A Symbolic-Numeric Algorithm for Computing the Alexander Polynomial of a Plane Curve Singularity

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***Abstract*—We report on a symbolic-numeric algorithm for computing the Alexander polynomial of each singularity of a plane complex algebraic curve defined by a polynomial with coefficients of limited accuracy, i.e. the coefficients are both exact and inexact data. We base the algorithm on combinatorial methods from knot theory which we combine with computational geometry algorithms in order to compute efficient and accurate results. Nonetheless the problem we are dealing with is ill-posed, in the sense that tiny perturbations in the coefficients of the defining polynomial cause huge errors in the computed results.**

I. Introduction

Plane complex algebraic curves play an important role in mathematical topics such as number theory, complex analysis or algebraic topology as discussed in [1]. For our study, we consider plane complex algebraic curves defined by polynomials whose coefficients are both exact data (i.e. integer or rational numbers) and inexact data (i.e. numerical values) as in [2]. In this context, when we refer to numerical values we mean indetermination of a given order with respect to computational operations. For instance, an inexact data represented by the numerical value 1.976 is interpreted as having attached an indetermination of order 10^{-3}, which means that the last digit is uncertain. In this setting, we are interested in the type of the singularities of the plane complex algebraic curve and in the way in which the type of these singularities changes when one slightly varies the coefficients of the polynomial. For this purpose, we compute the algebraic link of each singularity. From the algebraic link we compute the Alexander polynomial of each singularity. From the Alexander polynomial we may compute other information about the plane complex algebraic curve: the delta-invariant of each singularity and the genus of the plane complex algebraic curve.

In this paper, we give a precise symbolic-numeric algorithm for computing the Alexander polynomial of each singularity of a plane complex algebraic curve. We base the algorithm on computational geometry algorithms performed on a graph data structure [3], on combinatorial methods from knot theory [4], and on specific results concerning the singular points of complex hypersurfaces [5] and the Alexander polynomials of algebraic links [6], [7]. The results computed with this symbolic-numeric algorithm are interpreted in the frame of approximate algebraic computation, as described in [8], [9]. This interpretation has the advantage of ensuring that the computed results continuously depend on the input data.

In Section II we describe the mathematics required for computing the Alexander polynomial of the singularity of a plane complex algebraic curve. In Section III we discuss the ill-posedness of the problem, and we present a strategy called regularization which we use in order to handle this ill-posedness. In Section IV we give the algorithm for computing the Alexander polynomial of each singularity of a plane complex algebraic curve. In addition, we describe the library that contains the implementation of this algorithm, and present an example performed with the library. We end with giving the conclusion and future directions of research in Section V.

II. Mathematical Definition of the Alexander Polynomial

A. Plane Complex Algebraic Curves and Their Singularities

In this subsection, following [10], [11], [12], we define the objects of our study, i.e. the plane complex algebraic curves:

Definition 1: Let $\mathbb{C}$ to be the field of complex numbers, and $\mathbb{A}^2(\mathbb{C}) = \{(x, y) \in \mathbb{C}^2\}$ the affine plane over $\mathbb{C}$. Let $f(x, y) \in \mathbb{C}[x, y]$ to be an irreducible polynomial in x and y with coefficients in $\mathbb{C}$ of degree m. The set of zeroes of the polynomial $f(x, y)$ denoted with $\mathcal{C} = \{(x, y) \in \mathbb{A}^2(\mathbb{C}) | f(x, y) = 0\}$ is called the (affine) plane complex algebraic curve of degree m defined by $f(x, y)$.

In particular, we are interested in a special type of points of each plane complex algebraic curve, i.e the singular points, that we define as follows:

Definition 2: Let $\mathcal{C}$ be a plane complex algebraic curve of degree m defined by the irreducible polynomial $f(x, y) \in \mathbb{C}[x, y]$. The set of singular points (or simply singularities) of $\mathcal{C}$ is defined as $Sing(C) = \{(x_0, y_0) \in \mathbb{A}^2(\mathbb{C}) | f(x_0, y_0) = \dfrac{\partial f(x,y)}{\partial x}(x_0, y_0) = \dfrac{\partial f(x,y)}{\partial y}(x_0, y_0) = 0\}$.

Example 1: We consider $\mathcal{C}$ the plane complex algebraic curve defined by $f(x, y) = x^2 - y^5 \in \mathbb{C}[x, y]$, i.e. $\mathcal{C} = \{(x, y) \in \mathbb{C}^2 | x^2 - y^5 = 0\}$. Based on Definition 2, we compute

$Sing(\mathcal{C})$ by solving the following overdeterminate system of polynomial equations in $\mathbb{C}^2$:

$$\begin{cases} f(x_0, y_0) = x_0^2 - y_0^5 = 0 \\ \dfrac{\partial f(x,y)}{\partial x}(x_0, y_0) = 2x_0 = 0 \\ \dfrac{\partial f(x,y)}{\partial y}(x_0, y_0) = 5y_0^4 = 0 \end{cases} \tag{1}$$

We obtain $Sing(\mathcal{C}) = \{(0,0)\}$.

B. The Link of a Plane Curve Singularity

In this subsection, we define several notions that are required for introducing the link of a plane curve singularity. Firstly, we define the knots (links):

Definition 3: A knot is a piecewise linear or a differentiable simple closed curve in the 3-dimensional space $\mathbb{R}^3$. A link is a finite union of disjoint knots.

We add that the knots that make up a link are called the components of the link, and thus a knot is a link with one component.

