_{k_m}^{l_m}(x) && \text{if } k_m \neq 0,\\
C_{k_1,\ldots,k_m}^{l_1,\ldots,l_m}(z) &= \int_0^z dx\; f_{k_1}^{l_1}(x)\; C_{k_2,\ldots,k_m}^{l_2,\ldots,l_m}(x) && \text{if } (k_1,\ldots,k_m)\neq(0,\ldots,0),
\end{aligned}
$$

and $C_{\vec{a}}^{\vec{l}}(z)$ denotes cyclotomic harmonic polylogarithms of weigh $\mathsf{w} = m$. They form a shuffle algebra [9, 41] by multiplication

$$
C_{\vec{a_1}}^{\vec{a_2}}(z)\cdot C_{\vec{b_1}}^{\vec{b_2}}(z) = C_{\vec{a_1}}^{\vec{a_2}}(z) \sqcup\!\sqcup\, C_{\vec{b_1}}^{\vec{b_2}}(z) = \sum_{\left[\begin{smallmatrix}\vec{c}_2\\ \vec{c}_1\end{smallmatrix}\right]\in\left[\begin{smallmatrix}\vec{a}_2\\ \vec{a}_1\end{smallmatrix}\right]\sqcup\!\sqcup\left[\begin{smallmatrix}\vec{b}_2\\ \vec{b}_1\end{smallmatrix}\right]} C_{\vec{c_1}}^{\vec{c_2}}(z) \tag{3.21}
$$

of M^{w} elements at weight w, where M denotes the number of chosen letters from $\mathfrak{A}$. The shuffle symbol $\sqcup\!\sqcup$ implies all combinations of indices $a_{ij} \in \vec{a}_i$ and $b_{ij} \in \vec{b}_j$ leaving the order in both sets unchanged and the brackets [ ] pair the the upper and lower indices forming a unity, cf. (3.17).

The number of basis elements spanning the shuffle algebra are given by

$$
N^{\rm basic}(\mathsf{w}) = \frac{1}{\mathsf{w}}\sum_{d|\mathsf{w}} \mu\left(\frac{\mathsf{w}}{d}\right) M^d, \mathsf{w} \geq 1 \tag{3.22}
$$

according to the 1st Witt formula [41, 42]. Here μ denotes the Möbius function [43]. The number of basic cyclotomic harmonic polylogarithms in dependence of w and M is given in Table 1.

	letters						
weight	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8
2	1	3	6	10	15	21	28
3	2	8	20	40	70	112	168
4	3	18	60	150	315	588	1008
5	6	48	204	624	1554	3360	6552
6	9	116	670	2580	7735	19544	43596
7	18	312	2340	11160	39990	117648	299592
8	30	810	8160	48750	209790	729300	2096640

Table 1: Number of basic cyclotomic harmonic polylogarithms in dependence of the number of letters and weight.

Now the cyclotomic harmonic sums (2.1) can be represented as Mellin transforms of cyclotomic harmonic polylogarithms. In the example (2.10) the iterated integrals have to be rewritten. We express respective denominators in terms of products of cyclotomic polynomials, (A.2-A.16), and perform partial fractioning in the respective integration variable, (A.27-A.38). In our concrete example, the integrand of the y-integral in (2.9) has the representation

$$
\frac{y^4}{y^6+1} = \frac{1}{3}\left[f_4^0(y) - f_{12}^0(y) + 2f_{12}^2(y)\right] . \tag{3.23}
$$

With integration by parts one obtains the following Mellin transforms of argument $6N$ of cyclotomic harmonic polylogarithms $C_{k_1,...,k_m}^{l_1,...,l_m}(x)$ weighted by the letters $f_l^k(x)$ of the alphabet $\mathfrak{A}$:

$$\begin{aligned}
S_{\{3,2,2\},\{2,1,1\}}(1,-1;N) &= \frac{1}{6}(4-\pi)\int_0^1 dx x^3(x^{6N}-1)\left[6+f_1^0(x)-f_2^0(x)-2f_3^0(x)\right. \\
&\qquad \left.-f_3^1(x)-2f_6^0(x)+f_6^1(x)\right]C_0^0(x) \\
&\quad -2\int_0^1 dx x^3\left[(-1)^N x^{6N}-1\right]\left[3-f_4^0(x)-2f_{12}^0(x)+2f_{12}^2(x)\right]C_0^0(x) \\
&\quad -\frac{4}{3}\left[C_{0,4}^{0,0}(1)-C_{0,12}^{0,0}(1)+2C_{0,12}^{0,2}(1)\right]\int_0^1 dx x^3\left[(-1)^N x^{6N}-1\right] \\
&\qquad \times\left[3-f_4^0(x)-2f_{12}^0(x)+2f_{12}^2(x)\right] \\
&\quad +\frac{4}{3}\int_0^1 dx x^3\left[(-1)^N x^{6N}-1\right]\left[C_{0,4}^{0,0}(x)-C_{0,12}^{0,0}(x)+2C_{0,12}^{0,2}(x)\right] \\
&\qquad \times\left[3-f_4^0(x)-2f_{12}^0(x)+2f_{12}^2(x)\right]\ .
\end{aligned} \tag{3.24}$$

The constants $C_{k_1,...,k_m}^{l_1,...,l_m}(1)$ are discussed in Section 5. In particular one obtains

$$C_{0,4}^{0,0} = -\mathbf{C}, \tag{3.25}$$

with $\mathbf{C}$ the Catalan number [44], and $C_{0,12}^{0,0}$ and $C_{0,12}^{0,2}$ are linear combinations of $\psi'(1/12)$ and $\psi'(5/12)$. The latter numbers reduce further, cf. Section 5. Note that in all cases in which neither $f_1^0(z)$ nor $C_{1,k_2,...,k_m}^{0,l_2...,l_m}(z)$ are present, the z-independent terms in $[(\pm z^k)^N-1]$ can be integrated, since the other cyclotomic letters $f_l^k(z)$ and $C_{k,k_2,...,k_m}^{l,l_2...,l_m}(z)$ for $k>1$ are regular at $z=1$.

In general, linear combinations of the Mellin transforms

$$\mathbf{M}\left[f_c^d(x)\cdot C_{\vec{a}}^{\vec{b}}(x)\right](l\,N) = \int_0^1 dx\ x^{lN}\ (f_c^d(x))^u\cdot C_{\vec{a}}^{\vec{b}}(x), \quad u\in\{0,1\}. \tag{3.26}$$

with l being the least common multiple of $a_1,...,a_k$ allow to represent all cyclotomic harmonic sums. Summarizing, we can write

$$S_{\{a_1,b_1,c_1\},...,\{a_k,b_k,c_k\}}(N) = \sum_{n=1}^{s} e_n \int_0^1 dx\ x^{lN}(f_{\alpha_n}^{\beta_n}(x))^{u_n} C_{\vec{\gamma_n}}^{\vec{\delta_n}}(x)\ , \tag{3.27}$$

with $e_n\in\mathbb{R}$ and $u_n\in\{0,1\}$; here e_n is determined by polynomial expressions in terms of cyclotomic harmonic polylogarithms evaluated at 1 with rational coefficients. Within this transformation and in the following the letter $1/x$ plays a special role concerning the cyclotomic harmonic sums. We can exclude the case that the first letter in (3.26) is $1/x$ since it would just shift the Mellin index by one unit. In case of $\Phi_1(x)$ and related functions the $+$-regularization

$$\mathbf{M}\left[\left(\frac{f(x)}{x-1}\right)_+\right](N) = \int_0^1 dx\ \frac{x^N-1}{x-1}\ f(x) \tag{3.28}$$

is applied to (3.26). Only in case that $f_1(x)=1/(x-1)$ and $C_{\vec{a}}^{\vec{b}}(x)$ do not vanish in the limit $x\to 1$ a $+$-function must occur. Iterations of cyclotomic letters $f_k^l(x),\quad k\geq 1, 0\leq l<k$ for $k\geq 2$ have no singularities in $x\in[0,1]$.

We remark that the sketched transformation from cyclotomic harmonic sums to their Mellin transforms in terms of cyclotomic harmonic polylogarithms can be reversed, i.e., a given expression in terms of Mellin transforms of cyclotomic harmonic polylogarithms can be expressed in terms of cyclotomic harmonic sums. The explicit algorithms have been worked out in details for the alphabet

$$\mathfrak{A}' := \{f_0^0, f_1^0, f_2^0, f_4^0, f_4^1\} = \left\{\frac{1}{x}, \frac{1}{\Phi_1(x)}, \frac{1}{\Phi_2(x)}, \frac{1}{\Phi_4(x)}, \frac{x}{\Phi_4(x)}\right\} \subseteq \mathfrak{A}, \tag{3.29}$$

which allows one to express the cyclotomic harmonic sums $S_{\{a_1,b_1,c_1\},\ldots,\{a_l,b_l,c_l\}}(s_1,\ldots,s_l;N)$ with $a_i \in \{1,2\}$, $b_i \in \{0,1\}$ and $c_i \in \mathbb{N}_+$ in terms of Mellin transforms of cyclotomic harmonic polylogarithms in both directions; the implementation is available within the HarmonicSums package [27]. This transformation will be used, e.g., in Section 4.1.

The Mellin transforms (3.26, 3.28) obey difference equations of order l in N, which can be used to define these functions specifying respective initial values for l moments. In the following we illustrate this for the words $x^l/\Phi_k(x)$. One obtains

$$\sum_{n=0}^{N_k} c_{n,k}\phi_k(N+n-l) = \frac{1}{N+1}~, \tag{3.30}$$

where $\Phi_k(x)$ is given by

$$\Phi_k(x) = \sum_{n=0}^{N_k} c_{n,k}x^n~. \tag{3.31}$$

Here we define

$$\phi_1(0,N) = \int_0^1 dx~\frac{x^N-1}{x-1} \tag{3.32}$$

$$\phi_k(l,N) = \int_0^1 dx~x^N~f_k^l(x), \qquad \phi_k(N) = \phi_k(0,N), \qquad k \geq 2 \tag{3.33}$$

$$\phi_k(l,N)_+ = \int_0^1 dx~[x^N-1]~f_k^l(x), \qquad \phi_k(N)_+ = \phi_k(0,N)_+. \tag{3.34}$$

For $k < 105$ for all coefficients $c_{n,k} \in \{-1,0,1\}$ holds [45]. One derives the following first order difference equations (3.30) for Mellin transforms associated to the lowest order cyclotomic polynomials :

$$\phi_1(l,N+1) - \phi_1(l,N) = \frac{1}{N+l+1} \tag{3.35}$$

$$\phi_2(l,N+1) + \phi_2(l,N) = \frac{1}{N+l+1} \tag{3.36}$$

$$\phi_3(l,N+3) - \phi_3(l,N) = -\frac{1}{(N+l+1)(N+l+2)} \tag{3.37}$$

$$\phi_4(l,N+2) + \phi_4(l,N) = \frac{1}{N+l+1} \tag{3.38}$$

$$\phi_5(l,N+5) - \phi_5(l,N) = -\frac{1}{(N+l+1)(N+l+2)} \tag{3.39}$$

$$\phi_6(l,N+3) + \phi_6(l,N) = \frac{2(N+l)+3}{(N+l+1)(N+l+2)} \tag{3.40}$$

$$\begin{aligned}
\phi_7(l,N+7)-\phi_7(l,N) &= -\frac{1}{(N+l+1)(N+l+2)} && (3.41)\\
\phi_8(l,N+4)-\phi_8(l,N) &= \frac{1}{N+l+1} && (3.42)\\
\phi_9(l,N+9)-\phi_9(l,N) &= -\frac{3}{(N+l+1)(N+l+4)} && (3.43)\\
\phi_{10}(l,N+5)+\phi_{10}(l,N) &= \frac{3(N+l)+2}{(N+l+1)(N+l+2)} && (3.44)\\
\phi_{11}(l,N+11)-\phi_{11}(l,N) &= \frac{1}{(N+l+1)(N+l+2)} && (3.45)\\
\phi_{12}(l,N+6)+\phi_{12}(l,N) &= \frac{3(N+l)+2}{(N+l+1)(N+l+2)}, \quad \text{etc.} && (3.46)
\end{aligned}$$

Together with the corresponding initial values, these recurrence relations enable one to compute efficiently the values for N. In particular, due the special form of the recurrences, we get explicit representations in terms of finite sums. E.g., for $\phi_6(l,N)$, we obtain

$$\phi_6(l,3N)=(-1)^N\left(\sum_{i=1}^{N}\frac{(-1)^i(2(3i+l)-3)}{(3(i+l)-2)(3(i+l)-1)}+\phi_6(l,0)\right), \tag{3.47}$$

and

$$\begin{aligned}
\phi_6(l,3N+1) &= \phi_6(l+1,3N) && (3.48)\\
\phi_6(l,3N+2 &= \phi_6(l+2,3N)\ . && (3.49)
\end{aligned}$$

Looking at

$$\lim_{N\to\infty}\phi_6(l,N)=\lim_{N\to\infty}\int_0^1 x^{N+l}\frac{1}{x^2-x-1}dx=\lim_{N\to\infty}\int_0^1 x^{N+l}(1+x-x^3-x^4+\dots)dx=0$$

shows that

$$\phi_6(l,0)=-\sum_{i=1}^{\infty}\frac{(-1)^i[2(3i+l)-3]}{(3i+l-2)(3i+l-1)}.$$

Completely analogously, all the other functions (3.35–3.46) can be written in such a sum representation where the constants are the infinite versions of it multiplied with a minus sign. Note that these constants can be written as a linear combination of the infinite sums

$$\sigma_{\{a,b,s\}}=\sum_{k=1}^{\infty}\frac{s^k}{a\,k+b}$$

with $s\in\{-1,1\}$ and $a,b\in\mathbb{N}$ with $a\neq 0$. The relevant values for this article will be worked out explicitly in the Section 5.

The $\phi_k(l,N)_+$ functions are preferred if the recurrences (3.35–3.46) form telescoping equations. In this case, in particular if k is odd, the $\phi_k(l,r\,N)$ for properly chosen r can be related to sums without any extra constant, e.g.,

$$\phi_5(l,5N)_+=-\sum_{i=1}^{N}\frac{1}{(5i+l-4)(5i+l-3)}. \tag{3.50}$$

ϕ_1 and ϕ_2 can be related to the single cyclotomic harmonic sums at weight $\mathsf{w} = 1$ as follows

$$\phi_1(l,N)_+ = S_1(N+l), \tag{3.51}$$

$$\phi_2(l,N) = (-1)^N\Big[S_{-1}(N+l) - \ln(2) + S_{-1}(l)\Big]\ . \tag{3.52}$$

For later considerations we use $S_1(N)$ and $S_{-1}(N)$ instead of $\phi_1(N)$ and $\phi_2(N)$. In the following Section the single cyclotomic harmonic sums of weight $\mathsf{w} = 1$ are expressed in terms of the Mellin transforms $\phi_k(l, r\,N)$ for properly chosen r. In addition, the cyclotomic harmonic sums of higher weight and depth will be discussed.

4 Cyclotomic Harmonic Sums

We consider the extension of the finite nested harmonic sums [7, 8] to those generated by the cyclotomic harmonic polylogarithms discussed in Section 3. First the single cyclotomic harmonic sums are considered and explicit representations are given. We derive their analytic continuation to complex values of N. Next the algebraic, differential and three multiple argument relations of the cyclotomic harmonic sums are discussed. These relations are used to represent these sums over suitable bases. Finally we consider the nested sums over the alphabet $\{(\pm1)^k/k, (\pm1)^k/(2k+1)\}$ to higher weight deriving explicit relations up to $\mathsf{w} = 5$.

4.1 The Single Sums

The single cyclotomic harmonic sums are given by

$$\sum_{k=0}^{N} \frac{(\pm1)^k}{(l\cdot k+m)^n}\ . \tag{4.1}$$

Here N is either an even or an odd integer. In case one needs representations only for $N \in \mathbb{N}$ the following representations hold :

$$\sum_{k=0}^{\overline{N}} \frac{(-1)^k}{l\cdot k+m} = \left[\sum_{k=0}^{N} \frac{1}{(2l)\cdot k+m} - \frac{1}{(2l)\cdot k+m+l}\right], \tag{4.2}$$

$$\sum_{k=0}^{\overline{N}+1} \frac{(-1)^k}{l\cdot k+m} = \left[\sum_{k=0}^{N} \frac{1}{(2l)\cdot k+m} - \frac{1}{(2l)\cdot k+m+l}\right] - \frac{1}{(2l)N+l+m}\ , \tag{4.3}$$

with $\overline{N} = 2N$. However, one is interested in relations for general values of N, since for nested sums more and more cases have to be distinguished. The single sums can be expressed in terms of the Mellin transforms $\phi_k(l, kN)$, (3.32–3.33). Up to $\mathsf{w} = 6$ one obtains :

$$\sum_{k=1}^{N} \frac{1}{1+2k} = -\frac{2N}{2N+1} - \frac{S_1(N)}{2} + S_1(2N), \tag{4.4}$$

$$\sum_{k=1}^{N} \frac{(-1)^k}{1+2k} = (-1)^N\left[\frac{1}{2N+1} - \phi_4(2N)\right] + \sigma_{\{2,1,-1\}}, \tag{4.5}$$

$$\sum_{k=1}^{N} \frac{1}{1+3k} = -\frac{3N}{3N+1} - \frac{S_1(N)}{6} + \frac{1}{2}S_1(3N) - \frac{1}{2}\phi_3(3N)_+, \tag{4.6}$$

$$\sum_{k=1}^{N}\frac{(-1)^k}{1+3k}=\frac{1}{6}S_{-1}(N)-\frac{1}{2}S_{-1}(3N)+(-1)^N\left[\frac{1}{3N+1}-\frac{1}{2}\phi_6(3N)\right]+\frac{1}{3}\sigma_{\{1,0,-1\}}+\sigma_{\{3,1,-1\}}, \tag{4.7}$$

$$\sum_{k=1}^{N}\frac{1}{2+3k}=-\frac{3N}{2(3N+2)}-\frac{S_1(N)}{6}+\frac{1}{2}S_1(3N)+\frac{1}{2}\phi_3(3N)_+, \tag{4.8}$$

$$\sum_{k=1}^{N}\frac{(-1)^k}{2+3k}=-\frac{1}{6}S_{-1}(N)+\frac{1}{2}S_{-1}(3N)+(-1)^N\left[\frac{1}{3N+2}-\frac{1}{2}\phi_6(3N)\right]$$
$$+\frac{1}{3}\sigma_{\{1,0,-1\}}+\sigma_{\{3,1,-1\}}+\frac{1}{2}, \tag{4.9}$$

$$\sum_{k=1}^{N}\frac{1}{1+4k}=-\frac{2N}{4N+1}-\frac{1}{4}S_1(2N)+\frac{1}{2}S_1(4N)-\frac{1}{2}\phi_4(4N)+\frac{1}{2}\sigma_{\{2,1,-1\}}+\frac{1}{2(4N+1)}, \tag{4.10}$$

$$\sum_{k=1}^{N}\frac{(-1)^k}{1+4k}=(-1)^N\left[\frac{1}{4N+1}-\phi_8(4N)\right]+\sigma_{\{4,1,-1\}}, \tag{4.11}$$

$$\sum_{k=1}^{N}\frac{1}{3+4k}=-\frac{10N}{3(4N+3)}-\frac{1}{4}S_1(2N)+\frac{1}{2}S_1(4N)+\frac{1}{2}\phi_4(4N)-\frac{1}{2}\sigma_{\{2,1,-1\}}-\frac{3}{2(4N+3)}, \tag{4.12}$$

$$\sum_{k=1}^{N}\frac{(-1)^k}{3+4k}=(-1)^N\left[\frac{1}{4N+3}-\phi_8(2,4N)\right]+\sigma_{\{4,3,-1\}}, \tag{4.13}$$

$$\sum_{k=1}^{N}\frac{1}{1+5k}=-\frac{5N}{5N+1}-\frac{S_1(N)}{20}+\frac{1}{4}S_1(5N)-\frac{3}{4}\phi_5(5N)_+-\frac{1}{2}\phi_5(1,5N)_+-\frac{1}{4}\phi_5(2,5N)_+, \tag{4.14}$$

$$\sum_{k=1}^{N}\frac{(-1)^k}{1+5k}=(-1)^N\left[-\frac{3}{4}\phi_{10}(5N)+\frac{1}{2}\phi_{10}(1,5N)-\frac{1}{4}\phi_{10}(2,5N)+\frac{1}{5N+1}\right]$$
$$+\frac{1}{20}S_{-1}(N)-\frac{1}{4}S_{-1}(5N)+\frac{1}{5}\sigma_{\{1,0,-1\}}+\sigma_{\{5,1,-1\}}, \tag{4.15}$$

$$\sum_{k=1}^{N}\frac{1}{2+5k}=-\frac{5N}{2(5N+2)}-\frac{S_1(N)}{20}+\frac{1}{4}S_1(5N)+\frac{1}{4}\phi_5(5N)_+-\frac{1}{2}\phi_5(1,5N)_+-\frac{1}{4}\phi_5(2,5N)_+, \tag{4.16}$$

$$\sum_{k=1}^{N}\frac{(-1)^k}{2+5k}=(-1)^N\left[-\frac{1}{4}\phi_{10}(5N)-\frac{1}{2}\phi_{10}(1,5N)+\frac{1}{4}\phi_{10}(2,5N)+\frac{1}{5N+2}\right] \tag{4.17}$$
$$-\frac{1}{20}S_{-1}(N)+\frac{1}{4}S_{-1}(5N)-\frac{1}{5}\sigma_{\{1,0,-1\}}+\sigma_{\{5,2,-1\}},$$

$$\sum_{k=1}^{N}\frac{1}{3+5k}=-\frac{5N}{3(5N+3)}-\frac{S_1(N)}{20}+\frac{1}{4}S_1(5N)+\frac{1}{4}\phi_5(5N)_++\frac{1}{2}\phi_5(1,5N)_+-\frac{1}{4}\phi_5(2,5N)_+, \tag{4.18}$$

$$\sum_{k=1}^{N}\frac{(-1)^k}{3+5k}=(-1)^N\left[\frac{1}{4}\phi_{10}(5N)-\frac{1}{2}\phi_{10}(1,5N)-\frac{1}{4}\phi_{10}(2,5N)+\frac{1}{5N+3}\right]$$
$$+\frac{1}{20}S_{-1}(N)-\frac{1}{4}S_{-1}(5N)+\frac{1}{5}\sigma_{\{1,0,-1\}}+\sigma_{\{5,3,-1\}}, \tag{4.19}$$

$$\sum_{k=1}^{N}\frac{1}{4+5k} =-\frac{5N}{4(5N+4)}-\frac{S_1(N)}{20}+\frac{1}{4}S_1(5N)+\frac{1}{4}\phi_5(5N)_+ + \frac{1}{2}\phi_5(1,5N)_+ + \frac{3}{4}\phi_5(2,5N)_+, \tag{4.20}$$

$$\sum_{k=1}^{N}\frac{(-1)^k}{4+5k} =(-1)^N\left[-\frac{1}{4}\phi_{10}(5N)+\frac{1}{2}\phi_{10}(1,5N)-\frac{3}{4}\phi_{10}(2,5N)+\frac{1}{5N+4}\right] - \frac{1}{20}S_{-1}(N)+\frac{1}{4}S_{-1}(5N)+\frac{3}{5}\sigma_{\{1,0,-1\}}+\sigma_{\{5,1,-1\}}-\sigma_{\{5,2,-1\}}+\sigma_{\{5,3,-1\}}+\frac{7}{12}, \tag{4.21}$$

$$\sum_{k=1}^{N}\frac{1}{1+6k} =-\frac{6N}{6N+1}+\frac{S_1(N)}{12}-\frac{1}{6}S_1(2N)-\frac{1}{4}S_1(3N)+\frac{1}{2}S_1(6N)-\frac{1}{4}\phi_3(3N)_+-\frac{1}{2}\phi_3(6N)_+, \tag{4.22}$$

$$\sum_{k=1}^{N}\frac{(-1)^k}{1+6k} =(-1)^N\left[-\phi_{12}(6N)+\frac{1}{3}\phi_4(2N)+\frac{1}{6N+1}\right]+\sigma_{\{6,1,-1\}}, \tag{4.23}$$

$$\sum_{k=1}^{N}\frac{1}{5+6k} =-\frac{6N}{5(6N+5)}+\frac{S_1(N)}{12}-\frac{1}{6}S_1(2N)-\frac{1}{4}S_1(3N)+\frac{1}{2}S_1(6N) + \frac{1}{4}\phi_3(3N)_+ + \frac{1}{2}\phi_3(6N)_+, \tag{4.24}$$

$$\sum_{k=1}^{N}\frac{(-1)^k}{5+6k} =(-1)^N\left[\phi_{12}(6N)-\frac{2}{3}\phi_4(2N)-\phi_4(6N)+\frac{1}{6N+5}\right]+\frac{4}{3}\sigma_{\{2,1,-1\}}-\sigma_{\{6,1,-1\}}+\frac{2}{15}. \tag{4.25}$$

