Kick-off meeting of the LHCPhenoNet Initial Training Network

Multi-Summation for Particle Physics

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A warm up example:

$$\text{GIVEN } F(N)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\times$$
$$\times\underbrace{\left(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+N)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+N)\Gamma(k+j+2)}\right.
+\left.\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+N)\Gamma(k+j+2+\frac{\varepsilon}{2})}\right)}_{f(N,k,j)}.$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

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FIND the first coefficients of the ϵ-expansion

$$F(N)=F_0(N)+\varepsilon F_1(N)+\dots$$

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Step 1: Compute the first coefficients of the ϵ-expansion

$$f(N,k,j)=\qquad f_0(N,k,j)+\qquad \varepsilon f_1(N,k,j)+\dots$$

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A warm up example:

$$\text{GIVEN } F(N) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\times$$

$$\times\underbrace{\left(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+N)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+N)\Gamma(k+j+2)}\right.
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Step 2: Simplify the sums in

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f(N,k,j) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j) + \varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)+\dots$$

Arose in the context of

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Simplify the constant term

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\,(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

where

$$S_1(N)=\sum_{i=1}^{N}\frac{1}{i}$$

Simplify the constant term

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\,(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{j=0}^{a} f(N,k,j) =$$ ▸ Sigma

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FIND $g(j)$:

$$\boxed{f(N,k,j) = g(j+1) - g(j)}$$

Simplify the constant term

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!}+\frac{j!k!(j+k+N)!\,(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

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FIND $g(j)$:

$$\boxed{f(N,k,j)=g(j+1)-g(j)}$$

Sigma (based on a refined version of M. Karr's difference fields (1981)) computes

$$g(j)=\frac{(j+k+1)(j+N+1)j!k!(j+k+N)!\big(S_1(j)-S_1(j+k)-S_1(j+N)+S_1(j+k+N)\big)}{kN(j+k+1)!(j+N+1)!(k+N+1)!}$$

Simplify the constant term

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!}+\frac{j!k!(j+k+N)!\,(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{j=0}^{a} f(N,k,j) =$$ ▸ Sigma

FIND $g(j)$:

$$\boxed{f(N,k,j)=g(j+1)-g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\boxed{\sum_{j=0}^{a} f(N,k,j)=g(a+1)-g(0)}$$

Simplify the constant term

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!}}^{f(N,k,j)}$$
$$+\frac{j!k!(j+k+N)!\,(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)$$

$$\sum_{j=0}^{a} f(N,k,j)=\frac{(a+1)!(k-1)!(a+k+N+1)!(S_1(a)-S_1(a+k)-S_1(a+N)+S_1(a+k+N))}{N(a+k+1)!(a+N+1)!(k+N+1)!}$$
$$\underbrace{+\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}+\frac{(2a+k+N+2)a!k!(a+k+N)!}{(a+k+1)(a+N+1)(a+k+1)!(a+N+1)!(k+N+1)!}}_{a\to\infty}$$

Simplify the constant term

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\,(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{j=0}^{\infty} f(N,k,j) = \frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

Simplify the constant term

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\,(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{k=1}^{a}\sum_{j=0}^{\infty} f(N,k,j) = \sum_{k=1}^{a}\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

$=$ Sigma

Simplify the constant term

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!}}^{f(N,k,j)}$$
$$+\frac{j!k!(j+k+N)!\,(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}f(N,k,j)=\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$
$$=\frac{S_1(N)^2+S_2(N)}{2N(N+1)!}$$

where

$$S_2(N)=\sum_{i=1}^{N}\frac{1}{i^2}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$S(n)=\sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $S(n)$

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2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)S(n) + \dots + a_d(n)S(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products and sums
(Nörlund 24, Abramov/Petkovšek 94, Hendriks/Singer 99/Sigma 01)

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FIND all solutions expressible by indefinite nested products and sums
(Nörlund 24, Abramov/Petkovšek 94, Hendriks/Singer 99/Sigma 01)

3. Find a "closed form"

S(n)=combined solutions.