Secondly, we define the stereographic projection in $\mathbb{R}^3$ as a certain mapping that projects a sphere onto a plane. It is constructed as in Figure 1: we take a sphere; we draw a line from the north pole N of the sphere to a point $\hat{P}$ in the equator plane to intersect the sphere at a point P. The stereographic projection of $\hat{P}$ is P. In fact, the stereographic projection gives an explicit homeomorphism from the unit sphere minus the north pole to the Euclidean plane:

Definition 4: Two subsets $U \subset \mathbb{R}^k, V \subset \mathbb{R}^n$ are topologically equivalent or homeomorphic if and only if there exists a bijective function $\varphi : U \to V$ such that both φ and its inverse are continuous. In this case, φ is called an homeomorphism.

More generally, the stereographic projection may be applied to a n-sphere S^n in R^{n+1}:

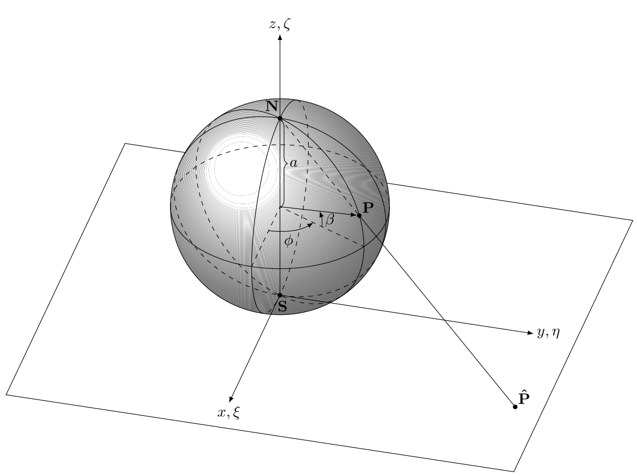


Fig. 1. Stereographic projection (generated with PGF/TikZ Latex packages by T. M. Trzeciak)

Definition 5: Consider a n-sphere $S^n = \{(x_1, x_2, ..., x_{n+1}) \subset \mathbb{R}^{n+1} | x_1^2 + x_2^2 + ... + x_{n+1}^2 = 1\}$ in $\mathbb{R}^{n+1}$, and $Q(0,0,0,...,1) \in S^n$ the north point of the n-sphere. If H is a hyperplane in $\mathbb{R}^{n+1}$ not containing Q, then the stereographic projection of the point $P \in S^n \setminus Q$ is the point $P^{'}$ of the intersection of the line QP with H. The stereographic projection is a homeomorphism from $S^n \setminus Q \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$.

We now define the link of a plane curve singularity:

Definition 6: Let $\mathcal{C} = \{(x,y) \in \mathbb{C}^2 | f(x,y) = 0\}$ be a plane complex algebraic curve defined by $f(x,y) \in \mathbb{C}[x,y]$ with an isolated singularity in the origin $(0,0) \in \mathbb{C}^2$, i.e. there is no other singularity on a sufficiently small neighborhood of $(0,0)$. Let $S_\epsilon = \{(x,y) \in \mathbb{C}^2 | x^2 + y^2 = \epsilon^2\}$ be the sphere centered in the origin of a sufficiently small radius $\epsilon \in \mathbb{R}_+^*$. Consider $X = \mathcal{C} \cap S_\epsilon \subset \mathbb{C}^2 \cong \mathbb{R}^4$, and $\pi_{(\epsilon,N)}$ the stereographic projection of the sphere S_ϵ in $\mathbb{R}^4$ from the north pole $N(0,\epsilon)$ of the sphere S_ϵ, which does not belong to the curve $\mathcal{C}$. Then $\pi_{(\epsilon,N)}(X) \subset \mathbb{R}^3$, i.e. the image of X through the stereographic projection, is called the link of the singularity $(0,0)$.

We define the equivalence of two links in the following way:

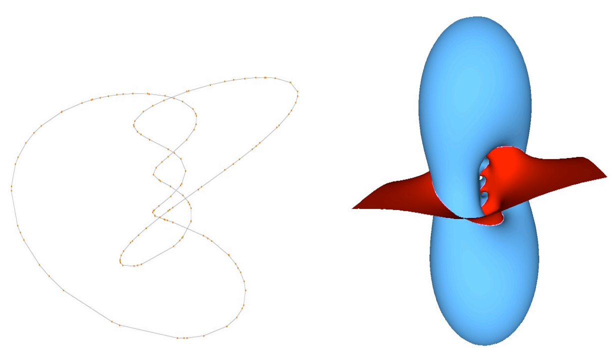
Definition 7: We say that two links are equivalent if there exists an orientation-preserving homeomorphism on $\mathbb{R}^3$ that maps one link onto the other. This equivalence is called (ambient) isotopy.

We introduce a certain type of links, i.e. the algebraic links:

Definition 8: A link is called algebraic if it is equivalent to the link of a plane curve singularity.

Remark 1: Under the same assumptions from Definition 6 and considering S^1 the unit circle, and $|\cdot|$ the absolute value function, Milnor fibration theorem states that the mapping $\phi : S_\epsilon \setminus X \to S^1, \phi(x,y) = \dfrac{f(x,y)}{|f(x,y)|}$ is a fibration, i.e. the complement $S_\epsilon \setminus X$ is a union of smooth surfaces, each being the preimage of one point.