Taking the sum representations such as (3.50) and (3.47) given by the recurrence relations (3.35–3.46) and the corresponding initial values $\phi_k(l,0)$ (which are expressible by the infinite sums $\sigma_{\{a,b,\pm 1\}}$, cf. Section 5), we used the summation package Sigma [17] to perform this transformation. As a consequence, the single cyclotomic harmonic sums can be expressed in terms of the sums

$$S_{-1}(N), S_{-1}(3N), S_{-1}(5N), S_1(N), S_1(2N), S_1(3N), S_1(4N), S_1(5N), S_1(6N),$$
$$\phi_3(3N)_+, \phi_3(6N)_+, \phi_4(2N), \phi_4(4N), \phi_4(6N), \phi_5(5N)_+, \phi_5(1,5N)_+,$$
$$\phi_5(2,5N)_+, \phi_6(3N), \phi_8(4N), \phi_8(2,4N), \phi_{10}(5N), \phi_{10}(1,5N), \phi_{10}(2,5N), \phi_{12}(6N).$$

In particular, in the way how this construction is carried out it follows by the summation theory of [17] that the sequences produced by these sums form an algebraic independent basis over the ring of sequences generated by the elements from $\mathbb{R}(N)[(-1)^N]$. Note that we exploited the algebraic relations (5.118–5.129) which implies that only the following constants

$$\sigma_{\{1,0,-1\}}, \sigma_{\{2,1,-1\}}, \sigma_{\{3,1,-1\}}, \sigma_{\{4,1,-1\}}, \sigma_{\{4,3,-1\}}, \sigma_{\{5,1,-1\}}, \sigma_{\{5,2,-1\}}, \sigma_{\{5,3,-1\}}\sigma_{\{6,1,-1\}}$$

from $\mathbb{R}$ appear in the representation found for the single cyclotomic harmonic sums with weight $\mathsf{w}=1$.

Due to (3.33, 3.34) the functions $\Phi_k(l,N), k \geq 1$ can be represented as factorial series [36,46]. Therefore they are meromorphic functions in N with poles at $-n$, $n \in \mathbb{N}$. This also applies to $\phi_k(0,N)$, (3.32). The latter function grows $\propto \ln(N)$ for $N \to \infty$, $|\arg(N)| < \pi$. The recursion relations (3.30) allow one to shift $\phi_k(l,N)$ in $N \to N+1$. To represent a function $\phi_k(l,N)$ for $N \in \mathbb{C}$ one needs to know its asymptotic representation in addition.

It is given in analytic form in terms of series involving the Stirling numbers of the 2nd kind [36, 37]. The corresponding representations read :

$$\begin{aligned}
\phi_1(0,N) &\sim \gamma + \ln(N) + \frac{1}{2N} - \frac{1}{12N^2} + \frac{1}{120N^4} - \frac{1}{252N^6} + \frac{1}{240N^8} \\
&\quad - \frac{1}{132N^{10}} + \frac{691}{32760N^{12}} + O\left(\frac{1}{N^{13}}\right)
\end{aligned} \tag{4.26}$$

$$\phi_2(0,N) \sim \frac{1}{2N} - \frac{1}{4N^2} + \frac{1}{8N^4} - \frac{1}{4N^6} + \frac{17}{16N^8} - \frac{31}{4N^{10}} + \frac{691}{8N^{12}} + O\left(\frac{1}{N^{13}}\right) \tag{4.27}$$

$$\phi_3(0,N) \sim \frac{1}{3N} - \frac{2}{9N^3} + \frac{2}{3N^5} - \frac{14}{3N^7} + \frac{1618}{27N^9} - \frac{3694}{3N^{11}} + O\left(\frac{1}{N^{13}}\right) \tag{4.28}$$

$$\phi_4(0,N) \sim \frac{1}{2N} - \frac{1}{2N^3} + \frac{5}{2N^5} - \frac{61}{2N^7} + \frac{1385}{2N^9} - \frac{50521}{2N^{11}} + O\left(\frac{1}{N^{13}}\right) \tag{4.29}$$

$$\begin{aligned}
\phi_5(0,N) &\sim \frac{1}{5N} + \frac{1}{5N^2} - \frac{1}{5N^3} - \frac{1}{N^4} + \frac{31}{25N^5} + \frac{67}{5N^6} - \frac{109}{5N^7} - \frac{361}{N^8} + \frac{3779}{5N^9} \\
&\quad + \frac{412751}{25N^{10}} - \frac{214093}{5N^{11}} - \frac{1150921}{N^{12}} + O\left(\frac{1}{N^{13}}\right)
\end{aligned} \tag{4.30}$$

$$\phi_6(0,N) \sim \frac{1}{N} - \frac{2}{N^3} + \frac{22}{N^5} - \frac{602}{N^7} + \frac{30742}{N^9} - \frac{2523002}{N^{11}} + O\left(\frac{1}{N^{13}}\right) \tag{4.31}$$

$$\begin{aligned}
\phi_7(0,N) &\sim \frac{1}{7N} + \frac{2}{7N^2} - \frac{16}{7N^4} - \frac{12}{7N^5} + \frac{56}{N^6} + \frac{3900}{49N^7} - \frac{20296}{7N^8} - \frac{5796}{N^9} \\
&\quad + \frac{1809992}{7N^{10}} + \frac{4582500}{7N^{11}} - \frac{35282968}{N^{12}} + O\left(\frac{1}{N^{13}}\right)
\end{aligned} \tag{4.32}$$

$$\begin{aligned}
\phi_8(0,N) &\sim \frac{1}{2N} + \frac{1}{2N^2} - \frac{3}{2N^3} - \frac{11}{2N^4} + \frac{57}{2N^5} + \frac{361}{2N^6} - \frac{2763}{2N^7} - \frac{24611}{2N^8} + \frac{250737}{2N^9} \\
&\quad + \frac{2873041}{2N^{10}} - \frac{36581523}{2N^{11}} - \frac{512343611}{2N^{12}} + O\left(\frac{1}{N^{13}}\right)
\end{aligned} \tag{4.33}$$

$$\begin{aligned}
\phi_9(0,N) &\sim \frac{1}{3N} + \frac{2}{3N^2} - \frac{2}{3N^3} - \frac{28}{3N^4} + \frac{34}{3N^5} + \frac{1172}{3N^6} - \frac{1862}{3N^7} - \frac{101428}{3N^8} + \frac{207394}{3N^9} \\
&\quad + \frac{14999012}{3N^{10}} - \frac{37996022}{3N^{11}} - \frac{3386034628}{3N^{12}} + O\left(\frac{1}{N^{13}}\right)
\end{aligned} \tag{4.34}$$

$$\begin{aligned}
\phi_{10}(0,N) &\sim \frac{1}{N} + \frac{1}{N^2} - \frac{5}{N^3} - \frac{17}{N^4} + \frac{151}{N^5} + \frac{871}{N^6} - \frac{11465}{N^7} - \frac{92777}{N^8} + \frac{1626151}{N^9} \\
&\quad + \frac{16922791}{N^{10}} - \frac{370714025}{N^{11}} - \frac{4715323337}{N^{12}} + O\left(\frac{1}{N^{13}}\right)
\end{aligned} \tag{4.35}$$

$$\begin{aligned}
\phi_{11}(0,N) &\sim \frac{1}{11N} + \frac{4}{11N^2} + \frac{6}{11N^3} - \frac{56}{11N^4} - \frac{282}{11N^5} + \frac{3064}{11N^6} + \frac{26646}{11N^7} - \frac{382616}{11N^8} \\
&\quad - \frac{4592442}{11N^9} + \frac{7618184}{N^{10}} + \frac{13945859346}{121N^{11}} - \frac{28200213176}{11N^{12}} + O\left(\frac{1}{N^{13}}\right)
\end{aligned} \tag{4.36}$$

$$\begin{aligned}
\phi_{12}(0,N) &\sim \frac{1}{N} + \frac{1}{N^2} - \frac{7}{N^3} - \frac{23}{N^4} + \frac{305}{N^5} + \frac{1681}{N^6} - \frac{33367}{N^7} - \frac{257543}{N^8} + \frac{6815585}{N^9} \\
&\quad + \frac{67637281}{N^{10}} - \frac{2237423527}{N^{11}} - \frac{27138236663}{N^{12}} + O\left(\frac{1}{N^{13}}\right)
\end{aligned} \tag{4.37}$$

$$\phi_5(2,N) \sim \frac{1}{5N} - \frac{2}{5N^3} + \frac{86}{25N^5} - \frac{338}{5N^7} + \frac{12094}{5N^9} - \frac{690866}{5N^{11}} + O\left(\frac{1}{N^{13}}\right) \quad (4.38)$$

$$\phi_8(2,N) \sim \frac{1}{2N} - \frac{2}{N^3} + \frac{40}{N^5} - \frac{1952}{N^7} + \frac{177280}{N^9} - \frac{25866752}{N^{11}} + O\left(\frac{1}{N^{13}}\right) \quad (4.39)$$

$$\phi_{10}(1,N) \sim \frac{1}{N} + \frac{1}{N^2} - \frac{5}{N^3} - \frac{17}{N^4} + \frac{151}{N^5} + \frac{871}{N^6} - \frac{11465}{N^7} - \frac{92777}{N^8} + \frac{1626151}{N^9} + \frac{16922791}{N^{10}} - \frac{370714025}{N^{11}} - \frac{4715323337}{N^{12}} + O\left(\frac{1}{N^{13}}\right) \quad (4.40)$$

$$\phi_{10}(2,N) \sim \frac{1}{N} - \frac{6}{N^3} + \frac{186}{N^5} - \frac{14166}{N^7} + \frac{2009946}{N^9} - \frac{458225526}{N^{11}} + O\left(\frac{1}{N^{13}}\right) . \quad (4.41)$$

The numerical accuracy of the asymptotic representations at a given suitably large value of N lowers with growing k, i.e., one has to choose larger values of N correspondingly to apply the asymptotic formulae. The above expansions can easily be extended to higher inverse powers of N. The recursion for $\phi_k(l, kN)$ is given in (3.30), resp. (3.35–3.46) in a more compact form. Due to these for any $N \in \mathbb{C}$ at which $\phi_k(l,N)$ is analytic one may map $\phi_k(l,N)$ to values $|N| \gg 1, \arg(N) < \pi$ and use the asymptotic representations.

The single cyclotomic harmonic sums of higher weight obey the representations

$$\sum_{k=0}^{N-1} \frac{1}{(lk+m)^n} = \frac{(-1)^{n-1}}{(n-1)!} \int_0^1 dx \ \ln^{n-1}(x) \ x^{m-1} \frac{x^{lN}-1}{x^l-1}, \quad m < l \quad (4.42)$$

$$\sum_{k=0}^{N-1} \frac{(-1)^k}{(lk+m)^n} = \frac{(-1)^n}{(n-1)!} \int_0^1 dx \ \ln^{n-1}(x) \ x^{m-1} \frac{(-x^l)^N-1}{x^l+1}, \quad m < l, \quad (4.43)$$

for $l, m, n \in \mathbb{N}_+$, which may be expressed in terms of cyclotomic letters again. We note that

$$\frac{\partial^n}{\partial m^n} \int_0^1 dx \ x^{m-1} \ f(x) = \int_0^1 dx \ x^{m-1} \ \ln^n(x) \ f(x) . \quad (4.44)$$

Therefore, (4.42, 4.43) can be expressed by the corresponding derivatives of $\phi_k(n, lN)$ for N and corresponding constants.

4.2 Cyclotomic harmonic polylogarithms at $x = 1$

We consider the cyclotomic harmonic polylogarithm $C_{a,\vec{b}}^{c,\vec{d}}(x)$. Its value at $x = 1$ is given by

$$C_{a,\vec{b}}^{c,\vec{d}}(1) = \int_0^1 dx f_a^c(x) C_{\vec{b}}^{\vec{d}}(x) . \quad (4.45)$$

Let $f_a^c(x) = x^l/\Phi_k(x)$, with $l < \deg(\Phi_k(x))$ and n be the smallest integer such that $\Phi_k(x)|(x^n-1)$, with

$$\frac{1}{\Phi_k(x)} = \frac{\prod_j \Phi_j(x)}{x^n-1} . \quad (4.46)$$

Since $x \in [0,1]$, the representation

$$\frac{x^l}{\Phi_k(x)} = -x^l \prod_k \Phi_k(x) \sum_{j=0}^{\infty} x^{jn} = \sum_{m=0}^{w} a_m(l) x^m \sum_{j=0}^{\infty} x^{jn}, \quad a_m(l) \in \mathbb{Z} \quad (4.47)$$

holds. Thus we get

$$\int_0^1 dx \frac{x^l}{\Phi_k(x)} = \sum_{m=0}^{w} a_m(l) \sum_{j=0}^{\infty} \frac{1}{jn+m+1} , \tag{4.48}$$

representing the value of the depth d=1 cyclotomic harmonic polylogarithms at $x = 1$ as a linear combination of the corresponding infinite cyclotomic sums, see Section 5.

We return now to Eq. (4.45). Integration by parts yields

$$C_{a,\vec{b}}^{c,\vec{d}}(1) = C_a^c(1) C_{\vec{b}}^{\vec{d}}(1) - \sum_{j=0}^{\infty} \sum_{m=0}^{w} a_m \int_0^1 dx \frac{x^{jn+m+1}}{jn+m+1} C_{\vec{b}}^{\vec{d}}(x) . \tag{4.49}$$

Since the Mellin transform of a cyclotomic harmonic polylogarithm can be represented by a linear combination of finite cyclotomic harmonic sums, the weighted infinite sum of the latter ones in (4.49) is thus given as a polynomial of infinite cyclotomic harmonic sums, Section 5.

4.3 Relations of Cyclotomic Harmonic Sums

The cyclotomic harmonic sums obey differentiation relations, cf. [7, 28], stuffle relations due to their quasi–shuffle algebra, cf. [9], and multiple argument relations. [3]

4.3.1 Differentiation

As has been illustrated in Section 3, cyclotomic harmonic sums can be represented as linear combinations of Mellin transforms (3.27) of cyclotomic harmonic polylogarithms. Based on this representation, the differentiation of these sums is defined by

$$\frac{\partial^m}{\partial N^m} S_{\{a_1,b_1,c_1\},\dots,\{a_k,b_k,c_k\}}(N) = \sum_{n=1}^{s} e_n \int_0^1 \; dx \; x^{lN} \; l^m \; \ln^m(x) (f_{\alpha_n}^{\beta_n}(x))^{u_n} C_{\vec{\gamma_n}}^{\vec{\delta_n}}(x). \tag{4.50}$$

The product $\ln^m(x) C_{\vec{\alpha_n}}^{\vec{\beta_n}}(x) = m! C_{0,\dots,0}^{0,\dots,0} C_{\vec{\alpha_n}}^{\vec{\beta_n}}(x)$ with m consecutive zeros may be transformed into a linear combination of cyclotomic harmonic polylogarithms using the shuffle relation (3.21). Finally, using the inverse Mellin transform, the derivative (4.50) of a cyclotomic harmonic sum w.r.t. N is given as a polynomial expression in terms of cyclotomic harmonic sums and cyclotomic harmonic polylogarithms at $x = 1$. Together with the previous section, the derivative (4.50) can be expressed as a polynomial expression with rational coefficients in terms of cyclotomic harmonic sums and their values at $N \to \infty$. The corresponding relations are denoted by (D). Further details on cyclotomic harmonic sums at infinity are given in Section 5.

A given finite cyclotomic harmonic sum is determined for $N \in \mathbb{C}$ by its asymptotic representation and the corresponding recursion from $N \to (N-1)$. Both the asymptotic representation and the recursion can be easily differentiated analytically. Therfore any differentiation of a cyclotomic harmonic sum w.r.t. N is closely related to the original sum. For this reason one may collect these derivatives in classes

$$S_{\{a_1,b_1,c_1\},\dots,\{a_k,b_k,c_k\}}^{(D)}(N) = \left\{ \frac{\partial^n}{\partial N^n} S_{\{a_1,b_1,c_1\},\dots,\{a_k,b_k,c_k\}}(N); n \in \mathbb{N} \right\} . \tag{4.51}$$

[3]For the relations given in this Section we mostly present the results, giving for a few cases the proofs in Appendix B. The other proofs proceed in a similar manner.

4.3.2 Stuffle Algebra

To derive the stuffle relations, cf. [11, 28], we consider the product of two denominator terms. For $a_1, a_2, b_1, b_2, c_1, c_2, i \in \mathbb{N}$ they are given by

$$\begin{aligned}\frac{1}{(a_1 i+b_1)^{c_1}(a_2 i+b_2)^{c_2}} &= (-1)^{c_1}\sum_{j=1}^{c_1}(-1)^j\binom{c_1+c_2-j-1}{c_2-1}\frac{a_1^{c_2}a_2^{c_1-j}}{(a_1b_2-a_2b_1)^{c_1+c_2-j}}\frac{1}{(a_1 i+b_1)^j}\\ &+ (-1)^{c_2}\sum_{j=1}^{c_2}(-1)^j\binom{c_1+c_2-j-1}{c_1-1}\frac{a_1^{c_2-j}a_2^{c_1}}{(a_2b_1-a_1b_2)^{c_1+c_2-j}}\frac{1}{(a_2 i+b_2)^j},\end{aligned} \tag{4.52}$$

and

$$\frac{1}{(a_1 i+b_1)^{c_1}(a_2 i+b_2)^{c_2}} = \left(\frac{a_1}{a_2}\right)^{c_2}\frac{1}{(a_1 i+b_1)^{c_1+c_2}}, \tag{4.53}$$

if $a_1b_2 = a_2b_1$.

The product of two cyclotomic harmonic sums has the following representation. Let $a_i, b_i, d_i, e_i, k, l, n \in \mathbb{N}_+$ and $c_i, f_i \in \mathbb{Z}\setminus\{0\}$. If $a_1e_1 \neq d_1b_1$ one has

$$\begin{aligned}S_{\{a_1,b_1,c_1\},\dots,\{a_k,b_k,c_k\}}(n)S_{\{d_1,e_1,f_1\},\dots,\{d_l,e_l,f_l\}}(n) &=\\ &\sum_{i=1}^{n}\frac{\operatorname{sign}(c_1)^i}{(a_1 i+b_1)^{|c_1|}}S_{\{a_2,b_2,c_2\},\dots,\{a_k,b_k,c_k\}}(i)S_{\{d_1,e_1,f_1\},\dots,\{d_l,e_l,f_l\}}(i)\\ &+\sum_{i=1}^{n}\frac{\operatorname{sign}(f_1)^i}{(d_1 i+e_1)^{|f_1|}}S_{\{a_1,b_1,c_1\},\dots,\{a_k,b_k,c_k\}}(i)S_{\{d_2,e_2,f_2\},\dots,\{d_l,e_l,f_l\}}(i)\\ &-\sum_{i=1}^{n}\Biggl((-1)^{|c_1|}\sum_{j_1=1}^{|c_1|}(-1)^j\binom{|c_1|+|f_1|-j-1}{|f_1|-1}\frac{a_1^{|f_1|}d_1^{|c_1|-j}}{a_1e_1-d_1b_1}\frac{1}{(a_1 i+b_1)^j}\\ &+(-1)^{|f_1|}\sum_{j=1}^{|f_1|}(-1)^j\binom{|c_1|+|f_1|-j-1}{|f_1|-1}\frac{a_1^{|f_1|-j}d_1^{|c_1|}}{d_1b_1-a_1e_1}\frac{1}{(d_1 i+e_1)^j}\Biggr)\\ &\times S_{\{a_2,b_2,c_2\},\dots,\{a_k,b_k,c_k\}}(i)S_{\{d_2,e_2,f_2\},\dots,\{d_l,e_l,f_l\}}(i),\end{aligned} \tag{4.54}$$

resp. for $a_1e_1 = d_1b_1$ one has

$$\begin{aligned}S_{\{a_1,b_1,c_1\},\dots,\{a_k,b_k,c_k\}}(n)S_{\{d_1,e_1,f_1\},\dots,\{d_l,e_l,f_l\}}(n) &=\\ &\sum_{i=1}^{n}\frac{\operatorname{sign}(c_1)^i}{(a_1 i+b_1)^{|c_1|}}S_{\{a_2,b_2,c_2\},\dots,\{a_k,b_k,c_k\}}(i)S_{\{d_1,e_1,f_1\},\dots,\{d_l,e_l,f_l\}}(i)\\ &+\sum_{i=1}^{n}\frac{\operatorname{sign}(f_1)^i}{(d_1 i+e_1)^{|f_1|}}S_{\{a_1,b_1,c_1\},\dots,\{a_k,b_k,c_k\}}(i)S_{\{d_2,e_2,f_2\},\dots,\{d_l,e_l,f_l\}}(i)\\ &-\left(\frac{a_1}{d_1}\right)^{|f_1|}\sum_{i=1}^{n}\frac{(\operatorname{sign}(c_1)\operatorname{sign}(f_1))^i}{(a_1 i+b_1)^{|c_1|+|f_1|}}\\ &\times S_{\{a_2,b_2,c_2\},\dots,\{a_k,b_k,c_k\}}(i)S_{\{d_2,e_2,f_2\},\dots,\{d_l,e_l,f_l\}}(i)\ .\end{aligned} \tag{4.55}$$

Subsequently, the relations given by (4.54, 4.55) are denoted by (A).

