GENOM3CK - A library for **GEN**us c**OM**putation of plane **C**omplex algebrai**C** **C**urves using **K**not theory

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Motivation

Why a library for genus computation of plane complex algebraic curves using knot theory (GENOM3CK)?

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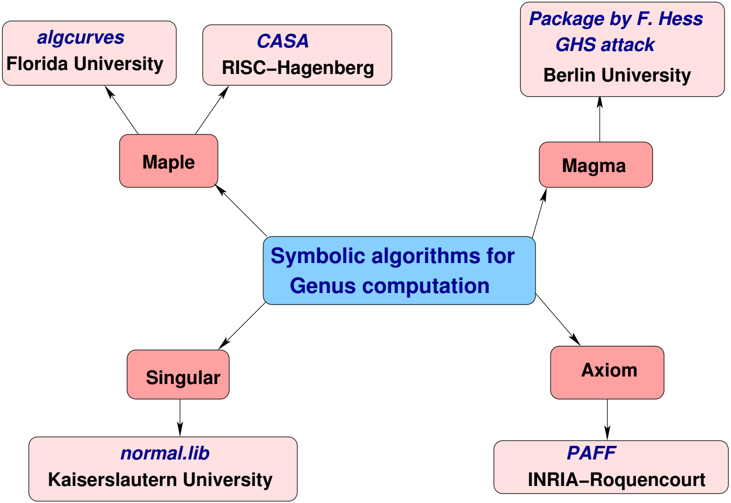
Motivation

At present, there exists several...



Motivation

But...

For moderate symbolic input data: they are too expensive

Disadvantages of Symbolic algorithms for Genus computation

For numeric input data: they are unusable

Motivation

For instance, in Maple using algcurves package...

> $with(algcurves);$

$[AbelMap, Siegel, Weierstrassform, algfun\_series\_sol, differentials, genus, homogeneous, homology, implicitize, integral\_basis, is\_hyperelliptic, j\_invariant, monodromy, parametrization, periodmatrix, plot\_knot, plot\_real\_curve, puiseux, singularities]$

> $f := x^2\, y + y^4$

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> $genus(f, x, y)$

$$-1$$

> $g := 1.02 \cdot x^2\, y + 1.12 \cdot y^4$

$$g := 1.02\, x^2\, y + 1.12\, y^4$$

> $genus(g, x, y)$

```
Error, (in content/polynom) general case of floats not handled
```

>

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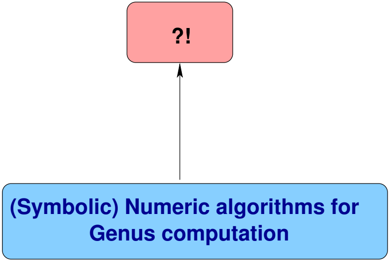
Motivation

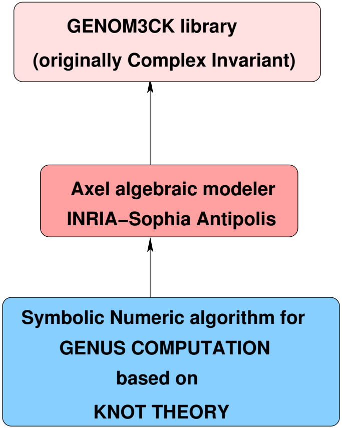
Thus we need...



Motivation

Hopefully...

- Symbolic-Numeric techniques for genus computation (initiated by J. Schicho).



- Other numeric method was reported (in the group of R. Sendra).

1. Motivation

2. Describing the library
   - Algorithm specifications
   - Short history
   - Interface

3. Testing the library
   - Setting the input data and parameters
   - Demo (Examples)

4. Conclusion

Algorithm specifications

- Input:
  - $F(x, y) \in \mathbb{C}[x, y]$ squarefree with exact and inexact coefficients;
  - $\mathcal{C} = \{(x, y) \in \mathbb{C}^2 | F(x, y) = 0\} \subseteq \mathbb{C}^2 \simeq \mathbb{R}^4$ of degree d;
  - $\epsilon \in \mathbb{R}_+^*$ input parameter.
- Output:
  - $Sing(\mathcal{C})$ set of singularities of $\mathcal{C}$;
  - A set of invariants of $\mathcal{C}$:
  - A set of operations from knot theory on each algebraic link:
- Method: shortly presented on the next slides.