Example 2: In Figure 2, we see the algebraic link and the Milnor fibration of the singularity $(0,0)$ of the plane complex algebraic curve $\mathcal{C}$ defined by $f(x,y) = x^2 - y^5 \in \mathbb{C}[x,y]$.



(a) Algebraic link of $(0,0)$ (b) Milnor fibration of $(0,0)$

Fig. 2. Output of the singularity $(0,0)$ of $f(x,y) = x^2 - y^5$ produced with GENOM3CK, see Section IV-B for more information

The equivalence class of the link of the singularity determines the homeomorphism class of the singularity, by the following theorem:

Theorem 1: (Milnor[5]) Let $V \subset \mathbb{C}^{n+1}$ be a hypersurface in $\mathbb{C}^{n+1}$, i.e. an algebraic variety defined by a single polynomial. Assume $\vec{0} \in V$ and $\vec{0}$ is an isolated singularity; S_ϵ is the sphere centered in $\vec{0}$ and of radius ϵ; and D_ϵ is the disk centered in $\vec{0}$ of radius ϵ. Then, for sufficiently small ϵ, $L = S_\epsilon \cap V$ is a $(2n-1)$-dimensional nonsingular set and $D_\epsilon \cap V$ is homeomorphic to the cone over L.

C. The Alexander Polynomial of an Algebraic Link

In this subsection, we define the Alexander polynomial of an algebraic link. Firstly we need to introduce some preliminary notions from knot theory. From now on, we consider only piecewise linear algebraic links. When we work with knots, we work with their projection in the 2-dimensional space:

Definition 9: A regular projection is a linear projection for which no three points on the knot project to the same point, and no vertex projects to the same point as any other point on the knot. A crossing point is an image of two knot points of such a regular projection from $\mathbb{R}^3$ to $\mathbb{R}^2$. Then:

1) A (link) diagram is the image under regular projection, together with the information on each crossing point telling which branch goes over and which goes under. Thus we speak about overcrossings and undercrossings.
2) A diagram together with an arbitrary orientation of each knot in the link is called an oriented diagram.

We define the elements of a diagram as follows:

Definition 10:

1) A crossing is called lefthanded (denoted with -1) if the underpass traffic goes from left to right or it is called righthanded (denoted with $+1$) if the underpass traffic goes from right to left.
2) An arc is the part of a diagram between two undercrossings (Figure 3). Whether lefthanded or righthanded, each crossing is determined by three arcs and we denote the overgoing arc with i, and the undergoing arcs with j and k (Figure 4). The number of arcs in a link diagram is equal to the number of crossings in the same link diagram.

The main problem in knot theory is to decide whether two different diagrams represent the same link or not, i.e. whether two links are equivalent or not. The difficult problem in knot theory is to show that two links are different up to (ambient) isotopy. In order to show that two links are not equivalent, we use the notion of link invariant defined in the following way:

Definition 11: A link invariant is a function from link diagrams to some discrete set ($\mathbb{Z}$ or $\mathbb{Z}[t]$) which is unchanged when we replace the link by an equivalent one.

For our purpose, we know that the Alexander polynomial is a complete invariant for algebraic links, i.e. it distinguishes between all the algebraic links ([13]). We compute the Alexander polynomial Δ_L of an algebraic link L in several stages as follows: from $D(L)$, the diagram of the algebraic link we compute $LM(L)$, the labeling matrix of L; from $LM(L)$ we compute $PM(L)$, the prealexander matrix of L; and from $PM(L)$ we compute Δ_L.

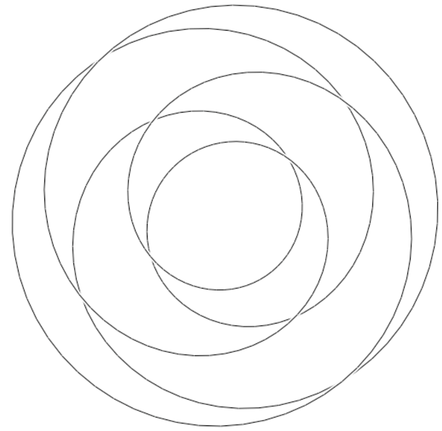


Fig. 3. Oriented counterclockwise diagram of the cinquefoil algebraic knot $x^2 - y^5$ with 8 arcs, 8 lefthanded crossings (produced with 3D-XplorMath-J Applet). We denote the crossings from the upperleft to the lowerright corner with $\{c_0, c_1, c_2, c_3\}$ and the crossings from the lowerleft to the upperright corner with $\{c_4, c_5, c_6, c_7\}$.

Fig. 4. Types of crossings: lefthanded (-1) and righthanded(+1).

Definition 12: Let $D(L)$ be an oriented link diagram with r components and p crossings $x_q : q \in \{0, ..., p-1\}$. We denote the arcs of $D(L)$ with the labels $\{0, ..., p-1\}$ and separately the crossings of $D(L)$ with $\{0, ..., p-1\}$. We denote the labeling matrix of $D(L)$ with $LM(L) \in \mathcal{M}(p, 4, \mathbb{Z})$. We define $LM(L) = (b_{ql})_{q,l}$ with $q \in \{0, ..., p-1\}, l \in \{1, ..., 4\}$ row by row for each crossing x_q as follows:

- at b_{q1} store the type of the crossing x_q ($+1$ or -1);
- at b_{q2} store the label of the arc i of x_q in $D(L)$;
- at b_{q3} store the label of the arc j of x_q in $D(L)$;
- at b_{q4} store the label of the arc k of x_q in $D(L)$.