4.3.3 Synchronization

A first multiple argument relation is implied as follows. Let $a, b, k \in \mathbb{N}$, $c \in \mathbb{Z} \setminus \{0\}$, $k \geq 2$. Then

$$S_{\{a,b,c\}}(k \cdot N) = \sum_{i=0}^{k-1} \text{sign}(c)^i \, S_{\{k \cdot a, b - a \cdot i, \text{sign}(c)^k |c|\}}(N) \ . \tag{4.56}$$

For $a_i, b_i, m, k \in \mathbb{N}$, $c_i \in \mathbb{Z} \setminus \{0\}$, $k \geq 2$, the general cyclotomic harmonic sums obey

$$S_{\{a_m,b_m,c_m\},\{a_{m-1},b_{m-1},c_{m-1}\},\ldots,\{a_1,b_1,c_1\}}(k \cdot N) = \sum_{i=0}^{m-1} \sum_{j=1}^{N} \frac{S_{\{a_{m-1},b_{m-1},c_{m-1}\},\ldots,\{a_1,b_1,c_1\}}(k \cdot j - i) \, \text{sign}(c_m)^{k \cdot j - i}}{(a_m(k \cdot j - i) + b_1)^{|c_m|}} \ . \tag{4.57}$$

Repeated application of (4.56, 4.57) allows to represent cyclotomic harmonic sums of argument kn by those of argument n. Subsequently, the resulting relations are denoted by (M).

4.3.4 Duplication Relations

The usual duplication relation [11] holds also for cyclotomic harmonic sums :

$$\sum_{\pm} S_{\{a_m,b_m,\pm c_m\},\{a_{m-1},b_{m-1},\pm c_{m-1}\},\ldots,\{a_1,b_1,\pm c_1\}}(2N) = 2^m S_{\{2a_m,b_m,c_m\},\ldots,\{2a_1,b_1,c_1\}}(N) \ . \tag{4.58}$$

The proof of (4.58) is given in Appendix B.

Similar to (4.58) one obtains a second duplication relation :

$$\sum_{d_i \in \{-1,1\}} d_m d_{m-1} \cdots d_1 S_{\{a_m,b_m,d_m c_m\},\{a_{m-1},b_{m-1},d_{m-1} c_{m-1}\},\ldots,\{a_1,b_1,d_1 c_1\}}(2N) = 2^m S_{\{2a_m,b_m - a_m,c_m\},\ldots,\{2a_1,b_1 - a_1,c_1\}}(N) ; \tag{4.59}$$

its proof is given in Appendix B. The resulting algebraic relations of (4.58) and (4.59) are denoted by (H_1) and (H_2), respectively. We remark that more general relations arise for generalized cyclotomic harmonic sums presented in Section 6, (6.12, 6.13).

4.4 Sums of Higher Depth and Weight

As an example we consider the class of cyclotomic sums, which occur in the physical application mentioned in Section 1. They are given by iteration of the summands

$$\frac{1}{k^{l_1}}, \qquad \frac{(-1)^k}{k^{l_2}}, \qquad \frac{1}{(2k+1)^{l_3}}, \qquad \frac{(-1)^k}{(2k+1)^{l_4}} \tag{4.60}$$

with $N \geq k \geq 1$, $l_i, k \in \mathbb{N}_+$. As pointed out earlier, this class of cyclotomic harmonic sums can be expressed by Mellin transforms in terms of cyclotomic harmonic polylogarithms generated by the alphabet (3.29).

We apply the relations in Section 4.2 to derive the corresponding bases for given weight w. The cyclotomic harmonic sums can be represented over the corresponding bases. Applying the relations using computer algebra we find the pattern for the number of basis elements up to w = 5 given in Table 2.

w	N_S	H_1	H_1,H_2	H_1,M	H_1,H_2,M	D	H_1,H_2,M,D	A	H_1,H_2,M,A	A,D	all
1	4	3	3	2	2	4	2	4	2	4	2
2	20	18	17	16	15	16	13	10	8	6	6
3	100	96	93	92	89	80	74	40	35	30	27
4	500	492	485	484	477	400	388	150	142	110	107
5	2500	2484	2469	2468	2453	2000	1976	624	607	474	465

Table 2: Reduction of the number of cyclotomic harmonic sums N_S over the elements (4.60) at given weight w by applying the three multiple argument relations (H_1, H_2, M), differentiation w.r.t. to the external sum index N, (D), and the algebraic relations (A). A sequence of symbols corresponds to the combination of these relations.

Due to the arguments given in Section 4.3.1 above, for any occurrence of a differential operator $(\partial^m/\partial N^m)S_{a_1,b_1,c_1},...,a_k,b_k,c_k(N)$ only one representative is counted.

The total number of sums of weight w for the alphabet (4.60) is

$$N_S(w) = 4\cdot 5^{w-1} . \tag{4.61}$$

We mention that the single application of any of the multiple argument relations (M, H_1, H_2) leads to the same number of basis sums

$$N_{H_1} = N_{H_2} = N_M . \tag{4.62}$$

This also applies to the combinations H_1M and H_2M

$$N_{H_1,M} = N_{H_2,M} . \tag{4.63}$$

Explicit counting relations for the number of basis elements given Table 2 can be derived :

$$N_A(w) = \frac{1}{w}\sum_{d|w}\mu\left(\frac{w}{d}\right)5^d \tag{4.64}$$

$$N_D(w) = N_S(w) - N_S(w-1) = 16\cdot 5^{w-2} \tag{4.65}$$

$$N_{H_1}(w) = N_S(w) - 2^{w-1} = 4\cdot 5^{w-1} - 2^{w-1} \tag{4.66}$$

$$N_{H_1H_2}(w) = N_S(w) - (2\cdot 2^{w-1} - 1) = 4\cdot 5^{w-1} - (2\cdot 2^{w-1} - 1) \tag{4.67}$$

$$N_{H_1M}(w) = N_S(w) - 2\cdot 2^{w-1} = 4\cdot 5^{w-1} - 2\cdot 2^{w-1} \tag{4.68}$$

$$N_{H_1H_2M}(w) = N_S(w) - (3\cdot 2^{w-1} - 1) = 4\cdot 5^{w-1} - (3\cdot 2^{w-1} - 1) \tag{4.69}$$

$$\begin{aligned} N_{AD}(w) &= N_A(w) - N_A(w-1) \\ &= \frac{1}{w}\sum_{d|w}\mu\left(\frac{w}{d}\right)5^d - \frac{1}{w-1}\sum_{d|w-1}\mu\left(\frac{w-1}{d}\right)5^d \end{aligned} \tag{4.70}$$

$$N_{AH_1H_2M}(w) = \frac{1}{w}\sum_{d|w}\mu\left(\frac{w}{d}\right)5^d - \left(3\cdot\frac{1}{w}\sum_{d|w}\mu\left(\frac{w}{d}\right)2^d - 1\right) \tag{4.71}$$

$$N_{DH_1H_2M}(w) = N_{H_1H_2M}(w) - N_{H_1H_2M}(w-1) = 16\cdot 5^{w-2} - 3\cdot 2^{w-2} \tag{4.72}$$

$$N_{ADH_1H_2M}(w) = N_{AH_1H_2M}(w) - N_{AH_1H_2M}(w-1) \tag{4.73}$$

$$= \frac{1}{w}\sum_{d|w}\mu\left(\frac{w}{d}\right)(5^d - 3\cdot 2^d) - \frac{1}{w-1}\sum_{d|w-1}\mu\left(\frac{w-1}{d}\right)(5^d - 3\cdot 2^d) . \tag{4.74}$$

Here μ denotes the Möbius function [43].

The analytic continuation of the cyclotomic harmonic sums can be performed as outlined for the case of the single sums in Section 4.1. Their representation as a Mellin transform of the cyclotomic harmonic polylogarithms (3.26) relates them to factorial series, except the case $c_1 = 1, s_1 = 1$, cf. (2.1). If a sequential set of first indices $c_i = 1, s_i = 1$ occurs, one may reduce the corresponding sum algebraically to convergent sums, separating factors, cf. [6]. I.e., also in the general case the poles of the cyclotomic harmonic sums are located at $-k, k \in \mathbb{N}$. The recursion relations of the cyclotomic harmonic sums (2.1) imply the shift relations $N \to N+1$ in a hierarchic manner, referring to the sums of lower depth.

To accomplish the analytic continuation in N, the asymptotic representations of the cyclotomic harmonic sums have to be computed. Since the cyclotomic harmonic sums are represented over respective bases, only the asymptotic representations for the basis elements have to be derived. One way consists in using iterated integration by parts

$$\int_0^1 dx\ x^N\ f(x) = \frac{f(1)}{N+1} - \frac{1}{N+1}\int_0^1 dx\ x^N\ [xf'(x)]\ , \tag{4.75}$$

with $f(x)$ a linear combination of cyclotomic polylogarithms. Here $f(x)$ is conveniently expressed in a power series to which we turn now.

We illustrate the principle steps considering the alphabet (3.29) related to (4.60), cf. (3.16, 3.17). In general the cyclotomic harmonic polylogarithms $C_{\vec{a}}^{\vec{b}}(x)$ over the alphabet (3.29) do not have a regular Taylor series expansion, cf. [26]. This is due to the effect that trailing zeroes in the index set may cause powers of $\ln(x)$. Hence the proper expansion is one in terms of both x and $\ln(x)$. For depth one and $0 < x < 1$ one obtains

$$C_0^0(x) = \ln(x) \tag{4.76}$$

$$C_2^0(x) = -\sum_{i=1}^{\infty}\frac{(-x)^i}{i} = -\sum_{i=1}^{\infty}\frac{(-x)^{2i}}{2i} + \sum_{i=1}^{\infty}\frac{(-x)^{2i+1}}{2i+1} \tag{4.77}$$

$$C_1^0(x) = -\sum_{i=1}^{\infty}\frac{x^i}{i} = \sum_{i=1}^{\infty}\frac{(-x)^{2i}}{2i} + \sum_{i=1}^{\infty}\frac{(-x)^{2i+1}}{2i+1} \tag{4.78}$$

$$C_4^0(x) = -\sum_{i=1}^{\infty}\frac{(-1)^i x^{2i-1}}{2i-1} \tag{4.79}$$

$$C_4^1(x) = \sum_{i=1}^{\infty}\frac{(-1)^i x^{2i}}{2i}\ . \tag{4.80}$$

Let $C_{\vec{a}}^{\vec{b}}(x)$ be a cyclotomic harmonic polylogarithm with depth d. Assume that its power series expansion is of the form

$$C_{\vec{a}}^{\vec{b}}(x) = \sum_{j=1}^{w}\sum_{i=1}^{\infty}\frac{\sigma^i x^{2i+c_j}}{(2i+c_j)^a} S_{\vec{n}_j}(i) \tag{4.81}$$

with the index sets $\vec{a}$ and $\vec{b}$ according to the iteration of the letters (4.76–4.80) and $\vec{n}_j$ a corresponding index set of the cyclotomic harmonic sum, $x \in (0,1)$, $w \in \mathbb{N}$ and $c_j \in \mathbb{Z}$.

Then the expansion of the cyclotomic harmonic polylogarithms of depth $d+1$ is obtained by using

$$C_{0,\vec{a}}^{0,\vec{b}}(x) = \sum_{j=1}^{w}\sum_{i=1}^{\infty}\frac{\sigma^i x^{2i+c_j}}{(2i+c_j)^{a+1}} S_{\vec{n}_j}(i) \tag{4.82}$$

$$C_{0,\vec{a}}^{1,\vec{b}}(x) = \sum_{j=1}^{w}\sum_{i=1}^{\infty}\frac{x^{2i+c_j+1}}{(2i+c_j+1)}S_{\{2,c_j,\sigma a\},\vec{n}_j}(i) + \sum_{j=1}^{w}\sum_{i=1}^{\infty}\frac{x^{2i+c_j+2}}{(2i+c_j+2)}S_{\{2,c_j,\sigma a\},\vec{n}_j}(i) \tag{4.83}$$

$$C_{2,\vec{a}}^{0,\vec{b}}(x) = \sum_{j=1}^{w}\sum_{i=1}^{\infty}\frac{x^{2i+c_j+1}}{(2i+c_j+1)}S_{\{2,c_j,\sigma a\},\vec{n}_j}(i) - \sum_{j=1}^{w}\sum_{i=1}^{\infty}\frac{x^{2i+c_j+2}}{(2i+c_j+2)}S_{\{2,c_j,\sigma a\},\vec{n}_j}(i) \tag{4.84}$$

$$C_{4,\vec{a}}^{0,\vec{b}}(x) = \sum_{j=1}^{w}\sum_{i=1}^{\infty}\frac{(-1)^i x^{2i+c_j+1}}{(2i+c_j+1)}S_{\{2,c_j,-\sigma a\},\vec{n}_j}(i) \tag{4.85}$$

$$C_{4,\vec{m}}^{1,\vec{m}}(x) = \sum_{j=1}^{w}\sum_{i=1}^{\infty}\frac{(-1)^i x^{2i+c_j+2}}{(2i+c_j+2)}S_{\{2,c_j,-\sigma a\},\vec{n}_j}(i)\ . \tag{4.86}$$

Sample proofs of these relations are given in Appendix B.

The analytic continuation of the cyclotomic harmonic sums at larger depths to $N \in \mathbb{C}$ is performed analogously to the case discussed in Section 4.1. Shifts parallel to the real axis are performed with the recurrence relation induced by (2.1). The asymptotic relations for $N \to \infty,\ |\arg(N)| < \pi$ can be derived analytically to arbitrary precision. Examples are :

$$\begin{aligned}
S_{\{2,1,2\},\{1,0,-2\}}(N) \sim & 4C_{f_4^0,f_0^0}(1)^2 + 4C_{f_4^0}(1)C_{f_0^0,f_4^0,f_0^0}(1) - 4C_{f_4^0,f_0^0,f_4^0,f_0^0}(1) + \Biggl(\frac{\pi^2}{2} - \frac{1}{N} \\
& + \frac{1}{N^2} - \frac{11}{12N^3} + \frac{3}{4N^4} - \frac{127}{240N^5} + \frac{5}{16N^6} - \frac{221}{1344N^7} + \frac{7}{64N^8} \\
& - \frac{367}{3840N^9} + \frac{9}{256N^{10}}\Biggr) C_{f_4^1,f_0^0}(1) + (-1)^N \Biggl(\frac{1}{16N^4} - \frac{1}{4N^5} + \frac{27}{64N^6} \\
& - \frac{1}{32N^7} - \frac{269}{256N^8} - \frac{11}{32N^9} + \frac{8699}{1024N^{10}}\Biggr) + O\left(\frac{1}{N^{11}}\right)
\end{aligned} \tag{4.87}$$

$$\begin{aligned}
S_{\{2,1,2\},\{2,1,-2\}}(N) \sim & -\frac{\pi^2}{8} + C_{f_4^0}(1)C_{f_0^0,f_4^1,f_0^0}(1) - C_{f_4^0,f_0^0,f_4^1,f_0^0}(1) + \frac{1}{4N} - \frac{1}{4N^2} + \frac{11}{48N^3} \\
& - \frac{3}{16N^4} + \frac{127}{960N^5} - \frac{5}{64N^6} + \frac{221}{5376N^7} - \frac{7}{256N^8} + \frac{367}{15360N^9} \\
& - \frac{9}{1024N^{10}} + (-1)^N \Biggl(\frac{1}{64N^4} - \frac{5}{64N^5} + \frac{25}{128N^6} - \frac{61}{256N^7} \\
& - \frac{77}{1024N^8} + \frac{221}{512N^9} + \frac{1545}{1024N^{10}}\Biggr) + C_{f_4^0,f_0^0}(1) \Biggl(C_{\frac{1}{4},f_0^0}(1) - \frac{\pi^2}{8} \\
& + \frac{1}{4N} - \frac{1}{4N^2} + \frac{11}{48N^3} - \frac{3}{16N^4} + \frac{127}{960N^5} - \frac{5}{64N^6} + \frac{221}{5376N^7} \\
& - \frac{7}{256N^8} + \frac{367}{15360N^9} - \frac{9}{1024N^{10}}\Biggr) + O\left(\frac{1}{N^{11}}\right)\ .
\end{aligned} \tag{4.88}$$

Given a cyclotomic harmonic sum with the iterative denominators (4.60) and given the number of desired terms, the corresponding expansion can be computed on demand by the HarmonicSums package.

5 Special Values

The values of the cyclotomic harmonic polylogarithms at argument $x = 1$ and, related to it, the associated cyclotomic harmonic sums at $N \to \infty$ occur in various relations of the finite cyclotomic harmonic sums and the Mellin transforms of cyclotomic harmonic polylogarithms. In this Section we investigate their relations and basis representations. The infinite cyclotomic harmonic sums extend the Euler-Zagier and multiple zeta values [11] and are related at lower weight and depth to other known special numbers. We first consider the single non-alternating and alternating sums up to cyclotomy $\mathsf{l} = 6$ at general weight w. Next the relations of the infinite cyclotomic harmonic sums associated to the summands (4.60) up to weight $\mathsf{w} = 6$ are worked out. Finally we investigate the sums of weight $\mathsf{w} = 1$ and 2 up to cyclotomy $\mathsf{l} = 20$.

5.1 Single Infinite Sums

We consider the single sums of the type

$$\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(lk+m)^n} , \tag{5.1}$$

with $l, m, n \in \mathbb{N}_+, l > m, n \geq 1$. These sums are linearly related to the colored harmonic sums by

$$\sum_{k=1}^{\infty} \frac{e_l^k}{k^n} = \sum_{m=1}^{l} e_l^m \sum_{k=0}^{\infty} \frac{1}{(lk+m)^n}, \qquad e_l = \exp\left(\frac{2\pi i}{l}\right) , \tag{5.2}$$

and similar linear relations for the nested sums. We will address the latter sums in Section 6.

5.1.1 Non-alternating Single Sums at $\mathsf{w} = 1$

For the non-alternating sums one obtains

$$\sum_{k=0}^{\infty} \left[\frac{1}{lk+m} - \frac{1}{lk}\right] = -\frac{1}{l}\left[\gamma_E + \psi\left(\frac{m}{l}\right)\right] , \tag{5.3}$$

and

$$\sigma_0 = \sum_{k=1}^{\infty} 1/k \tag{5.4}$$

denoting the divergence used to regulaize the first addend in the r.h.s. of (5.3). The corresponding sums, except σ_0, can always be regularized.

The reflection symmetry of the ψ-function interchanging the arguments x and $(1-x)$, [36], implies

$$\sum_{k=0}^{\infty} \left[\frac{1}{nk+l} - \frac{1}{nk+(n-l)}\right] = \frac{\pi}{n}\cot\left(\frac{l}{n}\pi\right) . \tag{5.5}$$

The digamma-function at positive rational arguments obeys [36, 47]

$$\psi\left(\frac{p}{q}\right) = -\gamma_E - \ln(2q) - \frac{\pi}{2}\cot\left(\frac{p\pi}{q}\right) + 2\sum_{k=1}^{[(q-1)/2]} \cos\left(\frac{2\pi kp}{q}\right)\ln\left[\sin\left(\frac{\pi k}{q}\right)\right] \tag{5.6}$$

$$\psi\left(\frac{1}{n}\right) = -n\left(\gamma_E + \ln(n)\right) - \sum_{k=2}^{n} \psi\left(\frac{k}{n}\right) . \tag{5.7}$$

Eq. (5.6) is used to remove dependencies in (5.3, 5.5). If the regular q–polygon is constructible, the trigonometric functions in (5.6) are algebraic numbers. This is the case for

$$q \in \{2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 34, 40, 48, 51, 60, 64, ...\} , \tag{5.8}$$

[48] [4]. Due to (5.6) the constants $\ln(d_i)$ with $d_i \neq 1$, the divisors of q, and logarithms of further algebraic numbers will occur. Theses are all transcendental numbers [50].

Let us list the first few of these relations, see also [51] :

$$-\frac{1}{2}\left[\gamma_E + \psi\left(\frac{1}{2}\right)\right] = \ln(2) \tag{5.9}$$

$$-\frac{1}{3}\left[\gamma_E + \psi\left(\frac{1}{3}\right)\right] = \frac{1}{2}\ln(3) + \frac{\pi}{18}\sqrt{3} \tag{5.10}$$

$$-\frac{1}{3}\left[\gamma_E + \psi\left(\frac{2}{3}\right)\right] = \frac{1}{2}\ln(3) - \frac{\pi}{18}\sqrt{3} \tag{5.11}$$

$$-\frac{1}{4}\left[\gamma_E + \psi\left(\frac{1}{4}\right)\right] = \frac{3}{4}\ln(2) + \frac{\pi}{8} \tag{5.12}$$

$$-\frac{1}{4}\left[\gamma_E + \psi\left(\frac{3}{4}\right)\right] = \frac{3}{4}\ln(2) - \frac{\pi}{8} \tag{5.13}$$

$$-\frac{1}{5}\left[\gamma_E + \psi\left(\frac{1}{5}\right)\right] = \frac{\sqrt{5}}{10}\left[\ln(2) - \ln(\sqrt{5}-1)\right] + \frac{1}{4}\ln(5) + \frac{\sqrt{25+10\sqrt{5}}}{50}\pi \tag{5.14}$$

$$-\frac{1}{5}\left[\gamma_E + \psi\left(\frac{2}{5}\right)\right] = -\frac{\sqrt{5}}{10}\left[\ln(2) - \ln(\sqrt{5}-1)\right] + \frac{1}{4}\ln(5) + \frac{\sqrt{10-2\sqrt{5}}}{40}\left[1 - \frac{\sqrt{5}}{5}\right]\pi \tag{5.15}$$

$$-\frac{1}{5}\left[\gamma_E + \psi\left(\frac{3}{5}\right)\right] = -\frac{\sqrt{5}}{10}\left[\ln(2) - \ln(\sqrt{5}-1)\right] + \frac{1}{4}\ln(5) - \frac{\sqrt{10-2\sqrt{5}}}{40}\left[1 - \frac{\sqrt{5}}{5}\right]\pi \tag{5.16}$$

$$-\frac{1}{5}\left[\gamma_E + \psi\left(\frac{4}{5}\right)\right] = \frac{\sqrt{5}}{10}\left[\ln(2) - \ln(\sqrt{5}-1)\right] + \frac{1}{4}\ln(5) - \frac{\sqrt{25+10\sqrt{5}}}{50}\pi \tag{5.17}$$

$$-\frac{1}{6}\left[\gamma_E + \psi\left(\frac{1}{6}\right)\right] = \frac{1}{3}\ln(2) + \frac{1}{4}\ln(3) + \frac{\pi}{12}\sqrt{3} \tag{5.18}$$

$$-\frac{1}{6}\left[\gamma_E + \psi\left(\frac{5}{6}\right)\right] = \frac{1}{3}\ln(2) + \frac{1}{4}\ln(3) - \frac{\pi}{12}\sqrt{3}, \quad \text{etc.} \tag{5.19}$$

5.1.2 Alternating Single Sums at $\mathsf{w} = 1$

The alternating sums at $\mathsf{w} = 1$ have the representation :

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{lk+m} = \frac{1}{2l}\left[\psi\left(\frac{m+l}{2l}\right) - \psi\left(\frac{m}{2l}\right)\right] , \tag{5.20}$$

[4] See [49] for special values of the trigonometric functions occurring in (5.6).

with the reflection relation

$$\sum_{k=0}^{\infty}\left[\frac{(-1)^k}{nk+l}-\frac{(-1)^k}{nk+(n-l)}\right] = \frac{1}{2n}\left[\psi\left(\frac{1}{2}-\frac{l}{2n}\right)+\psi\left(\frac{1}{2}+\frac{l}{2n}\right)-\psi\left(1-\frac{l}{2n}\right)-\psi\left(\frac{l}{2n}\right)\right]. \tag{5.21}$$