Simplify the constant term

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\overbrace{\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\left(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)\right)}{(j+k+1)!(j+N+1)!(k+N+1)!}}^{f(N,k,j)}\Big)$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(N,k,j) = \sum_{k=1}^{\infty}\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

$$= \frac{S_1(N)^2+S_2(N)}{2N(N+1)!}$$

where

$$S_2(N)=\sum_{i=1}^{N}\frac{1}{i^2}$$

Automatic machinery

"Background"

- The following examples arise in the context of 2– and 3–loop massive single scale Feynman diagrams with operator insertion.
- These are related to the QCD anomalous dimensions and massive operator matrix elements.
- At 2-loop order all respective calculations are finished:

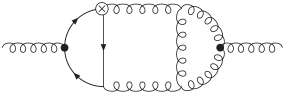
  M. Buza, Y. Matiounine, J. Smith, R. Migneron, W.L. van Neerven, Nucl. Phys. **B472** (1996) 611;
  I. Bierenbaum, J. Blümlein, S. Klein, Nucl. Phys. **B780** (2007) 40;
  I. Bierenbaum, J. Blümlein, S. Klein, C. Schneider, Nucl.Phys. **B803** (2008)

  and lead to representations in terms of harmonic sums.

Example 1: All N-Results for 3–Loop Ladder Graphs

Joint work with J. Ablinger (RISC), J. Blümlein (DESY),
A. Hasselhuhn (DESY), S. Klein (RWTH)

Consider, e.g., the diagram



(containing three massive fermion propagators)

$$\Downarrow$$

Around 1000 sums have to be calculated

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

Simple sum

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

$$||$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}}$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

$$||$$

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$$||$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^{r}(-j+N-2)!r!}{(N+1)(-j+N+r-1)(-j+N+r)!}+\frac{(-1)^{N+r}(j+1)!(-j+N-2)!(-j+N-1)_r r!}{(N-1)N(N+1)(-j+N+r)!(-j-1)_r(2-N)_j}\Big)}$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

$$||$$

$$\sum_{j=0}^{N-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+N-2)!r!}{(N+1)(-j+N+r-1)(-j+N+r)!}+\frac{(-1)^{N+r}(j+1)!(-j+N-2)!(-j+N-1)_r r!}{(N-1)N(N+1)(-j+N+r)!(-j-1)_r(2-N)_j}\Big)}$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

$$||$$

$$\sum_{j=0}^{N-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+N-2)!r!}{(N+1)(-j+N+r-1)(-j+N+r)!}+\frac{(-1)^{N+r}(j+1)!(-j+N-2)!(-j+N-1)_r r!}{(N-1)N(N+1)(-j+N+r)!(-j-1)_r(2-N)_j}\Big)}$$

$$||$$

$$\boxed{\Big(\frac{N^2-N+1}{(N-1)^2N^2(N+1)(2-N)_j}+\frac{\sum_{i=1}^{j}\frac{(2-N)_i}{(-i+N-1)^2(i+1)!}}{(N+1)(2-N)_j}+\frac{(-1)^{j+N}(-j-2)(-j+N-2)!}{(j-N+1)(N+1)^2N!}\Big)(j+1)!-\frac{1}{(N+1)^2(-j+N-1)}}$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

$$||$$

$$\sum_{j=0}^{N-2}\Bigg(\Bigg(\frac{N^2-N+1}{(N-1)^2N^2(N+1)(2-N)_j}+\frac{\sum_{i=1}^{j}\frac{(2-N)_i}{(-i+N-1)^2(i+1)!}}{(N+1)(2-N)_j}+$$

$$\frac{(-1)^{j+N}(-j-2)(-j+N-2)!}{(j-N+1)(N+1)^2N!}\Bigg)(j+1)!-\frac{1}{(N+1)^2(-j+N-1)}\Bigg)$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