Definition 13: Let $D(L)$ be an oriented link diagram with r components and p crossings $x_q : q \in \{0, ..., p-1\}$. We denote the arcs and the crossings of $D(L)$ as in Definition 12. We consider $LM(L)$ the labeling matrix of $D(L)$ as in Definition 12. We denote the prealexander matrix of L with $PM(L) \in \mathcal{M}(p, p, \mathbb{Z}[t_0, t_1, ..., t_{r-1}])$. We define $PM(L)$ row by row for each crossing x_q depending on $LM(L)$. For x_q we consider the variable t_s, where $s \in \{0, ..., r-1\}$ is the s-th knot component of $D(L)$, which contains the overgoing arc that determines the crossing x_q. Then:

- if x_q is righthanded, i.e. $b_{q1} = +1$ in $LM(L)$ then at position b_{q2} of $PM(L)$ store the label $1 - t_s$, at position b_{q3} store -1 and at position b_{q4} store t_s;
- if x_q is lefthanded, i.e. $b_{q1} = -1$ in $LM(L)$ then at

position b_{q2} of $PM(L)$ store the label $1-t_s$, at position b_{q3} store t_s and at position b_{q4} store -1;

- if two or all of the positions b_{q2}, b_{q3}, b_{q4} have the same value, then store the sum of the corresponding labels at the corresponding position. All other entries of the matrix are 0.

We define the Alexander polynomial of $D(L)$ depending on the number of knot components in L:

Definition 14: Let $D(L)$ be an oriented link diagram with r components and p crossings, $LM(L)$ be its labeling matrix as in Definition 12 and $PM(L)$ be its prealexander matrix as in Definition 13.

1) Univariate case, ([14]). The univariate Alexander polynomial $\Delta_L(t_0) \in \mathbb{Z}[t_0^{\pm 1}]$ is the normalized polynomial computed as the determinant of any $(p-1) \times (p-1)$ minor of the prealexander matrix of $D(L)$. A normalized polynomial is a polynomial in which the term of the lowest degree is a positive constant.
2) Multivariate case, ([7]). The multivariate Alexander polynomial $\Delta_L(t_0, ..., t_{r-1}) \in \mathbb{Z}[t_0^{\pm 1}, ..., t_{r-1}^{\pm 1}]$ is the normalized polynomial computed as the greatest common divisor of all the $(p-1) \times (p-1)$ minor determinants of the prealexander matrix of $D(L)$.

Example 3: We compute Δ_L for the cinquefoil algebraic knot from Figure 3. We denote the arcs with $\{0, ..., 7\}$. Then:

$$LM(L) = \left(\begin{array}{c|ccc} & type & label_i & label_j & label_k \\ \hline c_0 & -1 & 0 & 7 & 1 \\ c_1 & -1 & 1 & 4 & 3 \\ c_2 & -1 & 0 & 5 & 6 \\ c_3 & -1 & 1 & 2 & 0 \\ c_4 & -1 & 0 & 3 & 2 \\ c_5 & -1 & 1 & 0 & 5 \\ c_6 & -1 & 0 & 1 & 4 \\ c_7 & -1 & 1 & 6 & 7 \end{array}\right)$$

$$PM(L) = \left(\begin{array}{c|ccc} & label_i & label_j & label_k \\ \hline c_0 & 0 & 7 & 1 \\ -1 & 1-t_0 & t_0 & -1 \\ \hline c_1 & 1 & 4 & 3 \\ -1 & 1-t_0 & t_0 & -1 \\ \hline c_2 & 0 & 5 & 6 \\ -1 & 1-t_0 & t_0 & -1 \\ \hline ... & ... & ... & ... \\ \hline c_7 & 1 & 6 & 7 \\ -1 & 1-t_0 & -1 & t_0 \end{array}\right) =$$

$$= \left(\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1-t_0 & -1 & 0 & 0 & 0 & 0 & 0 & t_0 \\ 0 & 1-t_0 & 0 & -1 & t_0 & 0 & 0 & 0 \\ 1-t_0 & 0 & 0 & 0 & 0 & t_0 & -1 & 0 \\ -1 & 1-t_0 & t_0 & 0 & 0 & 0 & 0 & 0 \\ 1-t_0 & 0 & -1 & t_0 & 0 & 0 & 0 & 0 \\ t_0 & 1-t_0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1-t_0 & t_0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1-t_0 & 0 & 0 & 0 & 0 & t_0 & -1 \end{array}\right)$$

$$\det\Big(\text{Minor}_{77}\big(PM(L)\big)\Big) = -t_0^5 + t_0^4 - t_0^3 + t_0^2 - t_0$$

$$\Delta_L(t_0) = t_0^4 - t_0^3 + t_0^2 - t_0 + 1$$

D. Relation with the Genus

As an application, we may compute the genus of a plane complex algebraic curve in terms of the Alexander polynomial of its singularities:

Definition 15: Let $\mathcal{C}$ be a plane complex algebraic curve of degree m in $\mathbb{A}^2(\mathbb{C})$, and $\mathcal{C}^*$ the corresponding projective plane algebraic curve in $\mathbb{P}^2(\mathbb{C})$ defined as in [11]. We denote with $Sing(\mathcal{C}^*)$ the set of singularities of $\mathcal{C}^*$. The genus of $\mathcal{C}$ is defined as:

$$genus(\mathcal{C}) = \frac{(m-1)(m-2)}{2} - \sum_{s \in Sing(\mathcal{C}^*)} \delta_s,$$

with $genus(C) \in \mathbb{Z}$, and where $\delta_s \in \mathbb{N}$ denotes the delta-invariant of the singularity s.