For the sums at $\mathsf{w}=1$ one obtains :

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} = \frac{\pi}{4} \tag{5.22}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{3k+1} = \frac{1}{3}\left[\frac{\pi}{\sqrt{3}}+\ln(2)\right] \tag{5.23}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{3k+2} = \frac{1}{3}\left[\frac{\pi}{\sqrt{3}}-\ln(2)\right] \tag{5.24}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{4k+1} = \frac{1}{2\sqrt{2}}\left[\frac{\pi}{2}-\ln(\sqrt{2}-1)\right] \tag{5.25}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{4k+3} = \frac{1}{2\sqrt{2}}\left[\frac{\pi}{2}+\ln(\sqrt{2}-1)\right] \tag{5.26}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{5k+1} = \frac{1}{5}\left[1+\sqrt{5}\right]\ln(2)-\frac{1}{\sqrt{5}}\ln(\sqrt{5}-1)+\frac{1+\sqrt{5}}{5\sqrt{10+2\sqrt{5}}}\pi \tag{5.27}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{5k+2} = -\frac{1}{5}\left[1-\sqrt{5}\right]\ln(2)-\frac{1}{\sqrt{5}}\ln(\sqrt{5}-1)+\frac{\sqrt{5}-1}{5\sqrt{10-2\sqrt{5}}}\pi \tag{5.28}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{5k+3} = \frac{1}{5}\left[1-\sqrt{5}\right]\ln(2)+\frac{1}{\sqrt{5}}\ln(\sqrt{5}-1)+\frac{\sqrt{5}-1}{5\sqrt{10-2\sqrt{5}}}\pi \tag{5.29}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{5k+4} = -\frac{1}{5}\left[1+\sqrt{5}\right]\ln(2)+\frac{1}{\sqrt{5}}\ln(\sqrt{5}-1)+\frac{1+\sqrt{5}}{5\sqrt{10+2\sqrt{5}}}\pi \tag{5.30}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{6k+1} = \frac{\pi}{6}+\frac{1}{\sqrt{3}}\left[\frac{1}{2}\ln(2)-\ln(\sqrt{3}-1)\right] \tag{5.31}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{6k+5} = \frac{\pi}{6}-\frac{1}{\sqrt{3}}\left[\frac{1}{2}\ln(2)-\ln(\sqrt{3}-1)\right], \quad \text{etc.} \tag{5.32}$$

5.1.3 Single Infinite Sums of Higher Weight

These sums obey representations which are obtained by repeated differentiation of (5.3) and (5.20) for m :

$$\sum_{k=0}^{\infty}\frac{1}{(lk+m)^n} = \frac{1}{l^n}\zeta_{\mathrm{H}}\left(n,\frac{m}{l}\right) = \frac{1}{\Gamma(n)}\left(-\frac{1}{l}\right)^n\psi^{(n-1)}\left(\frac{m}{l}\right) \tag{5.33}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(lk+m)^n} = \frac{(-1)^{n-1}}{\Gamma(n)}\frac{1}{(2l)^n}\left[\psi^{(n-1)}\left(\frac{m+l}{2l}\right)-\psi^{(n-1)}\left(\frac{m}{2l}\right)\right]. \tag{5.34}$$

Here $\zeta_{\rm H}$ is the Hurwitz ζ–function [52] with the serial representation

$$\zeta_{\rm H}(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} ~. \tag{5.35}$$

We consider the representations of the polygamma functions at rational arguments for p/q, $p \nmid q, q \in \mathbb{N}_+, q \le 12$.[5] One obtains :

$$\psi^{(l)}\left(\frac{1}{2}\right) = (-1)^{l+1} l! \left(2^{l+1}-1\right)\zeta_{l+1}~, \tag{5.36}$$

with

$$\zeta_{2n} = (-1)^{n-1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n}, \tag{5.37}$$

where B_n denote the Bernoulli numbers [54, 55]. They are generated by

$$\frac{x}{\exp(x)-1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}x^n, \quad B_{2n+1} = 0 \text{ for } n > 1~. \tag{5.38}$$

For odd values of l no new basis elements occur due to (5.36).

The reflection formula [36]

$$\psi^{(n)}(1-z) = (-1)^n\left[\psi^n(z) + \pi\frac{d^n}{dz^n}\cot(\pi z)\right] \tag{5.39}$$

implies the relations for the argument p/q and $(q-p)/q$. Likewise, the multiplication formula for the Hurwitz zeta function [52] holds,

$$\zeta_{\rm H}(s,kz) = \frac{1}{k^s}\sum_{n=0}^{k-1}\zeta_{\rm H}\left(s, z+\frac{n}{k}\right), \quad k \in \mathbb{N}_+, \tag{5.40}$$

$$m^{l+1}\psi^{(l)}(mz) = \sum_{k=0}^{m-1}\psi^{(l)}\left(z+\frac{k}{m}\right), \tag{5.41}$$

with the special relation

$$n!(-1)^{n+1}\zeta_{n+1}\left[m^{n+1}-1\right] = \sum_{k=1}^{m-1}\psi^{(n)}\left(\frac{k}{m}\right)~. \tag{5.42}$$

One obtains, cf. also [56],

$$\psi^{(l)}\left(\frac{2}{3}\right) = (-1)^{l+1}l!\left(3^{l+1}-1\right)\zeta_{l+1} - \psi^{(l)}\left(\frac{1}{3}\right)~. \tag{5.43}$$

For even values of l, $\psi^{(l)}(1/3)$ and $\psi^{(l)}(2/3)$ are linear in $\pi^{l+1}\sqrt{3}$ and ζ_{l+1}, with

$$\psi^{(2)}\left(\frac{1}{3}\right) = -\frac{4}{9}\sqrt{3}\pi^3 - 26\zeta_3 \tag{5.44}$$

$$\psi^{(2)}\left(\frac{2}{3}\right) = \frac{4}{9}\sqrt{3}\pi^3 - 26\zeta_3 \tag{5.45}$$

$$\psi^{(4)}\left(\frac{1}{3}\right) = -\frac{16}{3}\sqrt{3}\pi^5 - 2904\zeta_5 \tag{5.46}$$

$$\psi^{(4)}\left(\frac{2}{3}\right) = \frac{16}{3}\sqrt{3}\pi^5 - 2904\zeta_5, \quad \text{etc.} \tag{5.47}$$

[5] Special examples were also considered in [53].

Therefore, only for odd values of l one new basis element due to $\psi^{(2l+1)}(1/3)$ contributes.

The two values at argument 1/4 and 3/4 are related, see e.g. [57],

$$\psi^{(2l-1)}\left(\frac{1}{4}\right) = \frac{4^{2l-1}}{2l}\left[\pi^{2l}(2^{2l}-1)|B_{2l}| + 2(2l)!\beta_D(2l)\right] \tag{5.48}$$

$$\psi^{(2l-1)}\left(\frac{3}{4}\right) = \frac{4^{2l-1}}{2l}\left[\pi^{2l}(2^{2l}-1)|B_{2l}| - 2(2l)!\beta_D(2l)\right] \tag{5.49}$$

$$\psi^{(2l)}\left(\frac{1}{4}\right) = -2^{2l-1}\left[\pi^{2l+1}|E_{2l}| + 2(2l)!(2^{2l+1}-1)\zeta_{2l+1}\right] \tag{5.50}$$

$$\psi^{(2l)}\left(\frac{3}{4}\right) = +2^{2l-1}\left[\pi^{2l+1}|E_{2l}| - 2(2l)!(2^{2l+1}-1)\zeta_{2l+1}\right] . \tag{5.51}$$

Here, E_{2l} denote the Euler numbers [58–61]. They are generated by

$$\frac{2}{\exp(x)+\exp(-x)} = \sum_{n=0}^{\infty}\frac{E_n}{n!}x^n, \quad E_{2n+1} = 0 \text{ for } n > 1 . \tag{5.52}$$

β_D is the Dirichlet β–function [52, 62]

$$\beta_D(l) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^l} = \mathrm{Ti}_l(1), l \in \mathbb{N}_+, \tag{5.53}$$

which is also given by the inverse tangent integral $\mathrm{Ti}_l(x)$ [63, 64], being related to Clausen integrals [65]. The special value for $l = 2$ yields Catalan's constant [44]

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} = \mathbf{C} . \tag{5.54}$$

At even values of l no new constants appear. Odd values contribute with $\mathrm{Ti}(2l)$.

For $z = 1/5, 2/5, 3/5, 4/5$ the relations

$$\psi^{(l)}\left(\frac{3}{5}\right) = (-1)^l\left[\psi^{(l)}\left(\frac{2}{5}\right) + \pi\frac{d^l}{dz^l}\cot(\pi z)|_{z=2/5}\right] \tag{5.55}$$

$$\psi^{(l)}\left(\frac{4}{5}\right) = (-1)^l\left[\psi^{(l)}\left(\frac{1}{5}\right) + \pi\frac{d^l}{dz^l}\cot(\pi z)|_{z=1/5}\right] \tag{5.56}$$

$$l!(-1)^{l+1}\zeta_{l+1}\left[5^{l+1}-1\right] = \psi^{(l)}\left(\frac{1}{5}\right) + \psi^{(l)}\left(\frac{2}{5}\right) + \psi^{(l)}\left(\frac{3}{5}\right) + \psi^{(l)}\left(\frac{4}{5}\right) \tag{5.57}$$

hold. For even values of l, $\psi^{(l)}(2/5)$ is thus dependent on $\psi^{(l)}(1/5)$.

For $z = 1/6, 5/6$ the relation

$$\psi^{(l)}\left(\frac{5}{6}\right) = l!(-1)^{l+1}\zeta_{l+1}\left[6^{l+1} - 3^{l+1} - 2^{l+1} + 1\right] - \psi^{(l)}\left(\frac{1}{6}\right) \tag{5.58}$$

holds. The reflection formula (5.39) relates $\psi^{(l)}(1/6)$ and $\psi^{(l)}(5/6)$ for even values of l to linear combinations of $\pi^{l+1}\sqrt{3}$ and ζ_{2l+1}. For odd l the ψ-values can be expressed by $\psi^{(l)}(1/6)$. The shift-formula by 1/2

$$2^{l+1}\psi^{(l)}(2z) = \psi^{(l)}(z) + \psi^{(l)}\left(z + \frac{1}{2}\right) \tag{5.59}$$

implies

$$\psi^{(l)}\left(\frac{1}{6}\right) = \left(2^{l+1}+1\right)\psi^{(l)}\left(\frac{1}{3}\right) + (-1)^l l!\left(3^{l+1}-1\right)\zeta_{l+1} \ , \tag{5.60}$$

and relates all values to $\psi^{(l)}(1/3)$.

For $z = 1/8, 3/8, 5/8, 7/8$ one obtains

$$\psi^{(l)}\left(\frac{1}{8}\right) + \psi^{(l)}\left(\frac{3}{8}\right) + \psi^{(l)}\left(\frac{5}{8}\right) + \psi^{(l)}\left(\frac{7}{8}\right) = l!(-1)^{l+1}\zeta_{l+1}4^{l+1}\left(2^{l+1}-1\right) \ . \tag{5.61}$$

Due to (5.41) and (5.39) the corresponding ψ-values can be expressed by $\psi^{(l)}(1/8)$

$$\psi^{(l)}\left(\frac{3}{8}\right) = 2^{l+1}\psi^{(l)}\left(\frac{3}{4}\right) - (-1)^l\left[\psi^{(l)}\left(\frac{1}{8}\right) + \pi\frac{d^l}{dz^l}\cot(\pi z)|_{z=1/8}\right] \tag{5.62}$$

$$\psi^{(l)}\left(\frac{5}{8}\right) = 2^{l+1}\psi^{(l)}\left(\frac{1}{4}\right) - \psi^{(l)}\left(\frac{1}{8}\right) \tag{5.63}$$

$$\psi^{(l)}\left(\frac{7}{8}\right) = (-1)^l\left[\psi^{(l)}\left(\frac{1}{8}\right) + \pi\frac{d^l}{dz^l}\cot(\pi z)|_{z=1/8}\right] \ . \tag{5.64}$$

For $z = 1/10, 3/10, 7/10, 9/10$ two reflection relations and

$$\psi^{(l)}\left(\frac{1}{10}\right) + \psi^{(l)}\left(\frac{3}{10}\right) + \psi^{(l)}\left(\frac{7}{10}\right) + \psi^{(l)}\left(\frac{9}{10}\right) = l!(-1)^{l+1}\zeta_{l+1}\left[10^{l+1}-5^{l+1}-2^{l+1}+1\right] \tag{5.65}$$

hold. By the shift relation (5.59) one obtains

$$\psi^{(l)}\left(\frac{1}{10}\right) = 2^{l+1}\psi^{(l)}\left(\frac{1}{5}\right) - \psi^{(l)}\left(\frac{3}{5}\right) \tag{5.66}$$

$$\psi^{(l)}\left(\frac{3}{10}\right) = 2^{l+1}\psi^{(l)}\left(\frac{3}{5}\right) - \psi^{(l)}\left(\frac{4}{5}\right) \tag{5.67}$$

$$\psi^{(l)}\left(\frac{7}{10}\right) = 2^{l+1}\psi^{(l)}\left(\frac{2}{5}\right) - \psi^{(l)}\left(\frac{1}{5}\right) \tag{5.68}$$

$$\psi^{(l)}\left(\frac{9}{10}\right) = 2^{l+1}\psi^{(l)}\left(\frac{4}{5}\right) - \psi^{(l)}\left(\frac{2}{5}\right) \ . \tag{5.69}$$

No new constants contribute.

For $z = 1/12, 5/12, 7/12, 11/12$ two reflection relations and

$$\psi^{(l)}\left(\frac{1}{12}\right) + \psi^{(l)}\left(\frac{5}{12}\right) + \psi^{(l)}\left(\frac{7}{12}\right) + \psi^{(l)}\left(\frac{11}{12}\right) = l!(-1)^{l+1}\zeta_{l+1}2^{l+1} \times\left(6^{l+1}-3^{l+1}-2^{l+1}+1\right) \tag{5.70}$$

hold. All values can be expressed by $\psi^{(l)}(1/12)$.

$$\psi^{(l)}\left(\frac{5}{12}\right) = (-1)^l\left[2^{l+1}\psi^{(l)}\left(\frac{1}{6}\right) - \psi^{(l)}\left(\frac{1}{12}\right) + \pi\frac{d^l}{dz^l}\cot(\pi z)|_{z=7/12}\right] \tag{5.71}$$

$$\psi^{(l)}\left(\frac{7}{12}\right) = 2^{l+1}\psi^{(l)}\left(\frac{1}{6}\right) - \psi^{(l)}\left(\frac{1}{12}\right) \tag{5.72}$$

$$\psi^{(l)}\left(\frac{11}{12}\right) = (-1)^l\left[\psi^{(l)}\left(\frac{1}{12}\right) + \pi\frac{d^l}{dz^l}\cot(\pi z)|_{z=1/12}\right], \quad \text{etc.} \tag{5.73}$$

Eq. (5.41) yields

$$\psi^{(l)}\left(\frac{5}{12}\right) = 3^{l+1}\psi^{(l)}\left(\frac{1}{4}\right) - \psi^{(l)}\left(\frac{3}{4}\right) - \psi^{(l)}\left(\frac{1}{12}\right) \ . \tag{5.74}$$

For odd values of l one obtains

$$\psi^{(l)}\left(\frac{1}{12}\right) = 2^l\psi^{(l)}\left(\frac{1}{6}\right) + \frac{1}{2}\left[3^{l+1}\psi^{(l)}\left(\frac{1}{4}\right) - \psi^{(l)}\left(\frac{3}{4}\right) + \pi\frac{d^l}{dz^l}\cot(\pi z)\Big|_{z=7/12}\right] \ . \tag{5.75}$$

Eq. (5.70) does not imply a further relation. $\psi^{(l)}(1/12)$ contributes as new constant for even values of l.

For even values of $l = 2q$ the Hurwitz ζ-function (5.5) obeys the representation [66]

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{1}{(2qk+2p-1)^n} &= \frac{1}{(2q)^n}\zeta_{\rm H}\left(n, \frac{2p-1}{2q}\right) \\ &= \frac{1}{q}\sum_{k=1}^{q}\left[C_n\left(\frac{k}{q}\right)\cos\left(\frac{(2p-1)k\pi}{q}\right) + S_n\left(\frac{k}{q}\right)\sin\left(\frac{(2p-1)k\pi}{q}\right)\right] \ ,\end{aligned} \tag{5.76}$$

where C_ν and S_ν are represented by the Legendre χ-function [67] (5.77),

$$\chi_\nu(z) = \frac{1}{2}\left[{\rm Li}_\nu(z) - {\rm Li}_\nu(-z)\right] \tag{5.77}$$

with [6]

$$\begin{aligned}C_\nu(x) &= {\sf Re}\chi_\nu\left(\exp(i\pi x)\right) \\ S_\nu(x) &= {\sf Im}\chi_\nu\left(\exp(i\pi x)\right) \ .\end{aligned} \tag{5.78}$$

For $\nu \in \mathbb{N}$, (5.78) can be represented by Euler polynomials [68] and powers of π,

$$C_{2n}\left(\frac{p}{q}\right) = \frac{(-1)^n}{4(2n-1)!}\pi^{2n}E_{2n-1}\left(\frac{p}{q}\right) \tag{5.79}$$

$$S_{2n+1}\left(\frac{p}{q}\right) = \frac{(-1)^n}{4(2n)!}\pi^{2n+1}E_{2n}\left(\frac{p}{q}\right), \ p,q \in \mathbb{N}_+, p \le q, \tag{5.80}$$

with

$$\frac{2\exp(xt)}{\exp(t)+1} = \sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!} \ . \tag{5.81}$$

Since

$$\psi^{(m-1)}\left(\frac{1}{l}\right) = (-1)^m l^m (m-1)!\sum_{k=0}^{\infty}\frac{1}{(lk+1)^m}, \tag{5.82}$$

in particular $\psi^{(l)}(1/8)$ and $\psi^{(l)}(1/12), l \in \mathbb{N}_+, l \ge 1$ consist of one term containing an integer power of π and a second term $\propto {\sf Re}({\sf Im})(\chi_{l+1}(r)), r \in \mathbb{Q}$ according to (5.76, 5.78).

[6] We corrected typos in Eq. (10) of Ref. [66].

In conclusion, the representation of the single sums of weight $\mathsf{w} \geq 2$ and $l \leq 6$ require the additional constants :

$$\zeta_{2k+1}, \psi^{(2k+1)}\left(\frac{1}{3}\right), \mathrm{Ti}_{2k}(1), \psi^{(k)}\left(\frac{1}{5}\right), \psi^{(2k+1)}\left(\frac{2}{5}\right), \psi^{(k)}\left(\frac{1}{8}\right), \psi^{(2k)}\left(\frac{1}{12}\right) \ , k \in \mathbb{N}_+ \ . \quad (5.83)$$

Finally we would like to comment on a relation in Ramanujan's notebooks [69] Chapter 9, (11.3), which was claimed to involve a cyclotomic harmonic sum,

$$G(1) = \frac{1}{8}\sum_{r=1}^{\infty}\frac{1}{r^3}\sum_{s=1}^{r}\frac{1}{2l-1} \quad (5.84)$$

$$H(1) = \frac{\pi}{4}\sum_{r=0}^{\infty}\frac{(-1)^r}{(4r+1)^3} - \frac{\pi}{3\sqrt{3}}\sum_{r=0}^{\infty}\frac{1}{(2r+1)^3} \ , \quad (5.85)$$

with

$$G(1) = H(1) \ . \quad (5.86)$$

This relation is unfortunately incorrect, cf. [69], p. 257. $G(1)$ may be given by the following representation in multiple zeta values introducing nested harmonic sums [70]

$$G(1) = \frac{7}{16}\sigma_{3,1} + \frac{1}{2}\sigma_{-3,1} = -\frac{53}{160}\zeta_2^2 - \frac{1}{4}\zeta_2\ln^2(2) + \frac{7}{8}\zeta_3\ln(2) + \frac{1}{24}\ln^4(2) + \mathrm{Li}_4\left(\frac{1}{2}\right)$$
$$\simeq 0.16227193947148339072 \quad (5.87)$$

using the representations in [11]. At present, no relation between the special constants used in (5.87) are known. On the other hand, $H(1)$ is given by

$$H(1) = -\frac{\pi}{2048}\left\{8\left[\pi^3 + 28\left(1+\frac{8}{9}\sqrt{3}\right)\zeta_3\right] + \psi^{(2)}\left(\frac{1}{8}\right)\right\}$$
$$\simeq 0.14402290986880995023 \ . \quad (5.88)$$

5.2 Infinite multiple sums at higher depth

As a further extension of the infinite cyclotomic harmonic sums [11] we consider the iterated summation of the terms (4.60). The corresponding sums diverge if the first indices have the pattern $c_i = 1, s_i = 1$, (2.1). However, these divergences can be regulated by polynomials in σ_0 and cyclotomic harmonic sums, which are convergent for $N \to \infty$, very similar to the case of the usual harmonic sums [7, 8]. We study the relations given in Section 4.3 supplemented by those of the shuffle algebra (SH) of the cyclotomic harmonic polylogarithms (at argument $x = 1$), Eq. (3.21). In Table 3 we present the number of basis elements obtained applying the respective relations up to weight $\mathsf{w} = 6$. The representation of all sums were computed by means of computer algebra in explicit form. We derive the cumulative basis, quoting only the new elements in the next weight. Up to $\mathsf{w} = 4$ we derive also suitable integral representations over known functions. One possible choice of basis elements is :

$\mathsf{w} = 1$:

$$\sigma_{\{1,0,1\}} = \sigma_0 \quad (5.89)$$

$$\sigma_{\{1,0,-1\}} = -\ln(2) \quad (5.90)$$

$$\sigma_{\{2,1,-1\}} = -1 + \frac{\pi}{4} \quad (5.91)$$

weight	N_S	A	SH	A + SH	A + sh + H_1	A + SH + H_1 + H_2	A + SH + H_1 + H_2 + M
1	4	4	4	4	4	3	3
2	20	10	13	3	3	2	1
3	100	40	46	6	6	5	3
4	500	150	163	10	10	9	6
5	2500	624	650	21	21	19	13
6	12500	2580	2635	36	36	34	25

Table 3: Basis representations of the infinite cyclotomic harmonic sums over the alphabet $\{(\pm 1)^k/k, (\pm 1)^k/(2k+1)\}$ after applying the stuffle (A), shuffle (SH) relations, their combination, and their application together with the three multiple argument relations (H_1, H_2, M), as far these lead not to new quantities. In the latter case we quote the cumulative number of basis elements appearing at the new weight.