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- Output:
  - $Sing(\mathcal{C})$ set of singularities of $\mathcal{C}$;
  - A set of invariants of $\mathcal{C}$:
    - algebraic link of each singularity (topological type);
    - Milnor fibration of each singularity;
    - Alexander polynomial of each algebraic link;
    - $\delta(s) \in \mathbb{N}$, δ-invariant of each singularity $s \in Sing(\mathcal{C})$;
    - $genus(\mathcal{C}) \in \mathbb{Z}$, genus of $\mathcal{C}$.
  - A set of operations from knot theory on each algebraic link:
    - diagram (crossings, arcs), type of crossings.
- Method: shortly presented on the next slides.

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$\mathcal{C} \subseteq \mathbb{R}^4$ with $Sing(\mathcal{C})$

Move each $s \in Sing(\mathcal{C})$ in 0

Let 0 singularity of $\mathcal{C} \subseteq \mathbb{R}^4$

$S_{(0,\epsilon)} \subseteq \mathbb{R}^4$ small sphere

$$X = \mathcal{C} \cup S_{(0,\epsilon)} \subseteq \mathbb{R}^4$$

$$f : S_{(0,\epsilon)} \setminus \{(0,0,0,\epsilon)\} \to \mathbb{R}^3$$

$$(a,b,c,d) \mapsto$$

$$(u = \frac{a}{\epsilon - d}, v = \frac{b}{\epsilon - d}, w = \frac{c}{\epsilon - d})$$

f is stereographic projection

GENOM3CK ⇜ (Axel) **Subdivision methods**

$\mathcal{C} \subseteq \mathbb{R}^4$ with $Sing(\mathcal{C})$

Move each $s \in Sing(\mathcal{C})$ in 0

Let 0 singularity of $\mathcal{C} \subseteq \mathbb{R}^4$

$\mathcal{C}$ defined by $\mathsf{F}(x, y) \in \mathbb{C}[x, y]$

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GENOM3CK (Axel)

Subdivision methods

For sufficiently small ϵ

$f(X) \subseteq \mathbb{R}^3$ differentiable algebraic link

$f(X) = \{(u, v, w) | \mathsf{ReF}(\dots) = \mathsf{ImF}(\dots) = 0\}$

Math

Adapted Milnor's research (Our)

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GENOM3CK (Axel) ← Subdivision methods

$f(X) \subseteq \mathbb{R}^3$ differentiable algebraic link

Math ← Adapted Milnor's research (Our)

$\widetilde{f}(X) \subseteq \mathbb{R}^3$ piecewise linear algebraic link

Let $\pi_{f(X)} \subseteq \mathbb{R}^2$ projection of $\widetilde{f}(X)$

GENOM3CK (Axel) ← Algo for $\widetilde{f}(X)$ i.e. topology of $f(X)$

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↓

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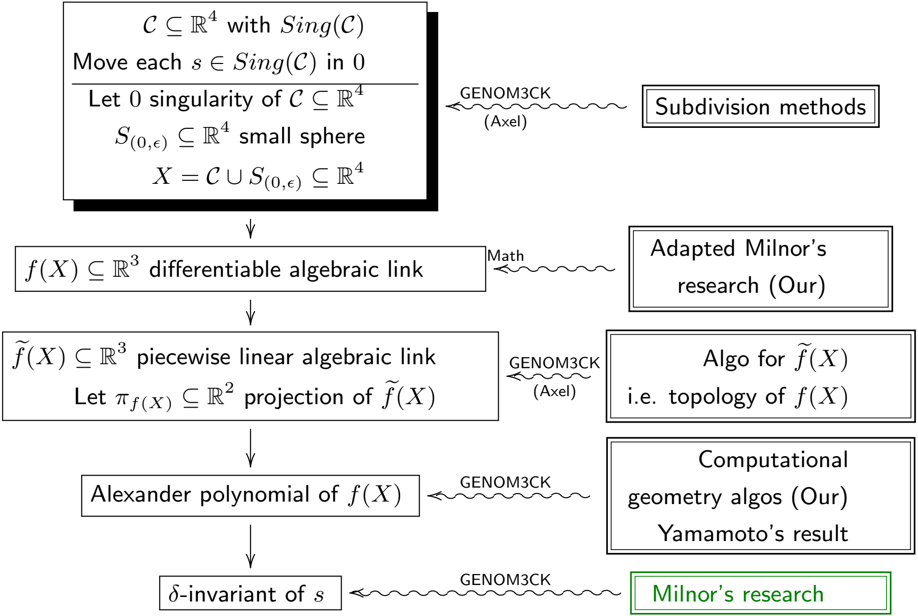
GENOM3CK (Axel) ← Algo for $\widetilde{f}(X)$ i.e. topology of $f(X)$