We notice that the computation of the genus reduces to the computation of the delta-invariant of each singularity, which we define in terms of the Alexander polynomial defined in Subsection II-C:

Definition 16: (based on Milnor[5]) Let $\Delta_L(t_0, \ldots, t_{r-1})$ be the Alexander polynomial of the link of the isolated singularity $s = (0,0)$ of a plane complex algebraic link. Let r be the number of variables in Δ_L and ρ the degree of Δ_L. If $r = 1$ then we define $\delta_s = \dfrac{\rho}{2}$, otherwise $\delta_s = \dfrac{\rho + r}{2}$.

III. Regularization Techniques for Dealing with Ill-Posedness of the Problem

In this subsection, we explain the notion of an ill-posed problem and we present a regularization method for dealing with such a problem. In particular, we apply these notions to the problem we solve, i.e. the computation of the Alexander polynomial of a plane curve singularity.

Firstly, we introduce basic notions from approximate algebraic computation following [15], [16], which we use for our problem. Approximate algebraic computation is a new promising and challenging field of mathematics, that developed in the recent years with important achievements such as for instance in [17], [18], [19].

The objects of approximate algebraic computation are polynomials with coefficients of limited accuracy, i.e. the coefficients may be exact data (integer or rational numbers) or inexact data (numerical values). In the polynomial $f(x,y) = x^3 - 1.865y^2 - y^3$, for 1.865 we associate a tolerance σ of 10^{-3} which means that the last digit is uncertain. When we apply exact computation on classical algebraic problems defined in terms of polynomials with coefficients of limited accuracy, we observe that tiny perturbations in the coefficients produce huge errors in the solution. This is the case in classical algorithms such as: the Euclidean algorithm for computing the greatest common divisor of polynomials, root computation of polynomials, factorization of polynomials, Groebner bases computation, etc. These algorithms (or rather the problem specifications addressed by them) are *ill-posed* in the sense of Hadamard, which means that the solution does not depend continuously on the input data, i.e. the solution is not stable under small changes of the input data. A major goal of approximate algebraic computation is to deal with this kind of ill-posed problems. In particular, the computation of the Alexander polynomial of each singularity of a plane complex algebraic curve, is an ill-posed problem in the sense of Hadamard discussed here.

A method called regularization has been introduced to solve ill-posed problems, that makes it possible to construct numerical methods that approximate solutions of ill-posed problems, which are stable under small changes of the input data. We adopt such a regularization method for the computation of the Alexander polynomial of a plane curve singularity, following [9]. In the rest of this subsection, we describe this regularization method.

We denote $\mathcal{D}$ the set of all squarefree polynomials in x and y with complex coefficients of degree $m \in \mathbb{N}$ such that the sum of squares of absolute values of the coefficients is 1. This is not a restriction because we are only interested in the zero sets of these polynomials, and this does not change if we multiply each polynomial by a scalar. The set $\mathcal{D}$ is a metric space by the Euclidean distance of coefficient vectors, denoted with $||-||$. We denote $\mathcal{P} = \{\mathbb{Z}[x_0] \cup \mathbb{Z}[x_0, x_1] \cup ... \cup \mathbb{Z}[x_0, ..., x_i] \cup ...\}$ the set of all normalized Alexander polynomials either in the x_0 variable, or in the x_0, x_1 variables, or in $x_0, x_1, ...x_i$ sequence of variables with $i \in \mathbb{N}$, etc.

We consider the function: $E : \mathcal{D} \to \mathcal{P},\ f \mapsto E(f)$. For the image of E we use the notation $I = \{E(f))|f \in \mathcal{D}\}$. We consider E as a function for computing the exact algorithm for the Alexander polynomial of a plane curve singularity. We notice that the function E is discontinuous. We consider the partial function $R\_ : \mathcal{D} \times \mathbb{R}_+ \to I,\ (f, \epsilon) \mapsto R_\epsilon(f)$. For $f \in \mathcal{D}$, we say that the function $f\_ : \mathbb{R}_{\geq 0} \to \mathcal{D}, \delta \mapsto f_\delta$ is a perturbation of f if and only if $||f_\delta - f|| < \delta$, for all $\delta \in \mathbb{R}_{\geq 0}$.

We say that R_ϵ is a regularization for $E : \mathcal{D} \to I$ if and only if for any perturbation function $f\_$, the following properties are fulfilled:

$$\forall f \in \mathcal{D} \lim_{\epsilon \to 0} R_\epsilon(f) = E(f), \tag{2}$$

$$\forall f \in \mathcal{D}\ \forall f_\delta \in \mathcal{D} \lim_{\delta \to 0} R_{\epsilon(\delta)}(f_\delta) = E(f), \tag{3}$$

for some function $\epsilon(-)$, which is independent of f.

We call (2) the convergence property for exact data, and (3) the convergence property for noisy data, where the parameter δ is called the error (or the noise) in the input data. The numerical parameter ϵ is called the regularization parameter of the function R_ϵ.