w = 2:

$$\sigma_{\{2,1,-2\}} = -1 + \mathbf{C} \tag{5.92}$$

w = 3:

$$\sigma_{\{1,0,3\}} = \zeta_3 \tag{5.93}$$

$$\sigma_{\{1,0,-2\},\{2,1,-1\}} = \frac{\pi^2}{12} - \frac{\pi^3}{48} + \frac{1}{2}\int_0^1 dx \frac{\sqrt{x}}{x+1}\mathrm{Li}_2(x) \tag{5.94}$$

$$\sigma_{\{2,1,-2\},\{1,0,-1\}} = -\mathbf{C}\ln(2) + \int_0^1 dx \frac{1}{1+x}\frac{\chi_2(\sqrt{x})}{\sqrt{x}} \tag{5.95}$$

w = 4:

$$\sigma_{\{1,0,-1\},\{1,0,1\},\{1,0,1\},\{1,0,1\}} = -\mathrm{Li}_4\left(\frac{1}{2}\right) \tag{5.96}$$

$$\sigma_{\{2,1,-4\}} = -1 - i\chi_4(i) \tag{5.97}$$

$$\sigma_{\{2,1,-3\},\{2,1,-1\}} = i\left(1 - \frac{\pi}{4}\right)\chi_3(i) + \frac{1}{2}\int_0^1 dx \frac{1}{x+1}\chi_3(\sqrt{x}) \tag{5.98}$$

$$\begin{aligned}\sigma_{\{1,0,-2\},\{1,0,-1\},\{2,1,-1\}} &= -\left(\frac{1}{4}\pi^2\ln(2) - \frac{5}{8}\zeta_3\right)\left(1 - \frac{\pi}{4}\right) \\ &\quad + \frac{1}{2}\int_0^1 \frac{\sqrt{x}}{1+x}\left[\left(\mathrm{Li}_2(-x) + \frac{\pi^2}{12}\right)\ln(1-x)\right. \\ &\quad \left. + \frac{1}{2}\mathrm{S}_{1,2}(x^2) - \mathrm{S}_{1,2}(x) - \mathrm{S}_{1,2}(-x)\right]\end{aligned} \tag{5.99}$$

$$\begin{aligned}\sigma_{\{1,0,-2\},\{2,1,-1\},\{1,0,1\}} &= -\frac{1}{2}\int_0^1 dx \frac{\sqrt{x}}{1+x}\left[\ln\left(1 - \sqrt{x}\right) - \ln\left(1 + \sqrt{x}\right)\right]\mathrm{Li}_2(x) \\ &\quad - \ln(2)\int_0^1 dx \frac{\sqrt{x}}{1+x}\mathrm{Li}_2(x) + \frac{\pi^2}{24}\left[\ln(2)\pi - 2\mathbf{C}\right]\end{aligned} \tag{5.100}$$

$$\begin{aligned}\sigma_{\{1,0,-2\},\{2,1,-1\},\{2,1,1\}} &= -\frac{1}{4}\int_0^1 dx \frac{\sqrt{x}}{1+x}\ln(1-x)\mathrm{Li}_2(x) \\ &\quad - \frac{1}{2}\left[1 - \ln(2)\right]\int_0^1 dx \frac{\sqrt{x}}{1+x}\mathrm{Li}_2(x)\end{aligned}$$

$$+\frac{\pi^2}{24}\left[\frac{\pi}{2}\left[1-\frac{1}{2}\ln(2)\right]-\mathbf{C}\right] . \qquad (5.101)$$

Here $S_{1,2}(x)$ denotes a Nielsen integral [71],

$$S_{1,2}(x)=\frac{1}{2}\int_0^x\frac{dz}{z}\ln^2(1-z) . \qquad (5.102)$$

At weight $\mathsf{w}=5{,}6$ we give only a few integral representations. They can in general be obtained form the Mellin transforms setting the kernel $x^N\to 1$. The following basis elements are obtained :

$\mathsf{w}=5$:

$$\sigma_{\{1,0,5\}}=\zeta_5, \qquad (5.103)$$

$$\sigma_{\{1,0,-1\},\{1,0,1\},\{1,0,1\},\{1,0,1\},\{1,0,1\}}=-\mathrm{Li}_5\left(\frac{1}{2}\right), \qquad (5.104)$$

$$\sigma_{\{1,0,-4\},\{2,1,-1\}}=\frac{7}{720}\pi^4-\frac{7}{2880}\pi^5+\frac{1}{2}\int_0^1 dx\frac{\sqrt{x}}{1+x}\mathrm{Li}_4(x), \qquad (5.105)$$

$$\sigma_{\{1,0,4\},\{2,1,-1\}}=-\frac{\pi^4}{90}+\frac{\pi^5}{360}+\frac{1}{2}\int_0^1 dx\frac{\sqrt{x}}{1+x}\mathrm{Li}_4(-x), \qquad (5.106)$$

$$\sigma_{\{2,1,-4\},\{1,0,-1\}},\ \sigma_{\{1,0,-3\},\{1,0,-1\},\{2,1,1\}},\ \sigma_{\{1,0,-3\},\{2,1,-1\},\{2,1,-1\}},\ \sigma_{\{1,0,3\},\{2,1,-1\},\{2,1,-1\}},$$
$$\sigma_{\{2,1,-3\},\{2,1,-1\},\{2,1,1\}},\ \sigma_{\{1,0,-2\},\{1,0,-1\},\{1,0,-1\},\{2,1,-1\}},\ \sigma_{\{1,0,-2\},\{1,0,-1\},\{2,1,-1\},\{1,0,-1\}},$$
$$\sigma_{\{1,0,-2\},\{2,1,-1\},\{1,0,1\},\{1,0,1\}},\ \sigma_{\{1,0,-2\},\{2,1,-1\},\{2,1,1\},\{1,0,1\}} . \qquad (5.107)$$

$\mathsf{w}=6$:

$$\sigma_{\{1,0,-5\},\{1,0,-1\}}=\sigma_{-5,-1}, \qquad (5.108)$$

$$\sigma_{\{1,0,-1\},\{1,0,1\},\{1,0,1\},\{1,0,1\},\{1,0,1\},\{1,0,1\}}=-\mathrm{Li}_6\left(\frac{1}{2}\right), \qquad (5.109)$$

$$\sigma_{\{2,1,-6\}}=-1-i\chi_6(i), \qquad (5.110)$$

$$\sigma_{\{1,0,-2\},\{2,1,-1\},\{2,1,1\},\{1,0,1\},\{1,0,1\}},\ \sigma_{\{1,0,-2\},\{2,1,-1\},\{1,0,1\},\{1,0,1\},\{1,0,1\}},$$
$$\sigma_{\{1,0,-2\},\{1,0,-1\},\{1,0,-1\},\{2,1,-1\},\{2,1,1\}},\ \sigma_{\{1,0,-2\},\{1,0,-1\},\{1,0,-1\},\{2,1,-1\},\{1,0,1\}},$$
$$\sigma_{\{1,0,-2\},\{1,0,-1\},\{1,0,-1\},\{1,0,-1\},\{2,1,-1\}},\ \sigma_{\{1,0,3\},\{2,1,-1\},\{2,1,-1\},\{1,0,1\}},$$
$$\sigma_{\{1,0,-3\},\{1,0,-1\},\{2,1,1\},\{1,0,1\}},\ \sigma_{\{1,0,-3\},\{1,0,-1\},\{1,0,1\},\{2,1,1\}},\ \sigma_{\{2,1,-3\},\{2,1,-1\},\{2,1,-1\},\{2,1,-1\}},$$
$$\sigma_{\{1,0,-3\},\{2,1,-1\},\{2,1,-1\},\{1,0,-1\}},\ \sigma_{\{1,0,-3\},\{2,1,-1\},\{1,0,-1\},\{2,1,-1\}},\ \sigma_{\{1,0,-3\},\{1,0,-1\},\{2,1,-1\},\{2,1,-1\}},$$
$$\sigma_{\{2,1,4\},\{1,0,-1\},\{2,1,-1\}},\ \sigma_{\{1,0,4\},\{2,1,-1\},\{1,0,-1\}},\ \sigma_{\{1,0,4\},\{1,0,-1\},\{2,1,-1\}},$$
$$\sigma_{\{2,1,-4\},\{1,0,-1\},\{2,1,1\}},\ \sigma_{\{2,1,-4\},\{1,0,-1\},\{1,0,1\}},\ \sigma_{\{1,0,-4\},\{2,1,-1\},\{2,1,1\}},$$
$$\sigma_{\{1,0,-4\},\{2,1,-1\},\{1,0,1\}},\ \sigma_{\{1,0,-4\},\{1,0,-1\},\{2,1,-1\}},\ \sigma_{\{2,1,-4\},\{2,1,-2\}},$$
$$\sigma_{\{2,1,-5\},\{2,1,-1\}} . \qquad (5.111)$$

Recently, infinite sums of a type proposed in [72, 73] [7] were studied in [75], Eqs. (11a-c). They can be expressed in terms of sums studied in this Section :

$$\mathcal{J}_1(r) \;=\; \frac{1}{2}\sum_{k=0}^{\infty}\frac{[\psi(k+1/2)-\psi(1/2)]}{(2k+1)^r}=\sigma_{\{2,1,r\}}-\sigma_{\{2,1,r+1\}}+\sigma_{\{2,1,r+1\},\{2,1,1\}} \qquad (5.112)$$

[7]For similar sums see [74].

$$\mathcal{J}_2(r) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{[\psi(k+1/2)-\psi(1/2)]}{(2k)^r} = \frac{1}{2^r}\sigma_{\{1,0,r\},\{2,-1,1\}} \tag{5.113}$$

$$\mathcal{M}(r) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{[\psi(k+1)+\gamma]}{(2k+1)^r} = \frac{1}{2}\sigma_{\{2,1,r\},\{1,0,1\}}\ . \tag{5.114}$$

Moreover, values of usual harmonic polylogarithms [26] at $x = 1$, $\int_0^1 dt[\mathrm{Li}_p(\pm t) - \mathrm{Li}_p(\pm 1)]/(1 \pm t)$, are discussed. In part these integrals refer to all three letters of the corresponding alphabet. The corresponding representations involve infinite harmonic sums of depth $\mathsf{d} > 1$ naturally, as e.g. for

$$\int_0^1 dx \frac{\mathrm{Li}_5(x)}{1+x} = H_{-1,0,0,0,1}(1) = -\frac{15}{16}\zeta_5 \ln(2) + \sigma_{-5,-1} \tag{5.115}$$

and in similar cases of higher weight, see Ref. [8, 11, 26].

5.3 Infinite multiple sums with more cyclotomic letters

Let us now consider more cyclotomic letters. We study the sums up to weight $\mathsf{w} = 2$ and cyclotomy $\mathsf{l} = 20$ [8], based on the sets of the non-alternating and alternating sums using the letters

$$\frac{(\pm 1)^k}{(lk+m)^n}, \qquad 1 \le n \le 2, 1 \le l \le 20, m < l\ . \tag{5.116}$$

We use the stuffle (quasi-shuffle) relations for the sums, the shuffle relations on the side of the associated cyclotomic harmonic polylogarithms, and the multiple argument relations for these sums, cf. Section 4.3. In some cases the latter request to include sums which are outside the above pattern. In this case the corresponding relations are not accounted for. At $\mathsf{w} = 1$ the respective numbers of basis elements is summarized in Table 4.

l	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
sums	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40
basis	2	3	4	5	6	6	8	9	8	10	12	10	14	14	11	17	18	14	20	18
new basis sums	2	1	2	2	4	1	6	4	4	3	10	2	12	5	3	8	16	4	18	6

Table 4: The number of the $\mathsf{w} = 1$ cyclotomic harmonic sums (5.116) up to $\mathsf{l} = 20$, the basis elements at fixed value of l, and the new basis elements in ascending sequence.

The reflection relation (5.39) for the ψ-functions for $x \leftrightarrow (1-x)$ implies that there are at most $l+1$ basis elements. We showed that the use of the above analytic representations and the shuffle, stuffle, and multiple argument relations lead to the same number of basis elements for $l \le 6$ in the non-alternating and alternating case.

The independent sums at $\mathsf{w} = 1$ up to $\mathsf{l} = 6$ are :

$$\sigma_{\{1,0,1\}}, \sigma_{\{1,0,-1\}}, \sigma_{\{2,1,-1\}}, \sigma_{\{3,1,1\}}, \sigma_{\{3,1,-1\}}, \sigma_{\{4,1,-1\}}, \sigma_{\{4,3,-1\}}, \sigma_{\{5,1,1\}}, \sigma_{\{5,1,-1\}},$$
$$\sigma_{\{5,2,-1\}}, \sigma_{\{5,3,-1\}}, \sigma_{\{6,1,-1\}}\ , \tag{5.117}$$

[8]Relations between colored nested infinite harmonic sums have been investigated also in Refs. [76, 77] recently.

see Sections 5.1.1, 5.1.2 for equivalent representations.

The dependent sums up to $\mathsf{l} = 6$ are

$$\sigma_{\{2,1,1\}} = -1 - \sigma_{\{1,0,-1\}} + \frac{1}{2}\sigma_{\{1,0,1\}} \tag{5.118}$$

$$\sigma_{\{3,2,1\}} = -\frac{1}{2} - \frac{1}{3}\sigma_{\{1,0,-1\}} - \sigma_{\{3,1,-1\}} + \sigma_{\{3,1,1\}} \tag{5.119}$$

$$\sigma_{\{3,2,-1\}} = \frac{1}{2} + \frac{2}{3}\sigma_{\{1,0,-1\}} + \sigma_{\{3,1,-1\}} \tag{5.120}$$

$$\sigma_{\{4,1,1\}} = -\frac{1}{2} - \frac{3}{4}\sigma_{\{1,0,-1\}} + \frac{1}{4}\sigma_{\{1,0,1\}} + \frac{1}{2}\sigma_{\{2,1,-1\}} \tag{5.121}$$

$$\sigma_{\{4,3,1\}} = -\frac{5}{6} - \frac{3}{4}\sigma_{\{1,0,-1\}} + \frac{1}{4}\sigma_{\{1,0,1\}} - \frac{1}{2}\sigma_{\{2,1,-1\}} \tag{5.122}$$

$$\sigma_{\{5,2,1\}} = \frac{1}{5}\sigma_{\{1,0,-1\}} + \sigma_{\{5,1,1\}} - \sigma_{\{5,2,-1\}} \tag{5.123}$$

$$\sigma_{\{5,3,1\}} = -\frac{1}{3} - \frac{1}{5}\sigma_{\{1,0,-1\}} - \sigma_{\{5,1,-1\}} + \sigma_{\{5,1,1\}} \tag{5.124}$$

$$\sigma_{\{5,4,1\}} = -\frac{7}{12} - \frac{2}{5}\sigma_{\{1,0,-1\}} - \sigma_{\{5,1,-1\}} + \sigma_{\{5,1,1\}} - \sigma_{\{5,3,-1\}} \tag{5.125}$$

$$\sigma_{\{5,4,-1\}} = \frac{7}{12} + \frac{4}{5}\sigma_{\{1,0,-1\}} + \sigma_{\{5,1,-1\}} - \sigma_{\{5,2,-1\}} + \sigma_{\{5,3,-1\}} \tag{5.126}$$

$$\sigma_{\{6,1,1\}} = -\frac{1}{6}\sigma_{\{1,0,-1\}} + \frac{1}{2}\sigma_{\{3,1,-1\}} + \frac{1}{2}\sigma_{\{3,1,1\}} \tag{5.127}$$

$$\sigma_{\{6,5,1\}} = -\frac{7}{10} - \frac{2}{3}\sigma_{\{1,0,-1\}} - \sigma_{\{3,1,-1\}} + \frac{1}{2}\sigma_{\{3,1,1\}} \tag{5.128}$$

$$\sigma_{\{6,5,-1\}} = \frac{2}{15} + \frac{4}{3}\sigma_{\{2,1,-1\}} - \sigma_{\{6,1,-1\}}, \text{ etc.} \tag{5.129}$$

The remaining sums are related to those given in (5.117–5.129). The following counting relations for the basis elements were tested up to $\mathsf{l} = 700$ using computer algebra methods. Let p, p_i, q be pairwise distinct primes > 2, and let k, k_i be positive integers. The number of basis elements at $\mathsf{w} = 1$ and cyclotomy l are given by

$$\varphi(l) = \begin{cases} l+1, & l = 1 \text{ or } l = 2^k \\ (p-1)p^{k-1}+2, & l = p^k \\ 2\varphi\left(2^{k-1}\prod_{i=1}^n p_i^{k_i}\right) - n - 1, & l = 2^k \prod_{i=1}^n p_i^{k_i} \\ (q-1)\varphi\left(\prod_{i=1}^n p_i^{k_i}\right) - n(q-2) - q + 3, & l = q\prod_{i=1}^n p_i^{k_i} \\ q\,\varphi\left(q^{k-1}\prod_{i=1}^n p_i^{k_i}\right) - (n+2)(q-1), & l = q^k \prod_{i=1}^n p_i^{k_i},\ k > 1\ . \end{cases} \tag{5.130}$$

Let us now consider the case $\mathsf{w} = 2$. Applying the relations given in Section 4.3 and the shuffle algebra of the cyclotomic harmonic polylogarithms at argument $x = 1$ the results given in Table 5 are obtained for the number of basis elements. Again we solved the corresponding linear systems using computer algebra methods and derived the representations for the dependent sums analytically.

The number of the weight $\mathsf{w} = 2$ infinite sums for cyclotomy l is

$$\mathsf{N_S} = 2l(2l+1)\ . \tag{5.131}$$

One may guess, based on the results for $\mathsf{l} \leq 20$, counting relations for the length of the bases listed in Table 5. We found for all but the last column :

$$\mathsf{N_A}(l) = l(2l+1) \tag{5.132}$$

l	N_S	SH	A	A + SH	A + SH + H_1	A + SH + H_1 + H_2	A + SH + H_1 + H_2 + M
1	6	4	3	1	1	1	1
2	20	13	10	3	3	2	1
3	42	27	21	7	6	6	5
4	72	46	36	12	11	10	3
5	110	70	55	19	17	17	16
6	156	99	78	27	25	24	5
7	210	133	105	37	34	34	33
8	272	172	136	48	45	44	12
9	342	216	171	61	57	57	52
10	420	265	210	75	71	70	22
11	506	319	253	91	86	86	85
12	600	378	300	108	103	102	21
13	702	442	351	127	121	121	120
14	812	551	406	147	141	140	49
15	930	585	465	169	162	162	145
16	1056	664	528	192	185	184	50
17	1190	748	595	217	209	209	208
18	1332	837	666	243	235	234	63
19	1482	931	741	271	262	262	261
20	1640	1030	820	300	291	290	74

Table 5: Number of basis elements of the w = 2 cyclotomic harmonic sums (5.116) up to cyclotomy l = 20 after applying the quasi-shuffle algebra of the sums (A), the shuffle algebra of the cyclotomic harmonic polylogarithms (SH), and the three multiple argument relations (H_1, H_2, M) of the sums.

$$N_{SH}(l) = \frac{(5l+3)l}{2} \tag{5.133}$$

$$N_{A,SH}(l) = \frac{6l^2+1-(-1)^l}{8} \tag{5.134}$$

$$N_{A,SH,H_1}(l) = \frac{6l^2-4l+7-(-1)^l}{8} \tag{5.135}$$

$$N_{A,SH,H_1,H_2}(l) = \frac{6l^2-4l+3(1-(-1)^l)}{8}\ . \tag{5.136}$$

The latter relation (5.136) has been derived prior to us by D. Broadhurst [9] :

$$N_{A,SH,H_1,H_2}(l) = \frac{3}{4}l^2 - \frac{1}{2}l + \text{if}(\text{modp(l,2)} = 0, 1, 3/4) \tag{5.137}$$

and the corresponding generating function

$$f(x) = \left[1 + \frac{x^3}{1+x}\right]\frac{1}{(1-x)^3} = \sum_{l=0}^{\infty} a(l)x^l\ . \tag{5.138}$$

We conjecture that in case of $N_{A,SH,H_1,H_2}(l)$ the (M)-relations lead to a reduction of one in the basis for l being a prime. Otherwise quite significant reductions are obtained for which we do not know an explicit counting relation. The corresponding sequence is also not recorded yet in the data base [78].

[9] We would like to thank D. Broadhurst for communicating this relation to us.

6 Generalized Harmonic Sums at Roots of Unity

In Section 5 we considered real representations for the infinite cyclotomic harmonic sums. These are related to the infinite generalized harmonic sums at the roots of unity. We define

$$\lim_{N\to\infty} S_{k_1,\ldots,k_m}(x_1,\ldots,x_m;N) = \sigma_{k_1,\ldots,k_m}(x_1,\ldots,x_m), \tag{6.1}$$

with $S_{k_1,\ldots,k_m}(x_1,\ldots,x_m;N)$ a generalized harmonic sum (1.3), see also [13,14], and $x_j \in \mathcal{C}_n, n \geq 1$, with $\mathcal{C}_n \in \{e_n | e_n^n = 1, e_n \in \mathbb{C}\}$; $k_1 \neq 1$ for $x_1 = 1$.

We seek the relations between the sums of $\mathsf{w} = 1{,}2$. They can be expressed in terms of polylogarithms by :

$$\begin{aligned}
\sigma_{\mathsf{w}}(x) &= \mathrm{Li}_{\mathsf{w}}(x), \qquad \mathsf{w} \in \mathbb{N},\ \mathsf{w} \geq 1 && (6.2)\\
\sigma_1(x) = \mathrm{Li}_1(x) &= -\ln(1-x) && (6.3)\\
\sigma_{1,1}(x,y) &= \mathrm{Li}_2(x) + \frac{1}{2}\ln^2(1-x) + \mathrm{Li}_2\left(-\frac{x(1-y)}{1-x}\right) && (6.4)\\
\sigma_{1,1}(x,x^*) &= \mathrm{Li}_2(x) + \frac{1}{2}\ln^2(1-x) && (6.5)\\
\mathrm{Li}_{\mathsf{w}}(x) &= \mathrm{Li}^*_{\mathsf{w}}(x^*) && (6.6)
\end{aligned}$$

and * denotes complex conjugation. Furthermore, the symmetric combination $\sigma_{1,1}(x,y) + \sigma_{1,1}(y,x)$ is given by [13, 14]

$$\sigma_{1,1}(x,y) + \sigma_{1,1}(y,x) = \ln(1-x)\ln(1-y) + \mathrm{Li}_2(xy)\ . \tag{6.7}$$

Knowing the representations at $\mathsf{w} = 1$ and the dilogarithms at the corresponding roots of unity one term $\sigma_{1,1}(y,x)$ may be expressed by (6.7).

In analyzing the functions $\sigma_{k_1,\ldots,k_m}(x_1,\ldots,x_m)$ with $x_k^l = 1$ one may use the real representations of nested cyclotomic sums for $N \to \infty$, accounting for the respective sub-cycles at d, $d|l$ and complex conjugation.

An example is given by

$$\left\{ e_{12}^k \Big|_{k=1}^{12} \right\} \equiv \left\{ e_{12}, e_6, e_4, e_3, e_{12}^5, e_2, {e_{12}^5}^*, e_3^*, e_4^*, e_6^*, e_{12}^*, e_1 \right\}\ . \tag{6.8}$$

Here we labeled the elements occurring in sub-cycles accordingly.