↓

Alexander polynomial of $f(X)$ is a complete invariant of $f(X)$

GENOM3CK ← Computational geometry algos (Our) Yamamoto's result



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Alexander polynomial of $f(X)$

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δ-invariant of s

← GENOM3CK ← Milnor's research

↓

$$genus(\mathcal{C}) = \frac{(d-1)(d-2)}{2} - \sum_{s \in Sing(\mathcal{C})} \delta(s)$$

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← GENOM3CK (Axel) ← **Subdivision methods**

↓

$f(X) \subseteq \mathbb{R}^3$ differentiable algebraic link

← Math ← **Adapted Milnor's research (Our)**

↓

$\widetilde{f}(X) \subseteq \mathbb{R}^3$ piecewise linear algebraic link

$\widetilde{f}(X) = \mathcal{S}_1 \cap \mathcal{S}_2 \cap (\mathcal{S}_1 + \mathcal{S}_2) \cap (\mathcal{S}_1 - \mathcal{S}_2)$

← GENOM3CK (Axel) ← **Algo for $\widetilde{f}(X)$ i.e. topology of $f(X)$**

↓

Alexander polynomial of $f(X)$

← GENOM3CK ← **Computational geometry algos (Our) Yamamoto's result**

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δ-invariant of s

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$$genus(\mathcal{C}) = \frac{(d-1)(d-2)}{2} - \sum_{s \in Sing(\mathcal{C})} \delta(s)$$

Short history: GENOM3CK

is written in Axel

C++, Qt Script for Applications (QSA)

is written in Mathemagix

C++

what is Axel?

- algebraic geometric modeler
- INRIA, Galaad team (2006)
- http://axel.inria.fr/

what is Mathemagix?

- computer algebra system
- http://www.mathemagix.org/

Short history: GENOM3CK

uses from Axel

- unique algebraic tools
  (for visualization of implicit algebraic curves in 3D)
- easy-to-use interface
- plugins that allow extension of the system

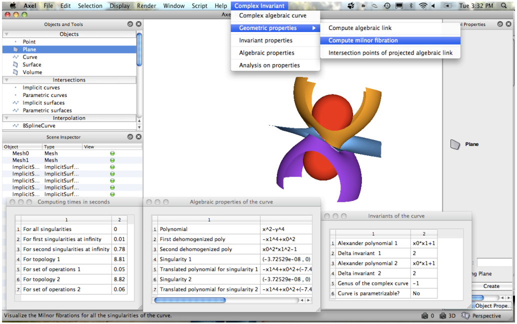
uses from Mathemagix

- subdivision techniques
  (for computing singularities)
- operations on polynomials, matrices, determinants, etc.

Interface

- part of Axel[a];
- main menu is `Complex Invariant`;
- contains 3 types of properties:
  - geometric;
  - invariant;
  - algebraic.
- contains computing time (at most polynomial).
- Examples in the Demo!

[a] Acknowledgements: Julien Wintz



Setting the input data and parameters

Input data and parameters:

- $F(x, y) \in \mathbb{C}[x, y]$ defining an input algebraic curve $\mathcal{C}$;

Restrictions!

- Introduce multiplication, power as $x * y$ and $x^\wedge n$.
- Introduce $F(x, y)$ in its expanded form.

Examples in the Demo!

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Setting the input data and parameters

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- $F(x,y) \in \mathbb{C}[x,y]$ defining an input algebraic curve $\mathcal{C}$;
- $\epsilon = \dfrac{n}{d} \in \mathbb{R}_+^*$ with $n, d \in \mathbb{N}^*$;

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- Introduce ϵ by introducing n, d.
- Choose ϵ small s.t. the algorithm is correct (heuristic methods).

Examples in the Demo!