In the following, we introduce the Milnor number depending on the Alexander polynomial of the singularity s of a plane complex algebraic curve, and on the algebraic link L of s: if L has one component, then the Milnor number equals the degree of the Alexander polynomial; otherwise (i.e. L has more than one component), the Milnor number equals the degree of the Alexander polynomial plus 1. The Milnor number measures the degeneracy of the singularity s of the plane complex algebraic curve. Thus, on $\mathcal{D}$ we consider the partial order $<$ induced by the Milnor number: $\forall p, q \in \mathcal{D}, p < q$ if and only if the Milnor number of p is less than the Milnor number of q. The partial order induced on $\mathcal{D}$ by the Milnor number makes the exact function E for computing the Alexander polynomial an upper semicontinuous function based on the result proved in [20] according to which: the Milnor number is an upper semicontinuous function of the coefficients of the defining polynomial of the plane complex algebraic curve.

Under these assumptions, we consider $A_\epsilon : \mathcal{D} \times \mathbb{R}_+ \to I$ as the symbolic numeric algorithm for computing the Alexander polynomial of the singularity s of a plane complex algebraic curve $\mathcal{C}$ defined by the polynomial $f(x, y) \in \mathbb{C}[x, y]$, constructed using the notions from Section II. We consider the parameter ϵ to be the radius of the sphere S_ϵ which we intersect with the zero set of $f(x, y)$, as described in Subsection II-B. We study whether $A\_$ is a regularization for the exact function $E : \mathcal{D} \to I$, as previously explained. The convergence property (2) for exact data holds for $A\_$, based on Theorem 1. The partial function $A\_$ is defined for all (f, ϵ) such that the intersection of the zero set of $f(x, y)$ with the sphere S_ϵ in $\mathbb{C}^2$ is nonsingular. The domain of definition of the partial function $A\_$ denoted with U is open and dense in $\mathcal{D} \times \mathbb{R}_{\geq 0}$, and $A\_$ is constant on each connected component of U (in other words $A\_$ is continuous).

We believe that the following "working hypothesis" is true, i.e. if E is an (upper) semicontinuous function from a compact set into a discrete partially ordered set, and $A\_$ is a partially continuous function defined on an open subset of $\mathcal{D} \times \mathbb{R}_{\geq 0}$ which satisfies the convergence property (2) for exact data, then $A\_$ is a regularization of E, i.e. $A\_$ satisfies also the convergence property (3) for noisy data. A similar statement can be found in [9], proposition 3.4. For our study, we considered the exact upper semicontinuous function for the computation of the Alexander polynomial to be defined as $E : \mathcal{D} \to I$. We notice that the set $\mathcal{D}$ is not compact, but it is possible to restrict the function E to compact subsets of $\mathcal{D}$. If we assume this "working hypothesis" true, then it follows that the algorithm A_ϵ, which we construct using the notions from Section II, is a regularization for the Alexander polynomial. We present this algorithm thoroughly in Subsection IV-A.

IV. Algorithm and Implementation

A. Description of the Algorithm

In this subsection we describe the algorithm for computing the Alexander polynomial of the singularities of a plane complex algebraic curve, constructed using the notions from Section II.

Remark 2: Once the Alexander polynomial of a plane curve singularity is known, the computation of the delta-invariant and the genus are not anymore subject to numerical errors, because we use discrete combinatorial algorithms combined with robust computational geometry algorithms for their computation. The Alexander polynomial itself is determined by the topology of the algebraic link. The computation of the topology of the link is unstable under tiny perturbations. Thus, we need to analyze the numerical behavior of the algebraic link under tiny perturbations. This information is captured by the Alexander polynomial, which is a complete invariant for algebraic links, i.e. different algebraic links have different Alexander polynomials.

For a plane complex algebraic curve $\mathcal{C}$ defined by the squarefree complex polynomial $f(x, y) \in \mathbb{C}[x, y]$, for a point $p \in \mathbb{C}^2$, and for $\epsilon \in \mathbb{R}_{\geq 0}$, we define the curve $L_{(\mathcal{C},p,\epsilon)}$ as the stereographic projection of the intersection of $\mathcal{C}$ with the sphere $S_\epsilon(p)$ of radius ϵ and origin p. If $L_{(\mathcal{C},p,\epsilon)}$ is a link, then we define the ϵ-Alexander polynomial of $\mathcal{C}$ at p as the Alexander polynomial of $L_{(\mathcal{C},p,\epsilon)}$. We give the algorithm for this computation, denoted in the following with ALEXPOLY$(f, \mathcal{C}, \epsilon)$.

Remark 3: If ϵ is sufficiently small, then $L_{(\mathcal{C},p,\epsilon)}$ will be a link and the ϵ-Alexander polynomial will be the Alexander polynomial of the singularity of $\mathcal{C}$ at p.

Algorithm 1 Alexander polynomial of the singularities of a plane algebraic curve ALEXPOLY$(f, \mathcal{C}, \epsilon)$

Input: $f(x, y) \in \mathbb{C}[x, y]$ a complex squarefree polynomial
$\mathcal{C} = \{(x, y) \in \mathbb{C}^2 | f(x, y) = 0\}$ a plane algebraic curve
$\epsilon \in \mathbb{R}^*_+$ a positive real number
Output: $\Delta_L(t_0, ..., t_{r-1})$ the ϵ-Alexander polynomial of each numerical singularity of $\mathcal{C}$.