The polylogarithms $\mathrm{Li}_1(e_l^k)$ and $\mathrm{Li}_2(e_l^k)$ obey :

$$\begin{aligned}
\mathsf{Im}\left[\mathrm{Li}_1(e_l^k)\right] &= \frac{l-2k}{2l}\pi && (6.9)\\
\mathsf{Re}\left[\mathrm{Li}_2(e_l^k)\right] &= \frac{6k(k-l)+l^2}{6l^2}\pi^2\ . && (6.10)
\end{aligned}$$

More generally, PSLQ [79] tests let conjecture that $\mathsf{Im}\left[\mathrm{Li}_n(e_l^k)\right] = r_{n,l,k}\pi^n$ for n odd and $\mathsf{Re}\left[\mathrm{Li}_n(e_l^k)\right] = r_{n,l,k}\pi^n$ for n odd with $r_{n,l,k} \in \mathbb{Q}$.

Now we extend Proposition 2.3 of Ref. [19], where we consider generalized harmonic sums $S_{k_1,\ldots,k_m}(x_1,\ldots,x_m;N)$ with $N \in \mathbb{N}, k_i \in \mathbb{N}_+, x_i \in \mathbb{C}, |x_i| \leq 1$. Let $l \in \mathbb{N}_+$ and

$$y_i^l = x_i \tag{6.11}$$

then

$$S_{k_1,\ldots,k_m}(x_1,\ldots,x_m;N) = \prod_{j=1}^m l^{k_j-1} \sum_{y_i^l = x_i} S_{k_1,\ldots,k_m}(y_1,\ldots,y_m;lN)\ . \tag{6.12}$$

Here the sum is over the lth roots of x_i for $i \in [1, m]$. Eq. (6.12) is called Distribution Relation. It follows from the Vieta's theorem [80] for (6.11) and properties of symmetric polynomials [81]. Eq. (6.12) contains the well–known duplication relation, cf. Eq. (2.15) [11].

If also $x_1 \neq 1$ for $k_1 = 1$ the limit

$$\sigma_{k_1,...,k_m}(x_1, ..., x_m) = \lim_{N \to \infty} S_{k_1,...,k_m}(x_1, ..., x_m; N) \tag{6.13}$$

exists. One may apply (6.12,6.13) to roots of unity x_i and y_j, i.e. $x_i = \exp(2\pi i n_i/m_i)$ and $y_{jk} = \exp(2\pi i k n_i/(m_i l)), k = 1...(l-1), n_i, m_i \in \mathbb{N}_+$. Let us now consider the cases $\mathsf{w} = 1, 2$ in more detail.

6.1 $\mathsf{w} = 1$

The first element is real and occurs at $\mathsf{l} = 2$

$$\mathrm{Li}_1(e_2) = -\ln(2), \tag{6.14}$$

representing the simplest alternating multiple zeta value, cf. e.g. [11]. At $\mathsf{l} = 3$ we get the complex conjugate numbers

$$\mathrm{Li}_1(e_3) = -\frac{1}{2}\ln(3) + \frac{\pi i}{6} \tag{6.15}$$

$$\mathrm{Li}_1(e_3^2) = -\frac{1}{2}\ln(3) - \frac{\pi i}{6}\ . \tag{6.16}$$

Due to (6.9), $\ln(3)$ and $i\pi$ are considered as basis elements from this level on. For all higher values of l one thus needs only to consider the real part of the $\mathsf{w} = 1$ sums and one may work with the real representations given in Section 5. Let us consider the example

$$\begin{aligned} \mathrm{Li}_1(e_4) &= -\ln(1 - e_4) \\ &= e_4 \sum_{k=1}^{\infty} \frac{1}{4k-3} + e_2 \sum_{k=1}^{\infty} \frac{1}{4k-2} + e_4^* \sum_{k=1}^{\infty} \frac{1}{4k-1} + e_1 \sum_{k=1}^{\infty} \frac{1}{4k} \\ &= e_4 \sum_{k=1}^{\infty} \left(\frac{1}{4k-3} - \frac{1}{4k}\right) + e_2 \sum_{k=1}^{\infty} \left(\frac{1}{4k-2} - \frac{1}{4k}\right) + e_4^* \sum_{k=1}^{\infty} \left(\frac{1}{4k-1} - \frac{1}{4k}\right)\ . \end{aligned} \tag{6.17}$$

Eq. (6.17) follows from

$$\sum_{k=1}^{n-1} e_n^k = 0\ . \tag{6.18}$$

The type of sums occurring in (6.17) leads to digamma-functions and one may use their relations given before to find the corresponding basis representations. The first terms are given by :

$$\mathrm{Li}_1(e_4) = -\frac{1}{2}\ln(2) + \frac{\pi i}{4} \tag{6.19}$$

$$\mathrm{Li}_1(e_4^3) = \mathrm{Li}_1^*(e_4) \tag{6.20}$$

$$\mathrm{Li}_1(e_5) = \frac{1}{2}\ln\left(\frac{\sqrt{5}+1}{2}\right) - \frac{1}{4}\ln(5) + i\frac{3}{10}\pi \tag{6.21}$$

$$\mathrm{Li}_1(e_5^2) = \frac{1}{2}\ln\left(\frac{\sqrt{5}-1}{2}\right) - \frac{1}{4}\ln(5) + i\frac{1}{10}\pi \tag{6.22}$$

$$\mathsf{Re}(\mathrm{Li}_1(e_5)) + \mathsf{Re}(\mathrm{Li}_1(e_5^2)) = -\frac{1}{2}\ln(5) \tag{6.23}$$

$$\mathrm{Li}_1(e_6) = \frac{\pi i}{3} \tag{6.24}$$

$$\mathsf{Re}(\mathrm{Li}_1(e_8)) = -\frac{1}{4}\ln(2) - \frac{1}{2}\ln(\sqrt{2}-1) \tag{6.25}$$

$$\mathsf{Re}(\mathrm{Li}_1(e_{12})) = \frac{1}{2}\ln(2) - \ln(\sqrt{3}-1)\ . \tag{6.26}$$

In Table 6 we summarize the number of basis elements.

l	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
basis	0	1	2	2	3	3	4	3	4	4	6	4	7	5	6	5	9	5	10	6
Ref. [82]	0	1	2	2	3	3	4	3	5	4	6									
new elements	0	1	2	0	2	0	3	1	2	0	5	1	6	0	2	2	8	0	9	2

.

Table 6: The number of the basis elements spanning the w=1 cyclotomic harmonic polylogarithms at lth roots of unity up to 20.

At cyclotomy $\mathsf{l} = 9$ we find one basis element less than reported in [82]. The new elements contributing at the respective level of cyclotomy for $\mathsf{l} \le 20$ are :

$$\mathsf{l}=2 \quad \ln(2) \tag{6.27}$$

$$\mathsf{l}=3 \quad \ln(3), \pi \tag{6.28}$$

$$\mathsf{l}=4 \quad - \tag{6.29}$$

$$\mathsf{l}=5 \quad \mathsf{Re}(\mathrm{Li}_1(e_5)), \mathsf{Re}(\mathrm{Li}_1(e_5^2)) \tag{6.30}$$

$$\mathsf{l}=6 \quad - \tag{6.31}$$

$$\mathsf{l}=7 \quad \mathsf{Re}(\mathrm{Li}_1(e_7^k))\big|_{k=1}^{3} \tag{6.32}$$

$$\mathsf{l}=8 \quad \mathsf{Re}(\mathrm{Li}_1(e_8)) \tag{6.33}$$

$$\mathsf{l}=9 \quad \mathsf{Re}(\mathrm{Li}_1(e_9)), \mathsf{Re}(\mathrm{Li}_1(e_9^2)) \tag{6.34}$$

$$\mathsf{l}=10 \quad - \tag{6.35}$$

$$\mathsf{l}=11 \quad \mathsf{Re}(\mathrm{Li}_1(e_{11}^k))\big|_{k=1}^{5} \tag{6.36}$$

$$\mathsf{l}=12 \quad \mathsf{Re}(\mathrm{Li}_1(e_{12})) \tag{6.37}$$

$$\mathsf{l}=13 \quad \mathsf{Re}(\mathrm{Li}_1(e_{13}))\big|_{k=1}^{6} \tag{6.38}$$

$$\mathsf{l}=14 \quad - \tag{6.39}$$

$$\mathsf{l}=15 \quad \mathsf{Re}(\mathrm{Li}_1(e_{15})), \mathsf{Re}(\mathrm{Li}_1(e_{15}^2)) \tag{6.40}$$

$$\mathsf{l}=16 \quad \mathsf{Re}(\mathrm{Li}_1(e_{16})), \mathsf{Re}(\mathrm{Li}_1(e_{16}^3)) \tag{6.41}$$

$$\mathsf{l}=17 \quad \mathsf{Re}(\mathrm{Li}_1(e_{17}))\big|_{k=1}^{8} \tag{6.42}$$

$$\mathsf{l}=18 \quad - \tag{6.43}$$

$$\mathsf{l}=19 \quad \mathsf{Re}(\mathrm{Li}_1(e_{19}))\big|_{k=1}^{9} \tag{6.44}$$

$$\mathsf{l}=20 \quad \mathsf{Re}(\mathrm{Li}_1(e_{20})), \mathsf{Re}(\mathrm{Li}_1(e_{20}^3))\ . \tag{6.45}$$

6.2 $\mathsf{w} = 2$

We first consider the relations for $\mathrm{Li}_2(e_n^k)$. The following well–known representations for the function holds, cf. [83, 84] :

$$\mathrm{Li}_2\left(e^{i\theta}\right) = \pi^2 \bar{B}_2\left(\frac{\theta}{2\pi}\right) + i\mathrm{Cl}_2(\theta)\ , \tag{6.46}$$

with

$$\mathrm{Cl}_2(\theta) = \sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k^2} \tag{6.47}$$

$$\bar{B}_2(x) = -\frac{1}{\pi^2}\sum_{k=1}^{\infty} \frac{\cos(2\pi k x)}{k^2}\ . \tag{6.48}$$

$\bar{B}_2$ denotes the second modified Bernoulli polynomial. Due to (6.10) only the imaginary parts have to be considered. The number π occurs at $\mathsf{w} = 1$, $\mathsf{l} = 3$ only. The first terms are given by :

$$\mathrm{Li}_2(e_1) = \frac{\pi^2}{6} \tag{6.49}$$

$$\mathrm{Li}_2(e_2) = -\frac{\pi^2}{12} \tag{6.50}$$

$$\mathsf{Im}(\mathrm{Li}_2(e_3)) = \frac{\sqrt{3}}{9}\left[\psi^{(1)}\left(\frac{1}{3}\right) - \frac{2}{3}\pi^2\right] \tag{6.51}$$

$$\mathsf{Im}(\mathrm{Li}_2(e_4)) = \mathbf{C} \tag{6.52}$$

$$\mathrm{Li}_2(e_4^3) = \mathrm{Li}_2^*(e_4) \tag{6.53}$$

$$\begin{aligned}\mathsf{Im}(\mathrm{Li}_2(e_5)) = 5\sqrt{10}\Bigg\{&\sqrt{1+\frac{1}{\sqrt{5}}}\left[\psi^{(1)}\left(\frac{1}{5}\right) - \pi^2\left(1+\frac{1}{\sqrt{5}}\right)\right] \\ &+\sqrt{1-\frac{1}{\sqrt{5}}}\left[\psi^{(1)}\left(\frac{2}{5}\right) - \pi^2\left(1-\frac{1}{\sqrt{5}}\right)\right]\Bigg\}\end{aligned} \tag{6.54}$$

$$\begin{aligned}\mathsf{Im}(\mathrm{Li}_2(e_5^2)) = \frac{i}{5\sqrt{10}}\Bigg\{&\sqrt{1-\frac{1}{\sqrt{5}}}\left[\psi^{(1)}\left(\frac{1}{5}\right) - \pi^2\left(1+\frac{1}{\sqrt{5}}\right)\right] \\ &-\sqrt{1+\frac{1}{\sqrt{5}}}\left[\psi^{(1)}\left(\frac{2}{5}\right) - \pi^2\left(1-\frac{1}{\sqrt{5}}\right)\right]\Bigg\}\end{aligned} \tag{6.55}$$

$$\mathrm{Li}_2(e_5^3) = \mathrm{Li}_2^*(e_5^2) \tag{6.56}$$

$$\mathrm{Li}_2(e_5^4) = \mathrm{Li}_2^*(e_5) \tag{6.57}$$

$$\mathsf{Im}(\mathrm{Li}_2(e_6)) = \frac{3}{2}\mathsf{Im}(\mathrm{Li}_2(e_3)) \tag{6.58}$$

$$\mathrm{Li}_2(e_6^5) = \mathrm{Li}_2^*(e_6) \tag{6.59}$$

$$\mathsf{Im}\mathrm{Li}_2(e_8) = \frac{\sqrt{2}}{32}\psi'\left(\frac{1}{8}\right) + \frac{1}{4}\left(1-2\sqrt{2}\right)\mathbf{C} - \frac{1}{16}\left(1+\sqrt{2}\right)\pi^2 \tag{6.60}$$

$$\mathsf{Im}\mathrm{Li}_2(e_{12}) = \frac{\sqrt{3}}{24}\psi'\left(\frac{1}{3}\right) + \frac{2}{3}\mathbf{C} - \frac{\sqrt{3}}{36}\pi^2\ . \tag{6.61}$$

The new basis elements spanning the dilogarithms of the lth roots of unity for $\mathsf{l} \le 20$ are :

$$
\begin{aligned}
\mathsf{l} = 1,2 &\quad - && (6.62)\\
\mathsf{l} = 3 &\quad \mathsf{Im}(\mathrm{Li}_2(e_3)) && (6.63)\\
\mathsf{l} = 4 &\quad \mathbf{C} && (6.64)\\
\mathsf{l} = 5 &\quad \mathsf{Im}(\mathrm{Li}_2(e_5)), \mathsf{Im}(\mathrm{Li}_2(e_5^2)) && (6.65)\\
\mathsf{l} = 6 &\quad - && (6.66)\\
\mathsf{l} = 7 &\quad \mathsf{Im}(\mathrm{Li}_2(e_7^k))\big|_{k=1}^{3} && (6.67)\\
\mathsf{l} = 8 &\quad \mathsf{Im}(\mathrm{Li}_2(e_8)) && (6.68)\\
\mathsf{l} = 9 &\quad \mathsf{Im}(\mathrm{Li}_2(e_9)), \mathsf{Im}(\mathrm{Li}_2(e_9^2)) && (6.69)\\
\mathsf{l} = 10 &\quad - && (6.70)\\
\mathsf{l} = 11 &\quad \mathsf{Im}(\mathrm{Li}_2(e_{11}^k))\big|_{k=1}^{5} && (6.71)\\
\mathsf{l} = 12 &\quad \mathsf{Im}(\mathrm{Li}_2(e_{12})) && (6.72)\\
\mathsf{l} = 13 &\quad \mathsf{Im}(\mathrm{Li}_2(e_{13}))|_{k=1}^{6} && (6.73)\\
\mathsf{l} = 14 &\quad - && (6.74)\\
\mathsf{l} = 15 &\quad \mathsf{Im}(\mathrm{Li}_2(e_{15})) && (6.75)\\
\mathsf{l} = 16 &\quad \mathsf{Im}(\mathrm{Li}_2(e_{16})), \mathsf{Im}(\mathrm{Li}_2(e_{16}^3)) && (6.76)\\
\mathsf{l} = 17 &\quad \mathsf{Im}(\mathrm{Li}_2(e_{17}))|_{k=1}^{8} && (6.77)\\
\mathsf{l} = 18 &\quad - && (6.78)\\
\mathsf{l} = 19 &\quad \mathsf{Im}(\mathrm{Li}_2(e_{19}))|_{k=1}^{9} && (6.79)\\
\mathsf{l} = 20 &\quad \mathsf{Im}(\mathrm{Li}_2(e_{20}))\ . && (6.80)
\end{aligned}
$$

Let us now turn to all convergent sums $\sigma_{1,1}(x,y)$, $x,y \in \mathcal{C}_n$. These sums belong to cyclotomy l if $x = e_l^{k_1}, y = e_l^{k_2}$ and $k_1, k_2 \in \mathbb{N}_+, k_1 < l, k_2 < l$.

We consider first the case $x \neq 1, y = 1$,

$$
\begin{aligned}
\sigma_{1,1}(x,1) &= \mathrm{Li}_2(x) + \frac{1}{2}\mathrm{Li}_1^2(x)\\
&= \frac{1}{2}\mathsf{Re}(\mathrm{Li}_1^2(x)) + \left(r_2 - \frac{1}{2}r_1^2\right)\pi^2 + i\left[r_1\pi\mathsf{Re}(\mathrm{Li}_1(x)) + \mathsf{Im}(\mathrm{Li}_2(x))\right]\ , \qquad (6.81)
\end{aligned}
$$

with

$$
\begin{aligned}
r_1 &= \mathsf{Im}(-\ln(1-x)) && (6.82)\\
r_2 &= \mathsf{Re}(\mathrm{Li}_2(x))\ . && (6.83)
\end{aligned}
$$

Including the basis elements of $\mathsf{w} = 1$ up to l, no new basis element is obtained. Furthermore,

$$
\begin{aligned}
\sigma_{1,1}(e_2,x) &= -\frac{1}{2}\pi^2 + \frac{1}{2}\ln^2(2) + \mathrm{Li}_2\left(\frac{1-x}{2}\right) && (6.84)\\
\sigma_{1,1}(x,e_2) &= -\mathrm{Li}_1(x)\ln(2) + \frac{1}{2}\mathrm{Li}_2(x^2) - \mathrm{Li}_2(x) + \frac{1}{2}\left[\pi^2 + \ln^2(2)\right] - \mathrm{Li}_2\left(\frac{1-x}{2}\right) && (6.85)
\end{aligned}
$$

hold. $\mathrm{Li}_2\left((1+x)/2\right)$ may yield a new basis element. In some cases besides x also $-x$ is element of the the cycle of the roots of unity which have to be considered. Here, however, $\mathrm{Li}_2\left((1+x)/2\right)$

is given by

$$\mathrm{Li}_2\left(\frac{1+x}{2}\right) = -\mathrm{Li}_2\left(\frac{1-x}{2}\right) + \frac{\pi^2}{6} - \mathrm{Li}_1(x)\mathrm{Li}_1(-x) - \ln^2(2) - \ln(2)\left[\mathrm{Li}_1(x) + \mathrm{Li}_1(-x)\right] \ . \tag{6.86}$$

Also the elements $x = e_n^k$ and $e_1 - x \equiv 1 - x$ occur in the cycles, for which

$$\mathrm{Li}_2(1-e_n^k) = -\mathrm{Li}_2(e_n^k) + 2\pi i\frac{k}{n}\mathrm{Li}_1(e_n^k) + \frac{\pi^2}{6} \tag{6.87}$$

holds.

For $\mathsf{l} = 2$ one obtains

$$\sigma_{1,1}(e_2, 1) = -\frac{1}{2}\pi^2 + \frac{1}{2}\ln^2(2) \ . \tag{6.88}$$

Because of (6.50) two basis elements contribute. If a corresponding special number has occured already at $\mathsf{w} = 1$ at the same value of l, it is not counted as new. π occurs at $\mathsf{w} = 2$, $\mathsf{l} = 1$, unlike for $\mathsf{w} = 1$ where it contributes first at $\mathsf{l} = 3$. No new basis element occurs at l=2. For w = 3,4 the new basis elements occur for $\mathrm{Li}_2(e_n^k)$ only. We apply the relations, cf. e.g. [64],

$$\mathrm{Li}_2\left(\frac{1}{1+x}\right) = \mathrm{Li}_2(-x) + \ln(x)\ln(1+x) - \frac{1}{2}\ln^2(1+x) + \zeta_2 \tag{6.89}$$

$$\mathrm{Li}_2\left(\frac{x}{1+x}\right) = -\mathrm{Li}_2(-x) - \frac{1}{2}\ln^2(1+x) \ . \tag{6.90}$$

At $\mathsf{l} = 5$ the above relations lead to corresponding reductions and the two functions

$$\mathrm{Li}_2\left(-\frac{e_5^3(1-e_5)}{1-e_5^3}\right), \qquad \mathrm{Li}_2\left(-\frac{e_5^2}{1+e_5}\right) \tag{6.91}$$

remain. For the first function the identity

$$\mathrm{Li}_2\left(-\frac{e_5^3(1-e_5)}{1-e_5^3}\right) = \mathrm{Li}_2\left(\frac{e_5^4}{1+e_5^4}\right) = -\mathrm{Li}_2(-e_5^4) - \frac{1}{2}\ln^2(1+e_5^4) \tag{6.92}$$

holds, through which the corresponding sum $\sigma_{1,1}(e_5^3, e_5)$ can be expressed by

$$\sigma_{1,1}(e_5^3, e_5) = \frac{1}{2}\mathrm{Li}_2^*(e_5^2) - \mathrm{Li}_2^*(e_5) - \frac{1}{2}\left(\mathrm{Li}_1^2(e_5)\right)^* + \mathrm{Li}_1(e_5)^*\mathrm{Li}_1(e_5^2)^* \ . \tag{6.93}$$

The second dilogarithm can be transformed in the following way :

$$\mathrm{Li}_2\left(-\frac{e_5^2}{1+e_5}\right) = -\mathrm{Li}_2\left(\frac{1+e_5+e_5^2}{1+e_5}\right) - \ln\left(-\frac{e_5^2}{1+e_5}\right)\ln\left(\frac{1+e_5+e_5^2}{1+e_5}\right) + \zeta_2 \ . \tag{6.94}$$

Furthermore,

$$\mathrm{Li}_2\left(\frac{1+e_5+e_5^2}{1+e_5}\right) = \mathrm{Li}_2(-e_5^3) \tag{6.95}$$

holds, through which

$$\mathrm{Li}_2\left(-\frac{e_5^2}{1+e_5}\right) = -\frac{1}{2}\mathrm{Li}_2(e_5) + \mathrm{Li}_2^*(e_5^2) + \frac{19}{150}\pi^2 - \frac{i\pi}{5}\left[\mathrm{Li}_1(e_5^2) - \mathrm{Li}_1(e_5)\right] \tag{6.96}$$

is obtained.

The representations at $\mathsf{w} = 6$ were given in [20], with $\mathrm{Li}_4(1/2)$ the new basis element. The Clausen function $\mathrm{Cl}_2(\pi/3)$ used there is given by

$$\mathrm{Cl}_n(x) = \begin{cases} \frac{i}{2}\left[\mathrm{Li}_n(\exp(-ix)) - \mathrm{Li}_n(\exp(ix))\right], & n \text{ even} \\ \frac{1}{2}\left[\mathrm{Li}_n(\exp(-ix)) + \mathrm{Li}_n(\exp(ix))\right], & n \text{ odd} \end{cases} \tag{6.97}$$

$$\mathrm{Cl}_2\left(\frac{\pi}{3}\right) = \mathrm{Ls}_2^{(0)}\left(\frac{\pi}{3}\right) = \frac{3}{2}\mathsf{Im}(\mathrm{Li}_2(e_3)) \tag{6.98}$$

with

$$\mathrm{Ls}_j^{(k)}(\theta) = -\int_0^\theta dt t^k \ln^{j-k-1}\left|2\sin\left(\frac{t}{2}\right)\right| , \tag{6.99}$$

cf. [85]. In Table 7 we summarize the number of basis elements found at $\mathsf{w} = 2$ using the relations in Section 4.3, the distribution relations (6.12, 6.13), and the relations for dilogarithms given above. We also list the number of basis elements for the class of dilogarithms at roots of unity, and in both cases the number of new elements beyond those being obtained at $\mathsf{w} = 1$ at the same value of l.

l	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Li_2 basis	1	1	2	2	3	2	4	3	4	3	6	3	7	4	5	5	9	4	10	5
Li_2 new	1	0	1	1	2	0	3	1	2	0	5	0	6	0	1	2	8	0	9	1
basis	1	2	3	3	5	5	8	7	10	10	16	12	21	17	21	21	33	23	40	29
Ref. [82]	1	1	1	1	2	2	4	4	7	6	10									
new elements	1	0	1	1	2	1	4	3	5	4	10	5	14	8	12	12	24	11	30	16

.