$$||$$

$$\sum_{j=0}^{N-2}\Bigg(\Bigg(\frac{N^2-N+1}{(N-1)^2N^2(N+1)(2-N)_j}+\frac{\sum_{i=1}^{j}\frac{(2-N)_i}{(-i+N-1)^2(i+1)!}}{(N+1)(2-N)_j}+$$
$$\frac{(-1)^{j+N}(-j-2)(-j+N-2)!}{(j-N+1)(N+1)^2N!}\Bigg)(j+1)!-\frac{1}{(N+1)^2(-j+N-1)}\Bigg)$$

$$||$$

$$\frac{-N^2-N-1}{N^2(N+1)^3}+\frac{(-1)^N\left(N^2+N+1\right)}{N^2(N+1)^3}-\frac{2S_{-2}(N)}{N+1}+\frac{S_1(N)}{(N+1)^2}+\frac{S_2(N)}{-N-1}$$

Note: $S_a(N)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}}$, $a\in\mathbb{Z}\setminus\{0\}$.

A typical sum

$$\sum_{j=0}^{N-2}\sum_{s=1}^{j+1}\sum_{r=0}^{N+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+N+s-2}{r}(N-j)!(s-1)!\sigma!S_1(r+2)}{(N-r)(r+1)(r+2)(-j+N+\sigma+1)(-j+N+\sigma+2)(-j+N+s+\sigma)!}$$

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$$\sum_{j=0}^{N-2}\sum_{s=1}^{j+1}\sum_{r=0}^{N+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+N+s-2}{r}(N-j)!(s-1)!\sigma!S_1(r+2)}{(N-r)(r+1)(r+2)(-j+N+\sigma+1)(-j+N+\sigma+2)(-j+N+s+\sigma)!}$$

$$=\frac{\left(2N^2+6N+5\right)S_{-2}(N)^2}{2(N+1)(N+2)}+S_{-2,-1,2}(N)+S_{-2,1,-2}(N)$$
$$+\dots$$

where, e.g.,

$$S_{-2,1,-2}(N)=\sum_{i=1}^{N}\frac{(-1)^i\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{(-1)^k}{k^2}}{j}}{i^2}$$

Vermaseren 98;Blümlein/Kurth 99

A typical sum

$$\sum_{j=0}^{N-2}\sum_{s=1}^{j+1}\sum_{r=0}^{N+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+N+s-2}{r}(N-j)!(s-1)!\sigma!S_1(r+2)}{(N-r)(r+1)(r+2)(-j+N+\sigma+1)(-j+N+\sigma+2)(-j+N+s+\sigma)!}$$

$$=\frac{\left(2N^2+6N+5\right)S_{-2}(N)^2}{2(N+1)(N+2)}+S_{-2,-1,2}(N)+S_{-2,1,-2}(N)$$

$$+\dots-S_{2,1,1,1}(-1,2,\frac{1}{2},-1;N)+S_{2,1,1,1}(1,\frac{1}{2},1,2;N)$$

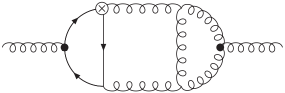
$$+\dots$$

where, e.g.,

145 S–sums occur

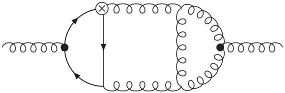
$$S_{2,1,1,1}(1,\frac{1}{2},1,2;N)=\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{\left(\frac{1}{2}\right)^j\sum_{k=1}^{j}\frac{\sum_{l=1}^{k}\frac{2^l}{l}}{k}}{j}}{i^2}$$

S. Moch, P. Uwer, S. Weinzierl 02



$\Downarrow$ Sigma.m

Around 1000 sums are calculated containing in total 533 S–sums



$\Downarrow$ Sigma.m

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$\Downarrow$ J. Ablinger's HarmonicSum.m