1) Compute numeric $Sing(\mathcal{C})$, the singularities of $\mathcal{C}$, by solving system (1) with subdivision methods from [21];
2) For each singularity $s_0 = (x_0, y_0) \in Sing(\mathcal{C})$ do:
   a) Translate (x_0, y_0) in $s = (0, 0)$ by a change of coordinates, i.e. $\mathcal{C} = \{(x, y) \in \mathbb{C}^2 | f(x + x_0, y + y_0) = 0\}$.
   b) Compute symbolic-numeric L, the link of the singularity $s = (0, 0)$, with the algorithm LINK$(f, \mathcal{C}, s, \epsilon)$.
   c) Compute symbolic-numeric:
      - $D(L)$, the diagram of L,
      - r, the number of components of $D(L)$,
      - p, the crossings of $D(L)$,
      with computational geometry and combinatorial algorithms from [22].
   d) Compute symbolic $LM(L)$, the labeling matrix of $D(L)$, with Definition 12.
   e) Compute symbolic $PM(L)$, the prealexander matrix of $D(L)$, with Definition 13.
   f) If $r = 1$ then:
      i) Compute M, any $(p-1)\times(p-1)$ minor of $PM(L)$;
      ii) Compute D, the determinant of the minor M;
      iii) Return $\Delta_L(t_0) = Normalize(D)$;
   g) If $r \geq 2$ then:
      i) Compute all the $(p-1)\times(p-1)$ minors of $PM(L)$;
      ii) Compute G, the greatest common divisor of all the computed minors in g).i);
      iii) Return $\Delta_L(t_0, ...t_{r-1}) = Normalize(G)$.

We describe the algorithm LINK$(f, \mathcal{C}, s, \epsilon)$ for computing the algebraic link L of the singularity s of the plane complex algebraic curve $\mathcal{C}$ defined by the squarefree complex polynomial $f(x, y) \in \mathbb{C}^2$. The parameter ϵ denotes the radius of the sphere $S_\epsilon \subset \mathbb{C}^2$ which we intersect with the zero set of $f(x, y)$, as described in Subsection II-B:

Algorithm 2 Link of a plane curve singularity $s = (0, 0)$ LINK$(f, \mathcal{C}, s, \epsilon)$

Input: $f(x, y) \in \mathbb{C}[x, y]$ a complex squarefree polynomial
$\mathcal{C} = \{(x, y) \in \mathbb{C}^2 | f(x, y) = 0\}$ a plane algebraic curve
$s = (0, 0)$ a numerical singularity of $\mathcal{C}$
$\epsilon \in \mathbb{R}^*_+$ a positive real number
Output: $G, \ H \in \mathbb{R}[u, v, w]$
where the common zero set of G, H equals $L_{(\mathcal{C},p,\epsilon)}$.

1) Substitute the variables $x = a + ib, y = c + id$:
$$f(x, y) \Leftrightarrow f(a, b, c, d) = R(a, b, c, d) + iI(a, b, c, d)$$
where $R, I \in \mathbb{R}[a, b, c, d]$.
2) Intersect the plane complex algebraic curve:
$$\mathcal{C} = \{(a, b, c, d) | R(a, b, c, d) = I(a, b, c, d) = 0\}$$
with an isolated singularity in the origin $(0, 0, 0, 0)$, with the sphere centered in the origin and of small radius ϵ:
$$S_\epsilon = \{(a, b, c, d) | a^2 + b^2 + c^2 + d^2 - \epsilon^2 = 0\} \ .$$
3) Obtain $X = \mathcal{C} \cap S_\epsilon \subset \mathbb{R}^4$;
4) Consider a point $N(0, 0, 0, \epsilon) \in S_\epsilon \setminus \mathcal{C}$;
5) Project X with the generalized stereographic projection:
$$\pi_{(\epsilon,N)} : S_\epsilon \setminus \{P\} \subset \mathbb{R}^4 \to \mathbb{R}^3 \ ,$$
$$(a, b, c, d) \to (u, v, w) = (\tfrac{a}{\epsilon-d}, \tfrac{b}{\epsilon-d}, \tfrac{c}{\epsilon-d}) \ .$$
6) Compute the inverse:
$$\pi^{-1}_{(\epsilon,N)} : \mathbb{R}^3 \to S^3 \setminus \{P\}$$
$$(u, v, w) \to (a, b, c, d) = (\tfrac{2u\epsilon}{n}, \tfrac{2v\epsilon}{n}, \tfrac{2w\epsilon}{n}, \tfrac{-\epsilon+u^2\epsilon+v^2\epsilon+w^2\epsilon}{n})$$
where $\ n = 1 + u^2 + v^2 + w^2 \ $.
7) Compute $\pi_{(\epsilon,N)}(X)$ using $\pi^{-1}_{(\epsilon,N)}$ and finding G, H:
$$\pi_{(\epsilon,N)}(X) = \left\{(u, v, w) | \begin{array}{l} G := R(\frac{2u\epsilon}{n}, \frac{2v\epsilon}{n}, \frac{2w\epsilon}{n}, \frac{m}{n}) = 0 \\ H := I(\frac{2u\epsilon}{n}, \frac{2v\epsilon}{n}, \frac{2w\epsilon}{n}, \frac{m}{n}) = 0 \end{array}\right\}$$
where $m = -\epsilon + u^2\epsilon + v^2\epsilon + w^2\epsilon$.
8) Return $\pi_{(\epsilon,N)}(X)$ as computed in step 6.