Table 7: The number of the basis elements spanning the dilogarithms resp. $\mathsf{w} = 2$ cyclotomic harmonic sums at lth roots of unity up to 20.

At $\mathsf{w} = 2$ the respective new basis elements are :

$$\mathsf{l} = 1 \quad \pi \tag{6.100}$$

$$\mathsf{l} = 2 \quad - \tag{6.101}$$

$$\mathsf{l} = 3 \quad \mathsf{Im}(\mathrm{Li}_2(e_3)) \tag{6.102}$$

$$\mathsf{l} = 4 \quad \mathbf{C} \tag{6.103}$$

$$\mathsf{l} = 5 \quad \mathsf{Im}(\mathrm{Li}_2(e_5)), \mathsf{Im}(\mathrm{Li}_2(e_5^2)) \tag{6.104}$$

$$\mathsf{l} = 6 \quad \mathrm{Li}_4\left(\frac{1}{2}\right) \tag{6.105}$$

$$\mathsf{l} = 7 \quad \mathsf{Im}(\mathrm{Li}_2(e_7^k))\Big|_{k=1}^3, \sigma_{1,1}(e_7, e_7^2) \tag{6.106}$$

$$\mathsf{l} = 8 \quad \mathsf{Im}(\mathrm{Li}_2(e_8)), \sigma_{1,1}(e_8, e_4), \sigma_{1,1}(e_8, e_8^3) \tag{6.107}$$

$$\mathsf{l} = 9 \quad \mathsf{Im}(\mathrm{Li}_2(e_9)), \mathsf{Im}(\mathrm{Li}_2(e_9^2)), \sigma_{1,1}(e_9, e_9^2), \sigma_{1,1}(e_9, e_3), \sigma_{1,1}(e_9^2, e_3) \tag{6.108}$$

$$\mathsf{l} = 10 \quad \sigma_{1,1}(e_5, e_2), \sigma_{1,1}(e_5^2, e_2), \sigma_{1,1}(e_{10}, e_5), \sigma_{1,1}(e_{10}, e_{10}^3) \tag{6.109}$$

$$\mathsf{l} = 11 \quad \mathsf{Im}(\mathrm{Li}_2(e_{11}^k))\Big|_{k=1}^5, \sigma_{1,1}(e_{11}, e_{11}^k)\Big|_{k=2}^4, \sigma_{1,1}(e_{11}^2, e_{11}^k)\Big|_{k=3}^4, \text{ etc.} \tag{6.110}$$

We mention that counting relations for majorants of the motivic numbers, which are claimed to be related to the bases of the sums $\sigma_{k_1,...,k_m}(x_1, ..., x_m), x_j \in \mathcal{C}_n$, were given in [77]. The dimension of the respective $\mathbb{Q}$-vector space is majorized by the expansion coefficients $D_n(\mathsf{w})$ of

$$G_{\mathsf{w}}(x) = \sum_{n=0}^{\infty} D_n(\mathsf{w}) x^n \tag{6.111}$$

with

$$\begin{aligned}
G_{\mathsf{1}}(x) &= \frac{1}{1-x^2-x^3} \\
G_{\mathsf{2}}(x) &= \frac{1}{1-x-x^2} \\
G_{\mathsf{k}}(x) &= \frac{1}{1-\frac{1}{2}\left[\varphi(k)+\nu\right]x-(\nu-1)x^2}, \quad k \geq 3 \ .
\end{aligned} \tag{6.112}$$

Here, $\varphi(k)$ denotes Euler's totient function [24] and ν is the number of prime factors of k.

7 Conclusions

In evaluating massive 3-loop integrals with local operator insertions nested sums occur, containing denominators which belong to the class of harmonic sums generated by cyclotomic polynomials. To simplify the computations, the relations between these quantities have to be know and used in computer algebra codes such as Sigma [17] and HarmonicSums [27] to allow for an efficient reduction in the corresponding summation problem. In integrals of similar structure we expect even more general terms (1.5) to occur.

The usual harmonic sums [7, 8] and polylogarithms [26] were thus generalized to cover the newly occurring structures. We started with the harmonic polylogarithms extending the usual alphabet of denominators ranging to $\Phi_2(x)$ to general cyclotomic polynomials $\Phi_n(x), n \geq 3$. The cyclotomic harmonic polylogarithms form a shuffle algebra. They have support $x \in [0, 1]$ and one may define a Mellin transform, usually of argument kN, with $k, N \in \mathbb{N}_+$ which span the finite cyclotomic harmonic sums (2.1) together with special values as the cyclotomic harmonic sums in the limit $N \to \infty$.

The cyclotomic harmonic sums are meromorphic functions with poles at the non-positive integers. They obey recurrence relations in terms of sums of lower depth and one may derive analytical asymptotic representations. In this way the cyclotomic harmonic sums are analytically continued from integer values of the sum index N to $N \in \mathbb{C}$. The cyclotomic harmonic sums form a quasi-shuffle algebra, cf. [9]. After the analytic continuation they obey differentiation relations, accounting for their values at $N \to \infty$. Furthermore, three multiple argument relations apply. Using these relations one may reduce the number of cyclotomic harmonic sums vastly growing with the weight to smaller bases. We study the case of the sums following from iteration of the denominators (4.60) up to weight $\mathsf{w} = 5$. Corresponding counting relations for the number of basis elements are obtained.

The values of the cyclotomic harmonic sums for $N \to \infty$, resp. the values of the cyclotomic harmonic polylogarithms at $x = 1$, are of special interest. They are linearly related to the infinite nested harmonic sums with roots of unity as numerator weight factors. Already in case of the single infinite cyclotomic harmonic sums a large number of new constants contribute beyond those know in the case of multiple zeta values and Euler-Zagier values [11]. The quasi-shuffle and multiple argument relations of the cyclotomic harmonic sums and the shuffle relations of the cyclotomic harmonic polylogarithms allow to derive basis representations induced by these relations. We studied in this respect the case of the infinite cyclotomic harmonic sums based on the iteration of the summands (4.60) to weight $\mathsf{w} = 6$ and the sums of weight $\mathsf{w} = 1,2$ for cyclotomy $\mathsf{l} \leq 20$. Using computer algebra methods the explicit representation of all infinite cyclotomic harmonic sums were derived as well. For wide classes of relations explicit counting relations of the basis elements were given. The corresponding representations were derived with the Mathematica-based computer algebra system HarmonicSums [27].

The present investigations can be readily extended to cyclotomic harmonic sums and polylogarithms of higher weight and cyclotomy, both in the case of finite values of N and for $N \to \infty$, using the present methods. The requested computational time and storage resources grow accordingly. This applies in particular to the derivation of the explicit representations of all sums over the corresponding bases.

We finally considered also the generalized harmonic sums of weight $\mathsf{w} = 1,2$ at lth roots of unity for $1 \leq l \leq 20$. They obey dilogarithmic reprentations. Besides the shuffle and stuffle relations, they obey distribution relations and the known relations for dilogarithms. We used these relations to derive corresponding basis representations. Compared to the case of the infinite cyclotomic harmonic sums these sums obey more symmetries. Thus at a given weight and cyclotomy l they are represented by a lower number of basis elements. We compared to results in literature.

Acknowledgment. We thank D. Broadhurst for providing us the relations (5.137) and (5.138). For discussions we would like to thank D. Broadhurst, F. Brown and D. Kreimer, and N.J.A. Sloan and M. Kauers for remarks on special issues. We would like to thank H. Kawamura for Ref. [54]. This work has been supported in part by DFG Sonderforschungsbereich Transregio 9, Computergestützte Theoretische Teilchenphysik, by the Austrian Science Fund (FWF) grants P20162-N18 and P203477-N18, and by the EU Network LHCPHENOnet PITN-GA-2010-264564.

A Appendix

In the following we summarize some technical aspects needed to represent expressions used in the present paper.

First we summarize some aspects on cyclotomic polynomials, [23]. We give the decompositions of the polynomials

$$x^l + 1, \quad l \in \mathbb{N}\backslash\{0\} \tag{A.1}$$

in terms of cyclotomic polynomials for $l \leq 20$:

$$\begin{aligned}
x^3 + 1 &= \Phi_2(x)\Phi_6(x) && \text{(A.2)}\\
x^5 + 1 &= \Phi_2(x)\Phi_{10}(x) && \text{(A.3)}\\
x^6 + 1 &= \Phi_4(x)\Phi_{12}(x) && \text{(A.4)}\\
x^7 + 1 &= \Phi_2(x)\Phi_{14}(x) && \text{(A.5)}\\
x^9 + 1 &= \Phi_2(x)\Phi_6(x)\Phi_{18}(x) && \text{(A.6)}\\
x^{10} + 1 &= \Phi_4(x)\Phi_{20}(x) && \text{(A.7)}\\
x^{11} + 1 &= \Phi_2(x)\Phi_{22}(x) && \text{(A.8)}\\
x^{12} + 1 &= \Phi_8(x)\Phi_{24}(x) && \text{(A.9)}\\
x^{13} + 1 &= \Phi_2(x)\Phi_{26}(x) && \text{(A.10)}\\
x^{14} + 1 &= \Phi_4(x)\Phi_{28}(x) && \text{(A.11)}\\
x^{15} + 1 &= \Phi_2(x)\Phi_6(x)\Phi_{10}(x)\Phi_{30}(x) && \text{(A.12)}\\
x^{17} + 1 &= \Phi_2(x)\Phi_{34}(x) && \text{(A.13)}\\
x^{18} + 1 &= \Phi_4(x)\Phi_{12}(x)\Phi_{36}(x) && \text{(A.14)}\\
x^{19} + 1 &= \Phi_2(x)\Phi_{39}(x) && \text{(A.15)}\\
x^{20} + 1 &= \Phi_8(x)\Phi_{40}(x)\ . && \text{(A.16)}
\end{aligned}$$

For odd values of n,

$$\Phi_{2n}(x) = \Phi_n(-x) \tag{A.17}$$

holds. The decomposition

$$x^{2k+1} - 1 = (x-1)\prod_i \Phi_i(x) \tag{A.18}$$

results thus into

$$x^{2k+1} + 1 = \Phi_2(x)\prod_i \Phi_{2i}(x)\ . \tag{A.19}$$

From

$$\Phi_{2n}(x) \mid (x^{2n} - 1) \qquad \text{and} \qquad \Phi_{2n}(x) \nmid (x^n - 1) \tag{A.20}$$

it follows

$$\Phi_{2n}(x) \mid (x^n + 1)\ . \tag{A.21}$$

If p is a prime and $p|n$ then [86]

$$\Phi_{pn}(x) = \Phi_n(x^p) \ . \tag{A.22}$$

For $n = 2^k, k \in \mathbb{N}_+$ it follows that all $\Phi_{2^k}(x)$ are cyclotomic polynomials. In (A.6,A.12,A.14) more factors than just $\Phi_{2n}(x)$ occur. They originate due to power-rescaling, i.e.,

$$\frac{x^{15}+1}{x^3+1} = \frac{y^5+1}{y+1} = \Phi_{10}(y) \tag{A.23}$$

$$\frac{x^{15}+1}{x^5+1} = \frac{y^3+1}{y+1} = \Phi_6(y) \ . \tag{A.24}$$

Therefore, all the factors of (x^5+1) and (x^3+1) have to emerge in the decomposition, and similarly for other N with more non-trivial divisors. For $N = 2^k \cdot n$ where the integer $n > 1$ is odd we get

$$\frac{x^{2^k \cdot n}+1}{x^{2^k}+1} = \frac{y^n+1}{y+1} = \Phi_{2n}(y) \ . \tag{A.25}$$

The argument remains valid if n is a product of odd primes. Therefore the only cyclotomic polynomials of the structure x^a+1 are those with $a = 2^k, k \in \mathbb{N}_+$.

For a proper definition of the cyclotomic harmonic polynomials which appear in sum representations like (2.6)

$$\frac{1}{x^l \pm 1}, \quad l \in \mathbb{N}_+ \tag{A.26}$$

we perform partial fractioning. In the following we provide the corresponding expressions in terms of the words $f_k^l(x)$ forming the cyclotomic harmonic polylogarithms up to $l = 6$:

$$(x-1)^{-1} = f_1^0(x) \tag{A.27}$$

$$(x+1)^{-1} = f_2^0(x) \tag{A.28}$$

$$(x^2-1)^{-1} = \frac{1}{2}\left[f_1^0(x) - f_2^0(x)\right] \tag{A.29}$$

$$(x^2+1)^{-1} = f_4^0(x) \tag{A.30}$$

$$(x^3-1)^{-1} = \frac{1}{3}\left[f_1^0(x) - 2f_3^1(x) - f_3^0(x)\right] \tag{A.31}$$

$$(x^3+1)^{-1} = \frac{1}{3}\left[f_2^0(x) - f_6^1(x) + 2f_6^0(x)\right] \tag{A.32}$$

$$(x^4-1)^{-1} = \frac{1}{4}\left[f_1^0(x) - f_2^0(x) - 2f_4^0(x)\right] \tag{A.33}$$

$$(x^4+1)^{-1} = f_8^0(x) \tag{A.34}$$

$$(x^5-1)^{-1} = \frac{1}{5}\left[f_1^0(x) - \frac{4}{5}f_5^0(x) - \frac{3}{5}f_5^1(x) - \frac{2}{5}f_5^2(x) - \frac{1}{5}f_5^3(x)\right] \tag{A.35}$$

$$(x^5+1)^{-1} = \frac{1}{5}\left[f_2^0(x) + \frac{4}{5}f_5^0(x) - \frac{3}{5}f_5^1(x) + \frac{2}{5}f_5^2(x) - \frac{1}{5}f_5^3(x)\right] \tag{A.36}$$

$$(x^6-1)^{-1} = \frac{1}{6}\left[f_1^0(x) - f_2^0(x) - 2f_3^0(x) - f_3^1(x) - 2f_6^0(x) + f_6^1(x)\right] \tag{A.37}$$

$$(x^6+1)^{-1} = \frac{1}{3}\left[f_4^0(x) + 2f_{12}^0(x) - f_{12}^2(x)\right] \quad \text{etc.} \tag{A.38}$$

B Appendix

Proof of Eq. (4.58).

We proceed by induction on m. Let $m = 1$:

$$\begin{aligned}
S_{\{a_1,b_1,c_1\}}(2n) + S_{\{a_1,b_1,-c_1\}}(2n) &= \sum_{i=1}^{2n} \frac{1}{(a_1 i + b_1)^{c_1}} + \sum_{i=1}^{2n} \frac{(-1)^i}{(a_1 i + b_1)^{c_1}} \\
&= \sum_{i=1}^{n} \left(\frac{1}{(a_1 2i + b_1)^{c_1}} + \frac{1}{(a_1(2i-1) + b_1)^{c_1}} \right) \\
&\quad + \sum_{i=1}^{n} \left(\frac{1}{(a_1 2i + b_1)^{c_1}} - \frac{1}{(a_1(2i-1) + b_1)^{c_1}} \right) \\
&= 2 \sum_{i=1}^{n} \frac{1}{(2a_1 i + b_1)^{c_1}} = 2S_{\{2a_1,b_1,c_1\}}(n).
\end{aligned}$$

In the following we use the abbreviation:

$$A(n) := \sum S_{\{a_m,b_m,\pm c_m\},\ldots,\{a_1,b_1,\pm c_1\}}(n).$$

Now we assume that (4.58) holds for m :

$$\begin{aligned}
&\sum S_{\{a_{m+1},b_{m+1},\pm c_{m+1}\},\ldots,\{a_1,b_1,\pm c_1\}}(2n) \\
&= \sum_{i=1}^{2n} \frac{1}{(a_{m+1} i + b_{m+1})^{c_{m+1}}} \sum S_{\{a_m,b_m,\pm c_m\},\ldots,\{a_1,b_1,\pm c_1\}}(i) \\
&\quad + \sum_{i=1}^{2n} \frac{(-1)^i}{(a_{m+1} i + b_{m+1})^{c_{m+1}}} \sum S_{\{a_m,b_m,\pm c_m\},\ldots,\{a_1,b_1,\pm c_1\}}(i) \\
&= \sum_{i=1}^{2n} \frac{A(i)}{(a_{m+1} i + b_{m+1})^{c_{m+1}}} + \sum_{i=1}^{2n} \frac{(-1)^i A(i)}{(a_{m+1} i + b_{m+1})^{c_{m+1}}} \\
&= \sum_{i=1}^{n} \left(\frac{A(2i)}{(2a_{m+1} i + b_{m+1})^{c_{m+1}}} + \frac{A(2i-1)}{((2i-1)a_{m+1} + b_{m+1})^{c_{m+1}}} \right) \\
&\quad + \sum_{i=1}^{n} \left(\frac{(-1)^{2i} A(2i)}{(2a_{m+1} i + b_{m+1})^{c_{m+1}}} + \frac{(-1)^{2i-1} A(2i-1)}{((2i-1)a_{m+1} + b_{m+1})^{c_{m+1}}} \right) \\
&= 2 \sum_{i=1}^{n} \frac{A(2i)}{(2a_{m+1} i + b_{m+1})^{c_{m+1}}} \\
&= 2 \sum_{i=1}^{n} \frac{1}{(2a_{m+1} i + b_{m+1})^{c_{m+1}}} 2^m S_{\{2a_m,b_m,c_m\},\ldots,\{2a_1,b_1,c_1\}}(i) \\
&= 2^{m+1} S_{\{2a_{m+1},b_{m+1},c_{m+1}\},\ldots,\{2a_1,b_1,c_1\}}(n) \qquad \square
\end{aligned}$$

Proof of Eq. (4.59).

We proceed by induction on m. Let $m = 1$:

$$\begin{aligned}
S_{\{a_1,b_1,c_1\}}(2n) - S_{\{a_1,b_1,-c_1\}}(2n) &= \sum_{i=1}^{2n} \frac{1}{(a_1 i + b_1)^{c_1}} + \sum_{i=1}^{2n} \frac{(-1)^i}{(a_1 i + b_1)^{c_1}} \\
&= \sum_{i=1}^{n} \left(\frac{1}{(a_1 2i + b_1)^{c_1}} + \frac{1}{(a_1(2i-1) + b_1)^{c_1}} \right) \\
&\quad - \sum_{i=1}^{n} \left(\frac{1}{(a_1 2i + b_1)^{c_1}} - \frac{1}{(a_1(2i-1) + b_1)^{c_1}} \right) \\
&= 2 \sum_{i=1}^{n} \frac{1}{((2i-1)a_1 + b_1)^{c_1}} = 2S_{\{2a_1,b_1-a_1,c_1\}}(n).
\end{aligned}$$

In the following we use the abbreviation:

$$A(n) := \sum d_m \cdots d_1 S_{\{a_m,b_m,d_m c_m\},\ldots,\{a_1,b_1,d_1 c_1\}}(n).$$

Note that

$$\begin{aligned}
A(2n-1) &= A(2n) - \sum d_m \cdots d_1 \frac{d_m^{2n} S_{\{a_{m-1},b_{m-1},d_{m-1}c_{m-1}\},\ldots,\{a_1,b_1,d_1 c_1\}}(2n)}{(a_m 2n + b_m)^{c_m}} \\
&= A(2n) - \sum d_{m-1} \cdots d_1 \frac{S_{\{a_{m-1},b_{m-1},d_{m-1}c_{m-1}\},\ldots,\{a_1,b_1,d_1 c_1\}}(2n)}{(a_m 2n + b_m)^{c_m}} \\
&\quad + \sum d_{m-1} \cdots d_1 \frac{S_{\{a_{m-1},b_{m-1},d_{m-1}c_{m-1}\},\ldots,\{a_1,b_1,d_1 c_1\}}(2n)}{(a_m 2n + b_m)^{c_m}} = A(2n).
\end{aligned}$$

Now we assume that (4.59) holds for m :

$$\begin{aligned}
&\sum d_{m+1} \cdots d_1 S_{\{a_{m+1},b_{m+1},d_{m+1}c_{m+1}\},\ldots,\{a_1,b_1,d_1 c_1\}}(2n) \\
&= \sum_{i=1}^{2n} \frac{1}{(a_{m+1} i + b_{m+1})^{c_{m+1}}} \sum d_m \cdots d_1 S_{\{a_m,b_m,d_m c_m\},\ldots,\{a_1,b_1,d_1 c_1\}}(i) \\
&\quad - \sum_{i=1}^{2n} \frac{(-1)^i}{(a_{m+1} i + b_{m+1})^{c_{m+1}}} \sum d_m \cdots d_1 S_{\{a_m,b_m,d_m c_m\},\ldots,\{a_1,b_1,d_1 c_1\}}(i) \\
&= \sum_{i=1}^{2n} \frac{A(i)}{(a_{m+1} i + b_{m+1})^{c_{m+1}}} - \sum_{i=1}^{2n} \frac{(-1)^i A(i)}{(a_{m+1} i + b_{m+1})^{c_{m+1}}} \\
&= \sum_{i=1}^{n} \left(\frac{A(2i)}{(2a_{m+1} i + b_{m+1})^{c_{m+1}}} + \frac{A(2i-1)}{((2i-1)a_{m+1} + b_{m+1})^{c_{m+1}}} \right) \\
&\quad - \sum_{i=1}^{n} \left(\frac{(-1)^{2i} A(2i)}{(2a_{m+1} i + b_{m+1})^{c_{m+1}}} + \frac{(-1)^{2i-1} A(2i-1)}{((2i-1)a_{m+1} + b_{m+1})^{c_{m+1}}} \right) \\
&= 2 \sum_{i=1}^{n} \frac{A(2i-1)}{((2i-1)a_{m+1} + b_{m+1})^{c_{m+1}}} = 2 \sum_{i=1}^{n} \frac{A(2i)}{((2i-1)a_{m+1} + b_{m+1})^{c_{m+1}}} \\
&= 2 \sum_{i=1}^{n} \frac{1}{(2a_{m+1} i + b_{m+1} - a_{m+1})^{c_{m+1}}} 2^m S_{\{2a_m,b_m-a_m,c_m\},\ldots,\{2a_1,b_1-a_1,c_1\}}(n)
\end{aligned}$$

$$= 2^{m+1} S_{\{2a_{m+1}, b_{m+1}-a_{m+1}, c_{m+1}\},\ldots,\{2a_1, b_1-a_1, c_1\}}(n) \square$$

Proof of Eq. (4.82–4.86).

We start with $C_{f_0^0,\vec{m}}(x)$. We consider integrals of the form

$$\begin{aligned}
\int_0^x \frac{1}{y}\sum_{i=1}^{\infty}\frac{\sigma^i y^{2i+c_j}}{(2i+c_j)^a} S_{\boldsymbol{n}}(i)dy &= \sum_{i=1}^{\infty}\frac{\sigma^i}{(2i+c_j)^a} S_{\boldsymbol{n}}(i)\int_0^x y^{2i+c_j-1}dy \\
&= \sum_{i=1}^{\infty}\frac{\sigma^i}{(2i+c_j)^a} S_{\boldsymbol{n}}(i)\frac{x^{2i+c_j}}{2i+c_j}dy \\
&= \sum_{i=1}^{\infty}\frac{\sigma^i x^{2i+c_j}}{(2i+c_j)^{a+1}} S_{\boldsymbol{n}}(i) \ .
\end{aligned}$$

Summing over j yields the desired result.