After elimination the following sums remain:

$$S_{-4}(N), S_{-3}(N), S_{-2}(N), S_1(N), S_2(N), S_3(N), S_4(N), S_{-3,1}(N),$$
$$S_{-2,1}(N), S_{2,-2}(N), S_{2,1}(N), S_{3,1}(N), S_{-2,1,1}(N), S_{2,1,1}(N)$$

For 3–loop ladder graphs we dealt (so far) with up to 6–fold sums. E.g.,

$$\sum_{l=2}^{N}\sum_{j=2}^{l}\sum_{k=1}^{j}\sum_{r=0}^{l-k}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{2(-1)^{j+k+l+r}(k-1)!\binom{j}{k}\binom{l}{j}\binom{l-k}{r}\binom{N}{l}}{(N-1)N(k+m+n+r+2)(k+m)!}$$
$$\frac{(k+m-1)!(N-j)!(l+r-2)!(n+r+1)!\,(k+m+n+r-1)!}{(-j+N+2)!(k+r-1)!(l+n+r-1)!\,(k+m+n+r+1)!}$$

$$\begin{aligned}
=\frac{1}{N(N+1)(N+2)}\Big(&2\big((3-2^{N+3})-(-1)^N\big)\zeta(3)\\
&+\tfrac{1}{6}S_1(N)^3+\tfrac{4(2N+3)}{(N+1)^2(N+2)}S_1(N)+\tfrac{8(2N+3)}{(N+1)^3(N+2)}\\
&-\frac{-56-40N-3N^2+2S_1(N)+3NS_1(N)+N^2S_1(N)}{2(1+N)(2+N)}S_2(N)\\
&+\tfrac{\left(16+12N+N^2\right)}{2(1+N)\ (2+N)}S_1(N)^2+\tfrac{1}{3}(-3N-17)S_3(N)\\
&-(-1)^NS_{-3}(N)+(-N-3)S_{2,1}(N)-2(-1)^NS_{-2,1}(N)\\
&+2^{N+4}S_{1,2}\big(\tfrac{1}{2},1,N\big)+2^{N+3}S_{1,1,1}\big(\tfrac{1}{2},1,1,N\big)\Big)
\end{aligned}$$

and ...

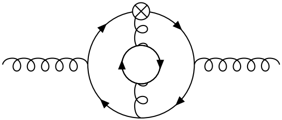
$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+N-3}\sum_{s=1}^{-l+N-q-3}\sum_{r=0}^{-l+N-q-s-3}\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{N-1}{j+2}\binom{-j+N-3}{q}\binom{-l+N-q-3}{s}\binom{-l+N-q-s-3}{r}r!(-l+N-q-r-s-3)!(s-1)!}{(-l+N-q-2)!}$$
$$\Bigg(\frac{\frac{2\ (-1)^{-j+k-l+N-q-3}(2S_1(-j+N-1)-S_1(-j+N-2)}{-j+N-1}-\frac{(-1)^{-j+k-l+N-q-3}S_1(k)}{-j+N-1}}{(N-q-r-s-2)(q+s+1)}$$
$$-\frac{(-1)^{-j+k-l+N-q-3}(S_1(-l+N-q-2)+S_1(-l+N-q-r-s-3)-2S_1(r+s))}{(-j+N-1)(N-q-r-s-2)(q+s+1)}$$
$$+\frac{2(-1)^{-j+k-l+N-q-3}(S_1(s-1)-S_1(r+s))}{(-j+N-1)(N-q-r-s-2)(q+s+1)}\Bigg)$$

$=$ polynomial expression in terms of 49 harmonic sums and S–sums

Example 2: 3-Loop All N-Results for the N_f Contributions

Joint work with J. Ablinger (RISC), J. Blümlein (DESY),
F. Wißbrock (DESY), S. Klein (RWTH)

E.g., for the diagram



768 sums are simplified.