We notice that after clearing out the denominators $G, H \in \mathbb{R}[u, v, w]$. Their common set of zeroes in $\mathbb{R}^3$ is equal to $\pi_{(\epsilon,N)}(X)$ the differentiable algebraic link of the singularity $(0, 0)$. In fact, $\pi_{(\epsilon,N)}(X)$ is an implicit algebraic curve in $\mathbb{R}^3$ with no singularities, given as the intersection of two implicit algebraic surfaces S_1, S_2 in $\mathbb{R}^3$ with defining polynomials G, H. The surfaces S_1, S_2 appear in the Milnor fibration of $\mathbb{R}^3 \setminus \pi_{(\epsilon,N)}(X)$ over S_ϵ. Using the library GENOM3CK implemented in Axel, we compute a piecewise linear approximation of this differentiable algebraic link. For an example, see Figure 2.

B. Implementation of the Algorithm

Algorithm 1 and **Algorithm 2** described in Subsection IV-A are implemented in the library GENOM3CK, a library we originally developed for computing the genus of a plane complex algebraic curve using knot theory. Together with its main functionality to compute the genus, the library computes other topological and algebraic invariants of each singularity of the plane complex algebraic curve. GENOM3CK is implemented in the free algebraic geometric modeler Axel [23], [24] (written in C++ and using Qt Script for Applications), and in the free computer algebra system Mathemagix [25]. Axel is a new system developed at INRIA-Sophia Antipolis, which provides for our purposes unique algebraic tools and visualization techniques to manipulate implicit algebraic curves and surfaces. Axel uses also libraries from the free computer algebra system Mathemagix [25], for instance a library for computing the singularities of a plane complex algebraic curve. The power of the Axel system comes from the fact that it allows its extension into "sub-programs" with new functionalities that are called plugins. We implement the proposed symbolic-numeric algorithms into one of Axel's plugins, which was further on transformed into a library. More information on the library is available at: http://people.ricam.oeaw.ac.at/m.hodorog/software.html.

Example 4: In Figure 5, we visualize the output of the library on the input plane complex algebraic curve $\mathcal{C}$ defined by $f(x,y) = x^2 - y^5$ for $\epsilon = 1.0$. The library computes: (i) the set of all distinct singularities both in the affine and in the projective space by using the subdivision method from Mathemagix. In this case, the set contains two singularities $Sing(\mathcal{C}) = \{s_1 = (-1.77636e-14, 0), s_2 = (0,0)\}$; (ii) the algebraic link of each singularity; (iii) information on the diagram of each algebraic link; (iv) the Milnor fibration of each singularity, i.e. the implicitly defined algebraic surfaces from the 3-dimensional space that define as their intersection the algebraic link. This operation is selected and displayed in Figure 5; (v) the Alexander polynomial of each singularity, i.e. $\Delta(s_1) = x_0^4 - x_0^3 + x_0^2 - x_0 + 1, \Delta(s_2) = x_0^8 - x_0^7 + x_0^5 - x_0^4 + x_0^3 - x_0^2 + x_0 - 1$, and the delta-invariant of each singularity, i.e. $\delta(s_1) = 2, \delta(s_2) = 4$; (vi) the genus of $\mathcal{C}$, i.e. $genus(\mathcal{C}) = 0$; (vii) and the computational time needed for the symbolic-numeric algorithms.

Remark 4: The test experiments performed with the library GENOM3CK indicate that the algorithm is a regularization for the Alexander polynomial as discussed in Section III. The precise proof for this statement is under construction.

V. Conclusion

We presented a symbolic-numeric algorithm for computing the Alexander polynomial of each singularity of a plane complex algebraic curve, that we completely and successfully automatized in the GENOM3CK library. Together with its main functionality to compute the Alexander polynomial of each singularity of a plane complex algebraic curve, the GENOM3CK library offers tools for computational operations in algebraic topology and geometry (i.e. algebraic link of each singularity of the plane algebraic curve, delta-invariant of each singularity of the plane algebraic curve, genus of the plane complex algebraic curve), and for computational operations in knot theory (i.e. information on the diagram of each algebraic link). The library also allows us to analyze the performance of the symbolic-numeric algorithm, that turns out to be efficient. For symbolic input data, the symbolic-numeric algorithm from the library computes certified and exact results. For numeric input data, based on the tests performed with the library GENOM3CK, the symbolic-numeric algorithm from the library computes a regularization of the problem. Thus, the library provides certified results for both symbolic and numeric input data, due to the efficient combination between the symbolic and numeric algorithms.

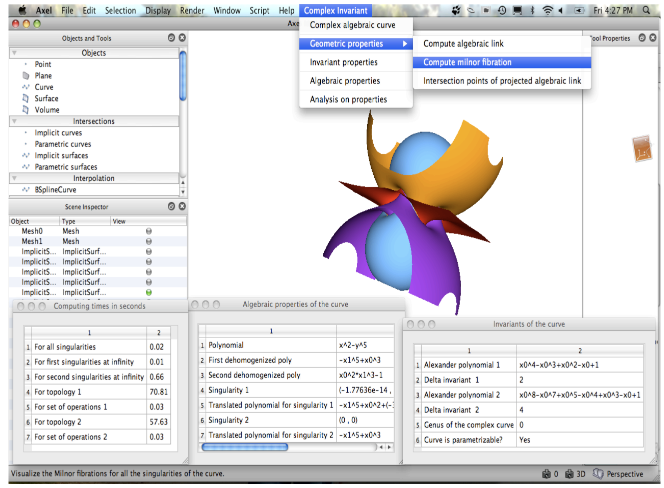


Fig. 5. GENOM3CK on the input curve defined by $F(x,y) = x^2 - y^5$

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