For $C_{f_4^0,\vec{m}}(x)$ we consider the integrals

$$\begin{aligned}
\int_0^x \frac{1}{1+y^2}\sum_{i=1}^{\infty}\frac{\sigma^i y^{2i+c_j}}{(2i+c_j)^a} S_{\boldsymbol{n}}(i)dy &= \int_0^x \sum_{i=1}^{\infty}(-1)^i y^{2i}\sum_{i=0}^{\infty}\frac{\sigma^{i+1} y^{2i+c_j+2}}{(2i+c_j+2)^a} S_{\boldsymbol{n}}(i+1)dy \\
&= \int_0^x \sum_{i=1}^{\infty}\sum_{k=0}^{i}(-1)^{i-k} y^{2i-2k}\frac{\sigma^{k+1} y^{2k+c_j+2}}{(2k+c_j+2)^a} S_{\boldsymbol{n}}(k+1)dy \\
&= \int_0^x \sum_{i=1}^{\infty}(-1)^{i+1} y^{2i+c_j+2}\sum_{k=0}^{i}\frac{(-\sigma)^{k+1}}{(2k+c_j+2)^a} S_{\boldsymbol{n}}(k+1)dy \\
&= \sum_{i=1}^{\infty}\frac{(-1)^{i+1} y^{2i+c_j+3}}{2i+c_j+3}\sum_{k=1}^{i+1}\frac{(-\sigma)^k}{(2k+c_j)^a} S_{\boldsymbol{n}}(k) \\
&= \sum_{i=1}^{\infty}\frac{(-1)^i x^{2i+c_j+1}}{(2i+c_j+1)} S_{\{2,c_j,-\sigma a\},\boldsymbol{n}}(i) \ .
\end{aligned}$$

Summing over j yields the desired result. The case $C_{f_4^1,\vec{m}}(x)$ follows analogously.

For $C_{f_2^0,\vec{m}}(x)$ we consider the integrals of the form

$$\begin{aligned}
\int_0^x \frac{1}{1+y}\sum_{i=1}^{\infty}\frac{\sigma^i y^{2i+c_j}}{(2i+c_j)^a} S_{\boldsymbol{n}}(i)dy &= \int_0^x \frac{1-y}{1-y^2}\sum_{i=1}^{\infty}\frac{\sigma^i y^{2i+c_j}}{(2i+c_j)^a} S_{\boldsymbol{n}}(i)dy \\
&= \int_0^x \frac{1}{1-y^2}\sum_{i=1}^{\infty}\frac{\sigma^i y^{2i+c_j}}{(2i+c_j)^a} S_{\boldsymbol{n}}(i)dy \\
&\quad -\int_0^x \frac{y}{1-y^2}\sum_{i=1}^{\infty}\frac{\sigma^i y^{2i+c_j}}{(2i+c_j)^a} S_{\boldsymbol{n}}(i)dy \\
&= \int_0^x \sum_{i=0}^{\infty} y^{2i}\sum_{i=0}^{\infty}\frac{\sigma^{i+1} y^{2i+c_j+2}}{(2i+c_j+2)^a} S_{\boldsymbol{n}}(i+1)dy
\end{aligned}$$

$$
\begin{aligned}
& -\int_0^x \sum_{i=0}^{\infty} y^{2i+1} \sum_{i=0}^{\infty} \frac{\sigma^{i+1} y^{2i+c_j+2}}{(2i+c_j+2)^a} S_{\boldsymbol{n}}\,(i+1) dy \\
= & \int_0^x \sum_{i=0}^{\infty} \sum_{k=0}^{i} y^{2i-2k} \frac{\sigma^{k+1} y^{2k+c_j+2}}{(2k+c_j+2)^a} S_{\boldsymbol{n}}\,(k+1) dy \\
& -\int_0^x \sum_{i=0}^{\infty} \sum_{k=0}^{i} y^{2i-2k+1} \frac{\sigma^{k+1} y^{2k+c_j+2}}{(2k+c_j+2)^a} S_{\boldsymbol{n}}\,(k+1) dy \\
= & \int_0^x \sum_{i=0}^{\infty} y^{2i+c_j+1} S_{\{2,c_j,\sigma a\},\boldsymbol{n}}\,(i+1)\, dy \\
& -\int_0^x \sum_{i=0}^{\infty} y^{2i+c_j+3} S_{\{2,c_j,\sigma a\},\boldsymbol{n}}\,(i+1)\, dy \\
= & \sum_{i=1}^{\infty} \frac{x^{2i+c_j+1}}{(2i+c_j+1)} S_{\{2,c_j,\sigma a\},\boldsymbol{n}}\,(i) \\
& -\sum_{i=1}^{\infty} \frac{x^{2i+c_j+2}}{(2i+c_j+2)} S_{\{2,c_j,\sigma a\},\boldsymbol{n}}\,(i) \quad .
\end{aligned}
$$

Summing over j yields the desired result. The case $C_{f_1^0,,\vec{m}}(x)$ follows analogously.

References

[1] G. 't Hooft, M. J. G. Veltman, Nucl. Phys. **B44** (1972) 189.

[2] G. 't Hooft, M. J. G. Veltman, Nucl. Phys. **B153** (1979) 365; Diagrammar, NATO Adv. Study Inst. Ser. B Phys. **4** (1974) 177;
G. Passarino, M. J. G. Veltman, Nucl. Phys. **B160** (1979) 151.

[3] P. Appell, Sur Les Fonctions Hypérgéometriques de Plusieurs Variables, (Gauthier-Villars, Paris, 1925);
P. Appell and J. Kampé de Fériet, Fonctions Hypérgéometriques; Polynômes d'Hermite, (Gauthier-Villars, Paris, 1926);
W.N. Bailey, Generalized Hypergeometric Series, (Cambridge University Press, Cambridge, 1935);
A. Erdélyi (ed.), Higher Transcendental Functions, Bateman Manuscript Project, Vol. **I**, (McGraw-Hill, New York, 1953);
L.J. Slater, Generalized Hypergeometric Functions, (Cambridge University Press, Cambridge, 1966);
H. Exton, Multiple Hypergeometric Functions and Applications, (Ellis Horwood Limited, Chichester, 1976);
H. Exton, Handbook of Hypergeometric Integrals, (Ellis Horwood Limited, Chichester, 1978).

[4] C. G. Bollini, J. J. Giambiagi, Nuovo Cim. **B12** (1972) 20;
J. F. Ashmore, Lett. Nuovo Cim. **4** (1972) 289;
G. M. Cicuta, E. Montaldi, Lett. Nuovo Cim. **4** (1972) 329.

[5] A.M. Legendre, Mémoires de l'Inst. de France **10** (1809), 416;
S.D. Poisson, Mémoires de l'Inst. de France (1811) pp. 57, 257;
C.F. Gauss, Comment. Gotting. Bd. **2** pp. 34.

[6] J. Blümlein, Comput. Phys. Commun. **180** (2009) 2218 [arXiv:0901.3106 [hep-ph]];
J. Blümlein, S. Klein, C. Schneider and F. Stan, arXiv:1011.2656 [cs.SC].

[7] J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018 [arXiv:hep-ph/9810241].

[8] J. A. M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037 [arXiv:hep-ph/9806280].

[9] M. E. Hoffman, J. Algebra, **194** (1997) 477; Nucl. Phys. (Proc. Suppl.) **135** (2004) 215 [arXiv:math/0406589].

[10] L. Euler, Novi Comm. Acad. Sci. Petropol. **20** (1775) 140, reprinted in Opera Omnia ser I vol. 15, (B.G. Teubner, Berlin, 1927), 217;
D. Zagier, in : First European Congress of Mathematics, Vol. II, (Paris, 1992), Progr. Math., **120**, (Birkhäuser, Basel–Boston, 1994), pp. 497.

[11] D. J. Broadhurst, [hep-th/9604128];
D. J. Broadhurst, D. Kreimer, Phys. Lett. **B393** (1997) 403-412. [hep-th/9609128];
J. M. Borwein, D. M. Bradley, D. J. Broadhurst and P. Lisonek, Trans. Am. Math. Soc. **353** (2001) 907 [arXiv:math/9910045];
J. Blümlein, D. J. Broadhurst and J. A. M. Vermaseren, Comput. Phys. Commun. **181** (2010) 582 [arXiv:0907.2557 [math-ph]] and references therein.

[12] J. A. M. Vermaseren, A. Vogt and S. Moch, Nucl. Phys. B **724** (2005) 3 [arXiv:hep-ph/0504242].

[13] S. Moch, P. Uwer, S. Weinzierl, J. Math. Phys. **43** (2002) 3363-3386. [hep-ph/0110083].

[14] J. Ablinger, J. Blümlein, and C. Schneider, in preparation.

[15] J. Ablinger, I. Bierenbaum, J. Blümlein, A. Hasselhuhn, S. Klein, C. Schneider and F. Wissbrock, Nucl. Phys. **B** (Proc. Suppl.) **205–206** (2010) 242, [arXiv:1007.0375 [hep-ph]].

[16] J. Ablinger, J. Blümlein, S. Klein, C. Schneider and F. Wissbrock, Nucl. Phys. B **844** (2011) 26 [arXiv:1008.3347 [hep-ph]].

[17] C. Schneider, J. Symbolic Comput. 43 (2008) 611, [arXiv:0808.2543v1]; Ann. Comb. 9 (2005) 75; J. Differ. Equations Appl. 11 (2005) 799; Ann. Comb. **14**(4) (2010) 533 [arXiv:0808.2596]; Proceedings of the Conference on Motives, Quantum Field Theory, and Pseudodifferential Operators, held at the Clay Mathematics Institute, Boston University, June 2–13, 2008, Clay Mathematics Proceedings **12** (2010) 285, Eds. A. Carey, D. Ellwood, S. Paycha, S. Rosenberg; Sém. Lothar. Combin. 56 (2007) 1, Article B56b, Habilitationsschrift JKU Linz (2007) and references therein;
J. Ablinger, J. Blümlein, S. Klein and C. Schneider, Nucl. Phys. (Proc. Suppl.) **B** (2010), 110, [arXiv:1006.4797 [math-ph]].

[18] See e.g. J. Zhao, arXiv:0707.1459 [mathNT]; C. R. Acad. Sci. Paris Ser. I 346 (2008), 1029 [arXiv:0810.1064[mathNT]]; Documenta Math. 15 (2010) 1; J. Algebra Appl. 9 (2010), 327 [arXiv:0904.0888[mathNT]];
cf. also [29], footnote 8.

[19] A.B. Goncharov, Mathematical Research Letters **5** (1998) 497.

[20] D. J. Broadhurst, Eur. Phys. J. C **8** (1999) 311 [arXiv:hep-th/9803091].

[21] M. Bigotte, G. Jacob, N.E. Oussous, and M. Petitot, Theor. Comp. Sci **273** (2002) 271.

[22] S. Weinzierl, J. Math. Phys. **45** (2004) 2656-2673. [hep-ph/0402131];
M. Y. Kalmykov, B. F. L. Ward, S. Yost, JHEP **0702** (2007) 040. [hep-th/0612240];
M. Y. Kalmykov, JHEP **0604** (2006) 056. [hep-th/0602028];
M. Y. Kalmykov, B. F. L. Ward, S. A. Yost, JHEP **0710** (2007) 048. [arXiv:0707.3654 [hep-th]];
T. Huber, D. Maitre, Comput. Phys. Commun. **178** (2008) 755-776. [arXiv:0708.2443 [hep-ph]];
M. Y. Kalmykov, B. A. Kniehl, Nucl. Phys. **B809** (2009) 365-405. [arXiv:0807.0567 [hep-th]].

[23] S. Lang, Algebra, (Springer, New York, 2002), 3rd edition.

[24] L. Euler, Novi Commentarii academiae scientiarum imperialis Petropolitanae **8** (1760/1, 1763), 74;
M. Takase, Euler's Theory of Numbers, in: Euler reconsidered, R. Baker, ed., (Kedrick Press, Heber City, UT, 2007), p. 377; `leonhardeuler.web.fc2.com/eulernumber_en.pdf`

[25] H. Poincaré, Acta Math. **4** (1884) 201;
J.A. Lappo-Danilevsky, Mémoirs sur la Théorie des Systèmes Différentielles Linéaires, (Chelsea Publ. Co, New York, NY, 1953);
K.T. Chen, Trans. A.M.S. **156** (3) (1071) 359.

[26] E. Remiddi and J. A. M. Vermaseren, Int. J. Mod. Phys. A **15** (2000) 725 [arXiv:hep-ph/9905237].

[27] J. Ablinger, code HarmonicSums, in preparation.

[28] J. Blümlein, Comput. Phys. Commun. **159** (2004) 19 [arXiv:hep-ph/0311046].

[29] J. Blümlein, Comput. Phys. Commun. **180** (2009) 2218 [arXiv:0901.3106 [hep-ph]].

[30] J. Blümlein, Structural Relations of Harmonic Sums and Mellin Transforms at Weight w=6, Proceedings of the Conference on Motives, Quantum Field Theory, and Pseudodifferential Operators, held at the Clay Mathematics Institute, Boston University, June 2–13, 2008, Clay Mathematics Proceedings **12** (2010) 167, Eds. A. Carey, D. Ellwood, S. Paycha, S. Rosenberg, [arXiv:0901.0837 [math-ph]].

[31] J. Ablinger, J. Blümlein, and C. Schneider, in preparation.

[32] J. Blümlein, Comput. Phys. Commun. **133** (2000) 76 [arXiv:hep-ph/0003100];
S. I. Alekhin and J. Blümlein, Phys. Lett. B **594** (2004) 299 [arXiv:hep-ph/0404034];
J. Blümlein and S. O. Moch, Phys. Lett. B **614** (2005) 53 [arXiv:hep-ph/0503188];
A. V. Kotikov, V. N. Velizhanin, [hep-ph/0501274];
S. Albino, Phys. Lett. **B674** (2009) 41 [arXiv:0902.2148 [hep-ph]].

[33] T. Gehrmann and E. Remiddi, Comput. Phys. Commun. **141** (2001) 296 [arXiv:hep-ph/0107173].

[34] J. Vollinga and S. Weinzierl, Comput. Phys. Commun. **167** (2005) 177 [arXiv:hep-ph/0410259].

[35] T. Gehrmann and E. Remiddi, Comput. Phys. Commun. **144** (2002) 200 [arXiv:hep-ph/0111255]. R. Bonciani, G. Degrassi and A. Vicini, arXiv:1007.1891 [hep-ph].

[36] N. Nielsen, Handbuch der Theorie der Gammafunktion, (Teubner, Leipzig, 1906); reprinted by (Chelsea Publishing Company, Bronx, New York, 1965).

[37] J. Stirling, METHODUS DIFFERENTIALIS SIVE TRACTATUS DE SUMMATIONE ET INTERPOLATIONE SERIERUM INFINITARUM, (London, 1730), p. 27.

[38] K. G. Chetyrkin, M. Steinhauser, Nucl. Phys. **B573** (2000) 617-651. [hep-ph/9911434].

[39] A. I. Davydychev, M. Y. Kalmykov, Nucl. Phys. **B605** (2001) 266 [hep-th/0012189].

[40] L. G. Almeida, C. Sturm, Phys. Rev. **D82** (2010) 054017 [arXiv:1004.4613 [hep-ph]];
J. A. Gracey, arXiv:1104.5382 [hep-ph].

[41] C. Reutenauer, Free Lie Algebras, (Calendron press, Oxford, 1993).

[42] E. Witt, Journ. Reine und Angew. Mathematik, **177** (1937) 152; Math. Zeitschr. **64** (1956) 195.

[43] A.F. Möbius, J. reine angew. Math. **9** (1832) 105;
G.W. Hardy and E.M. Wright, An Introduction to the Theory of Numbers, 5th Edition, (Calendron Press, Oxford, 1979).

[44] E. Catalan, Mémoires de l'Academie imperiale des sciences de Saint-Pétersbourg, Ser. **7** (1883) 31;
V. Adamchik, `http://www-2.cs.cmu.edu/~adamchik/articles/catalan/catalan.htm`

[45] A. Migotti, Sitzber. Math.-Naturwiss. Classe der Kaiser. Akad. der Wiss. Wien **87** (1883) 7;
P. Erdös, Bull. Amer. Math. Soc. **52** (1946) 179;
M. Endo, Comment. Math. Univ. St. Pauli **23** (1974/75) 121;
R. Thangadurai, On the coefficients of cyclotomic polynomials, in : Cyclotomic Fields and Related Topics, Pune, 1999, (Bhaskaracharya, Pune, 2000), 311.

[46] E. Landau, Über die Grundlagen der Theorie der Fakultätenreihen, S.-Ber. math.-naturw. Kl. Bayerische Akad. Wiss. München, **36** (1906) 151.

[47] C.F. Gauss, Comment. Gotting. **2** (1812) 33;
J.L.W.V. Jensen, Nyt Tidsskrift for Mathematik, **2B** (1891) 33.

[48] C.F. Gauss, Discquisitiones Arithmeticae, (Leipzig, Gerh. Fleischer, 1801), art. 365, 366;
É. Galois, Œuvre mathématiques publiées en 1846 dans le Journal de Liouville.

[49] `http://mathworld.wolfram.com/TrigonometryAnglesPi5.html` and similar pages.

[50] Ch. Hermite, Comptes Rendus Acad. Sci. Paris, **77** (1873) 18;
F. Lindemann, Math. Ann. **20** (1882) 213;
K. Weierstrass, Sitzungber. Königl. Preuss. Akad. Wissensch. zu Berlin, **2** (1885) 1067;
A. Baker, Transcendental Number Theory, (Cambridge University Press, Cambridge, 1975), and references therein.

[51] H.M. Srivastave and Junesang Choi, Series Associated with the Zeta and Related Functions, (Kluwer Academic Publishers, Dordrecht, 2001).

[52] A. Hurwitz, Zeitschr. f. Mathematik u. Physik **27** (1882) 86;
T.M. Apostol, Introduction to Analytic Number Theory, (Springer, Berlin, 1976), Chap. 12.

[53] M.W. Coffey, J. Math. Phys. **49** (2008) 043510;
D. Cvijovic, J. Math. Phys. **50** (2009) 023515.

[54] Seki Takakazu, Katsuyo Sampo (Syotsudo, Edo, 1712);
関 孝和 , 括要算法 , (書津堂 , 江 戸 ,1712)

[55] J. Bernoulli, Ars conjectandi, opus posthumum. Accedit Tractatus de seriebus infinitis, et epistola gallicé scripta de ludo pilae reticularis, (Brüder Thurneysen, Basel, 1713).

[56] J. Miller and V.S. Adamchik, J. Comp. and Applied Math., **100** (1998) 201.

[57] K. S. Kölbig, J. Comput. Appl. Math. **75** (1996) 43.

[58] L. Saalschütz, Vorlesungen über die Bernoullischen Zahlen, (Springer, Berlin, 1893).

[59] N. Nielsen, Traité élémentaire des nombres de Bernoulli, (Gauthier-Villars, Paris, 1923).

[60] L. Euler, Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum **II** (Typographeo Petro Galeatii, Ticini, 1787), cap. VIII, §§224, 226; first issue : (Academiae Imperialis Scientiarum, St. Petersburg, 1755).

[61] H.F. Scherk, Mathematische Abhandlungen, Erste Abhandlung - Betrifft die Bernoullischen und die Eulerschen Zahlen, (Berlin, 1825).

[62] M. Lerch, L'Enseignement Mathématique **5** (1903) 450;
A. Laurinčikas and R. Garunkštis, The Lerch Zeta-function, (Kluwer Academic Publishers, Dordrecht, 2010) and references therein.

[63] W. Spence, An Essay on Logarithmic Transcendents, (John Murray, London and Archibald Constable and Company, Edinburgh, 1809).

[64] L. Lewin, Dilogarithms and Associated Functions (Macdonald, London, 1958);
Polylogarithms and Associated Functions (North Holland, New York, 1981);
D. Zagier, in : Frontiers in Number Theory, Physics, and Geometry II: On Conformal Field Theories, Discrete Groups and Renormalization, Vol. 2, P. Cartier et al. (eds.), (Springer, Berlin, 2006), pp. 3.

[65] R. Clausen, J. reine angew. Math. **8** (1832) 298.

[66] D. Cvijović and J. Klinkowski, Mathematics of Computation, **68** (1999) 1623;
D. Cvijović, J. Math. Anal. Appl. **332** (2007) 1056.

[67] A.M. Legendre, Exercices de calcul intégral, tome **1** (1811) 247.

[68] see [60], p. 499 (edition of 1755) and [59].

[69] B.C. Berndt, Ramanujan's Notebooks, Part I, (Springer, Berlin, 1985).

[70] R. Sitaramachandrarao, J. Number Theory. **25** (1987) 1.

[71] N. Nielsen, Der Eulersche Dilogarithmus und seine Verallgemeinerungen, Nova Acta Leopoldina, Vol. XC, Nr. 3, Halle, 1909;
K.S. Kölbig, SIAM J. Math. Anal. **16** (1986) 1232.

[72] P.F. Jordan, Bull. Am. Math. Soc. **79** (1973) 681.

[73] M.S. Milgram, arXiv:math/0406338v2.

[74] O. M. Ogreid, P. Osland, J. Comput. Appl. Math. **98** (1998) 245, [hep-th/9801168];
M. W. Coffey, J. Comput. Appl. Math. **183** (2005) 84, [math-ph/0505051].

[75] B. J. Laurenzi, arXiv:1010.6229 [math-ph].

[76] P. Deligne, Publications Mathématiques de l'Institut des Hautes Études Scientifiques, (2010) 101.http://www.math.ias.edu/people/faculty/deligne/preprints.

[77] P. Deligne and A.B. Gocharov, Groupes fondamentaux motivique de Tate mixte, IAS preprint, http://www.math.ias.edu/people/faculty/deligne/preprints

[78] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, http://oeis.org/

[79] H.R.P. Ferguson and D.H. Bailey, RNR Techn. Rept. RNR-91-032, Jul. 14, 1992.

[80] F. Vieta, Opera mathematica, 1579; reprinted (Officina Bonaventurae & Abrahami Elzeviriorum, Leiden, 1646).

[81] H. Lugowski, J. Weinert, Grundzüge der Algebra, Vol. III, (Teubner, Leipzig, 1960).

[82] G. Racinet, Comptes rendus de l'Académie des sciences. Série 1, Mathématique, **333** (2001) no 1, 5, [arXiv:math.QA/0012024].

[83] F.W. Newman, The Cambridge and Dublin Mathematical Journal, **II** (1847) 77; 172; 293.

[84] A.N. Kirillov, Progr. Theor. Phys. Suppl. **118** (1995) 61 [hep-th/9408113].

[85] M. Y. Kalmykov, B. A. Kniehl, Nucl. Phys. Proc. Suppl. **205-206** (2010) 129 [arXiv:1007.2373 [math-ph]].

[86] T. Nagell, Introduction to Number Theory, (Wiley, New York, 1951).