Using Theorema in the Formalization of Theoretical Economics

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Overview

Motivation:

- Proofs in economics use typically undergraduate level proofs
- Proofs in economics are error prone (just as in other theoretical fields)
- Formalization should be achievable
- Automation (or minimization of user interactions) as goal

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Outline

- Basic Theory
- Pseudo Algorithm
- Examples
- Two Lemmas and Theorema
- Summary

Power Function

$\mathcal{X} \equiv \{\{x_i\}_{i \in I} | x_i \geq 0, \sum_{i \in I} x_i = 1\}$., the following axioms can be defined. A power function π satisfies

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WR	if $y_i \geq x_i \forall i \in C \subseteq I$ then $\pi(C, \boldsymbol{y}) \geq \pi(C, \boldsymbol{x})$; and
SR	if $\emptyset \neq C \subseteq I$ and $y_i > x_i \forall i \in C$ then $\pi(C, \boldsymbol{y}) > \pi(C, \boldsymbol{x})$.

Properties

Other important properties that power functions may have:

- **AN** if $\sigma : I \to I$ is a 1:1 onto function permuting the agent set, $i \in C \Leftrightarrow \sigma(i) \in C'$, and $x_i = x'_{\sigma(i)}$ then $\pi(C, \boldsymbol{x}) = \pi(C', \boldsymbol{x}')$.
- **CX** $\pi(C, \boldsymbol{x})$ is continuous in $\boldsymbol{x}$.
- **RE** if $i \notin C$ and $\pi(\{i\}, \boldsymbol{x}) > 0$ then $\pi(C \cup \{i\}, \boldsymbol{x}) > \pi(C, \boldsymbol{x})$.

Domination

Def$_{\succeq}$ An allocation $\boldsymbol{y}$ dominates an allocation $\boldsymbol{x}$, written $\boldsymbol{y} \succeq \boldsymbol{x}$, iff $\pi(W, \boldsymbol{x}) > \pi(L, \boldsymbol{x})$; where $W \equiv \{i | y_i > x_i\}$ and $L \equiv \{i | x_i > y_i\}$. W = win set & L lose set.

Def$_D$ For $\mathcal{Y} \subset \mathcal{X}$, let $D(\mathcal{Y}) \equiv \{\boldsymbol{x} \in \mathcal{X} | \exists \boldsymbol{y} \in \mathcal{Y} \text{ s.t. } \boldsymbol{y} \succeq \boldsymbol{x}\}$ be the dominion of $\mathcal{Y}$. $U(\mathcal{Y}) = \mathcal{X} \backslash D(\mathcal{Y})$, the set of allocations undominated by any allocation in $\mathcal{Y}$.

Core and stable set

Def$_{\mathcal{K}}$ The core, $\mathcal{K}$, is the set of undominated allocations, $U(\mathcal{X})$.

Def$_{\mathcal{S}}$ A set of allocations, $\mathcal{S} \subseteq \mathcal{X}$, is a stable set iff it satisfies

internal stability,	$\mathcal{S} \cap D(\mathcal{S}) = \emptyset$	(IS)
external stability,	$\mathcal{S} \cup D(\mathcal{S}) = \mathcal{X}$	(ES)

The conditions combine to yield $\mathcal{S} = \mathcal{X} \backslash D(\mathcal{S})$. The core necessarily belongs to any existing stable set.

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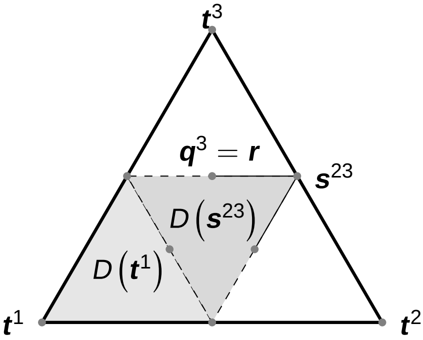

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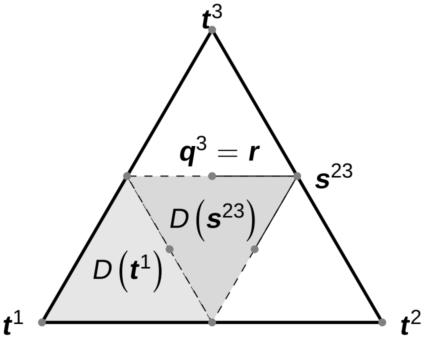
Stable Set: $S =$

$$\left\{ \begin{array}{l} (0,0,1),(0,1,0),(1,0,0), \\ (0,\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},0,\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2},0), \\ (\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{2}),(\tfrac{1}{4},\tfrac{1}{2},\tfrac{1}{4}),(\tfrac{1}{2},\tfrac{1}{4},\tfrac{1}{4}), \end{array} \right\}$$