Setting the input data and parameters

Input data and parameters:

- $F(x, y) \in \mathbb{C}[x, y]$ defining an input algebraic curve $\mathcal{C}$;
- $\epsilon = \frac{n}{d} \in \mathbb{R}_+^*$ with $n, d \in \mathbb{N}^*$;
- $B = [-a, a] \times [-b, b] \times [-c, c] \in \mathbb{R}^3, a, b, c \in \mathbb{N}^*$;

Restrictions!

- Introduce multiplication, power as $x * y$ and $x^\wedge n$.
- Introduce $F(x, y)$ in its expanded form.
- Introduce ϵ by introducing n, d.
- Choose ϵ small s.t. the algorithm is correct (heuristic methods).
- Introduce B by introducing a, b, c.
- Choose B big s.t. it contains all the singularities of $\mathcal{C}$ (heuristic methods).

Examples in the Demo!

Summary

- We have a symbolic-numeric algorithm (i.e. approximate algorithm) for plane complex algebraic curves, in the library GENOM3CK.
  About GENOM3CK (download, installation, documentation):
  http://people.ricam.oeaw.ac.at/m.hodorog/software.html
- ✉ Support: madalina.hodorog@oeaw.ac.at.

Run GENOM3CK in two ways:

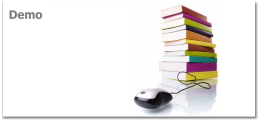
➠ click on the icon of Axel (see output).

➠ run command at terminal (see intermediate computations):

```
~/pathToAxelLinux/build$ ./bin/axel
~/pathToAxelMacOS/src$ ./Axel.app/Contents/MacOSs/Axel
```

Demo (Numeric and Symbolic Examples)

Equation	Box
$x^2 - y^2, \epsilon = 1.0$	$[-4, 4, -6, 6, -6, 6]$
$x^2 - y^3, \epsilon = 1.0$	$[-4, 4, -6, 6, -6, 6]$
$x^3 - y^3, \epsilon = 1.0$	$[-4, 4, -6, 6, -6, 6]$
$-x^3 - 1.875xy + y^2, \epsilon = 0.25$	$[-4, 4, -6, 6, -6, 6]$
$1.02x^2y + 1.12y^4, \epsilon = 0.25$	$[-4, 4, -6, 6, -6, 6]$



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Conclusion and future work

✓ DONE:

- automatization of an approximate algorithm for complex curves in GENOM3CK;

× TO DO's:

Conclusion and future work

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- automatization of an approximate algorithm for complex curves in GENOM3CK;

✗ TO DO's:

- prove the properties of the approximate algorithm (i.e. convergency, continuity);
- deform the input curve and keep the invariants constant.

Conclusion and future work

✓ DONE:

- automatization of an approximate algorithm for complex curves in GENOM3CK;
- describe algorithm with principles from regularization theory;

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Conclusion and future work

✓ DONE:

- automatization of an approximate algorithm for complex curves in GENOM3CK;
- describe algorithm with principles from regularization theory;
- integrate symbolic, numeric, graphical capablities into a single library GENOM3CK (use of Axel);
- provide a natural graphical user interface (use of QSA);
- users can visualize the ongoing computations and the results;

✗ TO DO's:

- prove the properties of the approximate algorithm (i.e. convergency, continuity);
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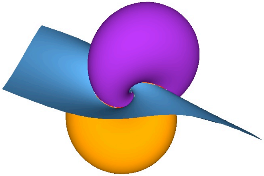
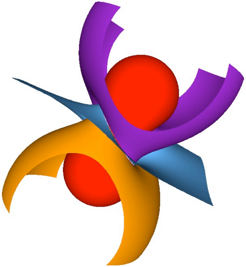
✗ TO DO's:

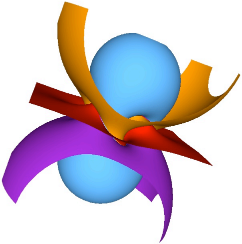
- prove the properties of the approximate algorithm (i.e. convergency, continuity);
- deform the input curve and keep the invariants constant.
- include other operations, i.e. from knot theory, algebraic geometry.

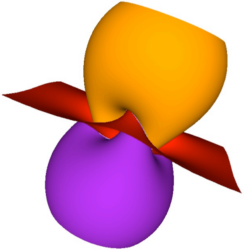




"...in programming mathematical elegance is not a dispensable luxury but a matter of life and death"(E.W. Dijkstra, 1978).

Thank you for your attention.





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