A Symbolic Summation Approach to Feynman Integrals

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We discuss new algorithmic strategies for multisums arising in the computation of Feynman parameter integrals, $F(N, \varepsilon)$, for $N \in \mathbb{N}$, $\varepsilon \in \mathbb{R}$, and present examples of typical computations coming from the two-loop integrals described in [3] which we encountered during our work on [5]. The genesis of these integrals is described in [4] in detail. Our summation methods have been efficiently implemented in a symbolic toolbox within Mathematica containing F. Stan's package `FSums` [10, Chapter 3] and C. Schneider's package `EvaluateMultiSums`, which rely on `Sigma` [8], J. Ablinger's `HarmonicSums` [1], and K. Wegschaider's `MultiSum` [12].

The first step of our procedure consists of rewriting Feynman parameter integrals as multisums over hypergeometric terms which fit the input class of our summation algorithms. The sum representations are of the form

$$Sum(\mu, \alpha) = \sum_{\kappa_1 \in \mathcal{R}_1} \cdots \sum_{\kappa_r \in \mathcal{R}_r} \mathcal{F}(\mu, \kappa_1, \ldots, \kappa_r . \alpha), \tag{1}$$

where the summand $\mathcal{F}(\mu, \kappa . \alpha)$ is proper hypergeometric in all discrete variables μ_i from $\mu = (\mu_1, \ldots, \mu_p)$ and in all summation variables κ_j, while the elements of $\alpha \in \mathbb{C}^l$ are additional parameters. For these nested sums, the summation range $\mathcal{R} \subseteq \mathbb{Z}^r$ does not satisfy a finite support condition. In this context, the package `FSums` takes the sum (1) as input and calls the package `MultiSum` to obtain a recurrence for its summand, using the WZ-summation strategy [13]. After summing over the algorithmically computed recurrence for the summand $\mathcal{F}$, we determine an inhomogeneous recurrence relation for $Sum(\mu, \alpha)$. The inhomogeneous part of this recurrence will contain special instances of the multisum (1) of lower nested depth. Applying the same method on these new sums recursively, we get new recurrences. The recurrences computed at the end of this procedure will have only hypergeometric terms on their right hand sides.

For the next step of the method we use procedures from the `Sigma` package. Namely, these last inhomogeneous difference equations can be viewed in special difference fields introduced by M. Karr [7] and extended significantly by C. Schneider. In this setting, it is possible to find solutions of such recurrences [2, 9]. Plugging in these answers into the recurrences from the previous level, we can recursively compute all solutions of the initial recurrence satisfied by the multisum (1) in terms of indefinite nested sums and products. Combining those solutions we find an alternative representation of (1) which –due the special input class of Feynman integrals– can be transformed to harmonic sums [6, 11, 1] including as special case the harmonic numbers $S_r(N) = \sum_{i=1}^{N} i^{-r}$. E.g., for a given sum

$$\sum_{j_0=0}^{N-3}\sum_{j_1=0}^{N-3-j_0} \frac{(-1)^{j_1}\Gamma(5-\varepsilon)\Gamma\left(5+\frac{\varepsilon}{2}\right)}{(2+j_0+j_1)\Gamma\left(2-\frac{\varepsilon}{2}\right)\Gamma\left(3-\frac{\varepsilon}{2}\right)} \frac{\Gamma\left(2-\frac{\varepsilon}{2}+j_0\right)\Gamma\left(3-\frac{\varepsilon}{2}+j_1\right)\Gamma(4+j_0+j_1)\Gamma(N-j_0-1)}{\Gamma(1+j_1)\Gamma(5-\varepsilon+j_0+j_1)\Gamma\left(5+\frac{\varepsilon}{2}+j_0+j_1\right)\Gamma(N-j_0-j_1-2)}$$

in terms of the Γ-function our method derives the first coefficients of its ε-expansion

$$\frac{72\left(N^3-8N^2-27N-30\right)}{(N+1)^2(N+2)(N+3)} + \frac{288\left(N^2+N+6\right)S_1(N)}{N(N+1)(N+2)(N+3)} + \varepsilon\Bigg(\frac{72\left(N^2+N+6\right)S_1(N)^2}{N(N+1)(N+2)(N+3)} + \frac{72\left(N^2+9N+6\right)S_2(N)}{N(N+1)(N+2)(N+3)}$$
$$-\frac{12\left(43N^5+229N^4+581N^3+1031N^2+696N-180\right)S_1(N)}{N(N+1)^2(N+2)^2(N+3)^2} + \frac{3\left(11N^6+218N^5+760N^4+346N^3-2295N^2-4644N-2844\right)}{(N+1)^3(N+2)^2(N+3)^2}\Bigg) + O(\varepsilon^2).$$

This novel application of summation methods to Feynman integrals brings an algorithmic perspective to involved problems from theoretical particle physics [5] that arise, e.g., in [3]. We aim at optimizing and extending our procedures such that even larger problems of this type can be handled.

References

[1] J. Ablinger. A computer algebra toolbox for harmonic sums related to particle physics. Master's thesis, RISC, J. Kepler University, February 2009.

[2] S.A. Abramov and M. Petkovšek. D'Alembertian solutions of linear differential and difference equations. In J. von zur Gathen, editor, *Proc. ISSAC'94*, pages 169–174. ACM Press, 1994.

[3] I. Bierenbaum, J. Blümlein, S. Klein, and C. Schneider. Two-loop massive operator matrix elements for unpolarized heavy flavor production to $O(\epsilon)$. *Nucl. Phys. B*, 803(31-2):1–41, 2008.

[4] J. Blümlein. Structural relations of harmonic sums and Mellin transforms up to Weight $w = 5$. *Comput.Phys.Commun.*, 180:2218–2249, 2009.

[5] J. Blümlein, S. Klein, C. Schneider, and F. Stan. Symbolic summation methods for two loop Feynman parameter integrals. In preparation.

[6] J. Blümlein and S. Kurth. Harmonic sums and Mellin transforms up to two-loop order. *Phys. Rev.*, D60, 1999.

[7] M. Karr. Summation in finite terms. *J. ACM*, 28(2):305–350, 1981.

[8] C. Schneider. Symbolic summation assists combinatorics. *Sem. Lothar. Combin.*, 56:1–36, 2007.

[9] C. Schneider. Solving parameterized linear difference equations in terms of indefinite nested sums and products. *J. Differ. Equations Appl.*, 11(9):799–821, 2005.

[10] F. Stan. *Algorithms for Special Functions: Computer Algebra and Analytical Aspects.* PhD thesis, RISC, J. Kepler University Linz, June 2010.

[11] J.A.M. Vermaseren. Harmonic sums, Mellin transforms and integrals. *Int. J. Mod. Phys.*, A14:2037–2976, 1999.

[12] K. Wegschaider. Computer generated proofs of binomial multi-sum identities. Master's thesis, RISC, J. Kepler University, May 1997.

[13] H.S. Wilf and D. Zeilberger. An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities. *Invent. Math.*, 108(3):575–633, 1992.