Simple example:

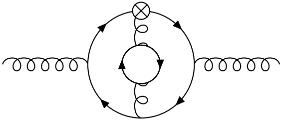
$$\sum_{j=1}^{N-2} \frac{j(j+1)(j+2)(N-j)(j-1)!^2(-j+N-1)!^2}{-j+N-1}$$
$$= \frac{\big(-N^3-5N^2-4N+6\big)(N!)^2}{(N-1)^2N^2}$$
$$+\frac{3}{2}\frac{(N!)^2\big(N^3+6N^2+11N+6\big)}{(N-1)N(2N+1)\binom{2N}{N}}\sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}.$$

Not expressible in terms of harmonic sums or S-sums!

The final expression is given in terms of 703 indefinite nested sums and products. Typical examples are:

$$\sum_{i=1}^{N} \frac{\displaystyle\sum_{j=1}^{i} \frac{\binom{2j}{j}}{j}}{\binom{2\,i}{i}}, \quad \sum_{i=1}^{N} \frac{S_1(i) \displaystyle\sum_{j=1}^{i} \frac{\binom{2j}{j}}{j}}{\binom{2i}{i}};$$

Sigma finds all algebraic relations among them. We get:



$$\Downarrow$$

$$
\begin{aligned}
&-\frac{20S(1,N)^4}{27(N+1)(N+2)}+\frac{32\left(6N^3+61N^2-21N+24\right)S_1(N)^3}{81N^2(N+1)(N+2)}-\frac{16\left(48N^5+746N^4+2697N^3+2746N^2+1104N+240\right)S_1(N)^2}{81N^2(N+1)^2(N+2)^2}\\
&+\frac{32\left(264N^7+4046N^6+21591N^5+52844N^4+74856N^3+66812N^2+30576N+2640\right)S_1(N)}{243N^2(N+1)^3(N+2)^3}\\
&-\frac{4\left(48N^2+101N+96\right)S_2(N)^2}{9N(N+1)(N+2)}-\frac{32\left(363N^7+6758N^6+41285N^5+121235N^4+190235N^3+150758N^2+46964N+2904\right)}{243N(N+1)^4(N+2)^3}\\
&+\Big(-\frac{40S_1(N)^2}{9(N+1)(N+2)}+\frac{32\left(6N^3+61N^2-21N+24\right)S_1(N)}{27N^2(N+1)(N+2)}\\
&-\frac{16\left(124N^5+198N^4-2387N^3-6162N^2-3632N-480\right)}{81N^2(N+1)^2(N+2)^2}\Big)S_2(N)+\\
&+\left(-\frac{32\left(9N^3-623N^2+894N+276\right)}{81N^2(N+1)(N+2)}-\frac{160S_1(N)}{27(N+1)(N+2)}\right)S_3(N)-\frac{8\left(56N^2+169N+112\right)S_4(N)}{9N(N+1)(N+2)}\\
&+\left(\frac{64S_1(N)}{3(N+1)(N+2)}-\frac{128\left(N^3+9N^2-10N-6\right)}{9N^2(N+1)(N+2)}\right)S_{2,1}(N)+\frac{64S_{3,1}(N)}{3(N+1)(N+2)}+\frac{64\left(3N^2+7N+6\right)}{3N(N+1)(N+2)}S_{2,1,1}(N)\\
&+\zeta(2)\left(\frac{8S_1(N)^2}{3(N+1)(N+2)}+\frac{16\left(3N^3-N^2+30N+12\right)S_1(N)}{9N^2(N+1)(N+2)}-\frac{16\left(3N^3+2N^2+17N+6\right)}{9N(N+1)^2(N+2)}-\frac{8\left(4N^2+9N+8\right)S_2(N)}{3N(N+1)(N+2)}\right)+\\
&+\zeta(3)\left(\frac{448}{9(N+1)(N+2)}-\frac{448S_1(N)}{9(N+1)(N+2)}\right)
\end{aligned}
$$

Concluding remarks

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Concluding remarks

- The Sigma-package is ready for many classes of 3–loop problems

- J. Ablinger's harmonic sum package facilitates this task (J. Blümlein 2009; J. Ablinger, J. Blümlein, C.S. 2011)
    - Algebraic relations of harmonic sums (up to weight 8) (covering also differentiation/half integer relations)