The stable set in $n = 3$ with AN, CX, and RE

if $\pi\left(\{i\}, \boldsymbol{t}^i\right) \geq \pi\left(\{j,k\}, \boldsymbol{t}^i\right)$ **then**
  $\mathcal{S}_0 = \mathcal{D}_0$
  if $M^i = \emptyset$ **then**
    return "no stable set exists"
  else
    $\mathcal{S}_1 = U^2(\mathcal{S}_0) = \mathcal{S}_0 \cup \bigcup_{i=1}^{3} \mathcal{S}^i$
    if $\mathcal{S}_1 \cup D(\mathcal{S}_1) \neq \mathcal{X}$ **then**
      $\mathcal{S} = \mathcal{S}_2 = U^2(\mathcal{S}_1) = \mathcal{S}_1 \cup \mathcal{P}$
    else
      $\mathcal{S} = \mathcal{S}_1$
    end if
  end if
else
  $\mathcal{S} = \mathcal{D}_1 \backslash \mathcal{D}_0$
end if
return $\mathcal{S}$

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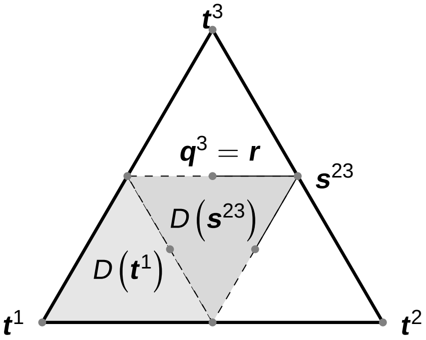

Strength In Numbers with $\nu > 1$

$$\mathrm{SIN}\pi_{\nu}[C, x] := \sum_{i \in C} (x_i + \nu)$$

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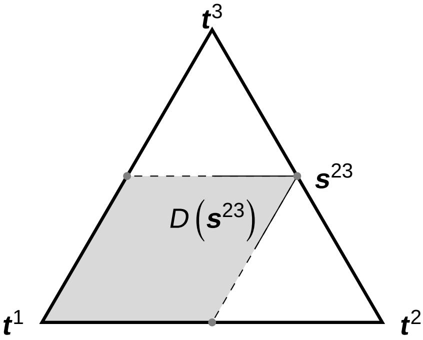

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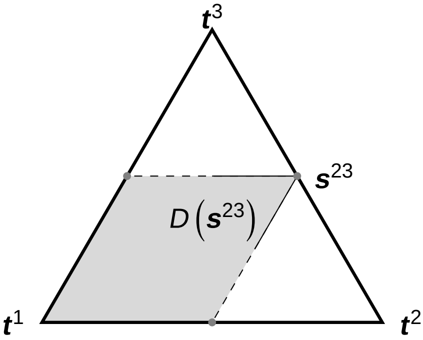

Strength In Numbers with $0 < \nu < 1$

$$\mathrm{SIN}\pi_\nu[C, x] := \sum_{i \in C} (x_i + \nu)$$

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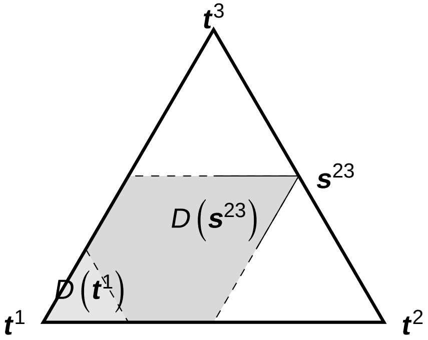
no stable set exists



Proof of a Lemma

(One Lemma of 14 lemmas, 12 theorems, and 4 corollaries)

Lemma

When $n = 3$: *1.* $\mathcal{K} = \emptyset$ *implies* $\mathbf{t}^i \in D\left(\mathbf{s}^{jk}\right)$ *for distinct* $i, j, k \in I$*.*

Proof.

1. As $\mathcal{K} = \emptyset$, no agent can defend its holdings against both others, so that $\pi\left(\{i\}, \mathbf{t}^i\right) < \pi\left(\{j, k\}, \mathbf{t}^i\right)$ for distinct i, j and k. As $\{j, k\}$ prefers $\mathbf{s}^{jk}$ to $\mathbf{t}^i$, this ensures that $\mathbf{s}^{jk} \succ \mathbf{t}^i$.

□

Summary (Part I)

The pseudo algorithm:

- Non-computational in several aspects
- Evaluation by a mixture of reasoning and computing. Can compute the stable set of WIP, SIN, assumed the corresponding lemmas are available.
- Plan: Extend the computational part, e.g., represent infinite set in a finite way. Use underlying Mathematica to compute solutions of equations.

Summary (Part II)

- Axiomatic approach in theoretical economics valuable (eliminate errors, even without full proof)
- Good field with non-trivial but not very deep mathematics.
- Formalisation in Theorema is easy and fast even for beginners.
- Automation at least partially possible. Reasoning requires more expert knowledge and work.
- Theorema offers mixture of reasoning and computation. Very useful for determining stable sets.