Weight	Number of							
	All	N_A	N_D	N_H	N_{AD}	N_{AH}	N_{DH}	N_{ADH}
1	2	2	2	1	2	1	1	1
2	6	3	4	4	1	2	3	1
3	18	8	12	14	5	6	10	4
4	54	18	36	46	10	15	32	9
5	162	48	108	146	30	42	100	27
6	486	116	324	454	68	107	308	65
7	1458	312	972	1394	196	294	940	187
8	4374	810	2916	4246	498	780	2852	486

Algebraic relations for S–sums: up to weight 6 (so far)

Concluding remarks

- The Sigma-package is ready for many classes of 3–loop problems
- J. Ablinger's harmonic sum package facilitates this task (J. Blümlein 2009; J. Ablinger, J. Blümlein, C.S. 2011)
  - asymptotic expansion of harmonic sums

$$
\begin{aligned}
S(3,2,1,n) = {} & \tfrac{143}{210}\zeta_2^3+3\zeta_3^2+\Big(\tfrac{-1}{n^2}+\tfrac{1}{n^3}-\tfrac{1}{2n^4}+\tfrac{1}{6n^6}-\tfrac{1}{6n^8}+\tfrac{3}{10n^{10}}-\tfrac{5}{6n^{12}}+\tfrac{691}{210n^{14}}-\tfrac{35}{2n^{16}}+\tfrac{3617}{30n^{18}}-\tfrac{43867}{42n^{20}}\Big)\zeta_3\\
&+\Big(\tfrac{1}{3n^3}-\tfrac{5}{8n^4}+\tfrac{37}{60n^5}-\tfrac{7}{24n^6}-\tfrac{37}{420n^7}+\tfrac{13}{80n^8}+\tfrac{118}{945n^9}-\tfrac{11}{42n^{10}}-\tfrac{785}{2772n^{11}}+\tfrac{169}{240n^{12}}\\
&+\tfrac{42376}{45045n^{13}}-\tfrac{725}{264n^{14}}-\tfrac{1664693}{386100n^{15}}+\tfrac{105723}{7280n^{16}}+\tfrac{19976612}{765765n^{17}}-\tfrac{399}{4n^{18}}-\tfrac{58623353743}{290990700n^{19}}+\tfrac{3519341}{4080n^{20}}\Big)(\log(n)+\gamma)\\
&+\tfrac{4}{9n^3}-\tfrac{15}{32n^4}+\tfrac{107}{1800n^5}+\tfrac{17}{48n^6}-\tfrac{3341}{9800n^7}-\tfrac{583}{9600n^8}+\tfrac{2464639}{9525600n^9}+\tfrac{24449}{141120n^{10}}-\tfrac{20380229}{42688800n^{11}}\\
&-\tfrac{10399}{17280n^{12}}+\tfrac{60101665187}{43286443200n^{13}}+\tfrac{643811}{232320n^{14}}-\tfrac{3703037408669}{649296648000n^{15}}-\tfrac{37263089359}{2248646400n^{16}}\\
&+\tfrac{585708761937371}{18764673127200n^{17}}+\tfrac{77481991}{617760n^{18}}-\tfrac{1482883119715209058 1}{677404699891920 00n^{19}}-\tfrac{102648938023}{87393600n^{20}}+O\Big(\frac{1}{n^{21}}\Big)
\end{aligned}
$$

Needed for limits (see first example) and for analytic continuation of harmonic sums

Concluding remarks

- The Sigma-package is ready for many classes of 3–loop problems
- J. Ablinger's harmonic sum package facilitates this task (J. Blümlein 2009; J. Ablinger, J. Blümlein, C.S. 2011)
- Alternative summation methods are available: e.g.,
  (J. Blümlein, M. Kauers, S. Klein, C.S; 2009)
  (J. Blümlein, S. Klein, C.S., F. Stan; 2010)