RISC/SCIEnce Training School in Symbolic Computation 2010

Sigma - A package for multi-summation

Carsten Schneider

RISC-Linz, Austria

Some literature

- For hypergeometric terms:

  R.W. Gosper. *Decision procedures for indefinite hypergeometric summation*, Proc. Nat. Acad. Sci. U.S.A.", pp 40-42, 1978.

Some literature

- For hypergeometric terms:

  R.W. Gosper. *Decision procedures for indefinite hypergeometric summation*, Proc. Nat. Acad. Sci. U.S.A.", pp 40-42, 1978.

  $A = B$, M. Petkovšek, H. S. Wilf, and D. Zeilberger, 1996. Available at `http://www.math.upenn.edu/~wilf/AeqB.html`

Some literature

- For hypergeometric terms:

  R.W. Gosper. *Decision procedures for indefinite hypergeometric summation*, Proc. Nat. Acad. Sci. U.S.A.", pp 40-42, 1978.

  $A = B$, M. Petkovšek, H. S. Wilf, and D. Zeilberger, 1996. Available at `http://www.math.upenn.edu/~wilf/AeqB.html`

- For difference fields:

  M. Karr. Summation in finite terms. *J. ACM*, 28:305–350, 1981.

Some literature

- For hypergeometric terms:

  R.W. Gosper. *Decision procedures for indefinite hypergeometric summation*, Proc. Nat. Acad. Sci. U.S.A.", pp 40-42, 1978.

  $A = B$, M. Petkovšek, H. S. Wilf, and D. Zeilberger, 1996. Available at `http://www.math.upenn.edu/∼wilf/AeqB.html`

- For difference fields:

  M. Karr. Summation in finite terms. *J. ACM*, 28:305–350, 1981.

  C. Schneider. Symbolic summation assists combinatorics. *Sém. Lothar. Combin.*, 56:1–35, 2007. Available at `http://www.mat.univie.ac.at/∼slc/`

Warmup example

FIND a closed form for

$$\sum_{k=1}^{n} \mathsf{H}_k = \boxed{?},$$

where $\mathsf{H}_n := \sum_{k=1}^{n} \frac{1}{k}$.

Telescoping

GIVEN $f(k) = H_k$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Telescoping

GIVEN $f(k) = H_k$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Sigma computes

$$g(k) = (H_k - 1)k.$$

Telescoping

GIVEN $f(k) = H_k$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Summing this equation over k from 1 to n gives

$$\boxed{\sum_{k=1}^{n} \mathsf{H}_k} = \boxed{g(n+1) - g(1)}$$

$$= (H_{n+1} - 1)(n+1).$$

Telescoping

GIVEN $f(k) = H_k$.
FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Summing this equation over k from 1 to n gives

$$\boxed{\sum_{k=1}^{n} \mathrm{H}_k} = \boxed{g(n+1) - g(1)}$$

$$= (H_{n+1} - 1)(n+1).$$

Try it out!

Telescoping

FIND a closed form for

$$\sum_{k=1}^{n} H_{\mathsf{k}}.$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

.

Telescoping

FIND a closed form for

$$\sum_{k=1}^{n} H_{k}.$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$
$$\sigma(k) = k + 1, \qquad\qquad \mathcal{S}\,k = k + 1,$$

.

Telescoping

FIND a closed form for

$$\sum_{k=1}^{n} H_k.$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

$$\sigma(h) = h + \frac{1}{k+1}, \qquad \mathcal{S}\, H_k = H_k + \frac{1}{k+1}.$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

Sigma computes

$$g = (h-1)k \in \mathbb{F}.$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

Sigma computes

$$g = (h-1)k \in \mathbb{F}.$$

This gives

$$\boxed{g(k+1) - g(k) = H_k}$$

with

$$g(k) = (H_k - 1)k.$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

Sigma computes

$$g = (h-1)k \in \mathbb{F}.$$

This gives

$$\boxed{g(k+1) - g(k) = H_k}$$

with

$$g(k) = (H_k - 1)k.$$

Hence,

$$(H_{n+1} - 1)(n+1) = \sum_{k=1}^{n} H_k.$$

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

Denominator bound: COMPUTE a polynomial $0 \neq d \in \mathbb{Q}(k)[h]$:

$d = 1$

$$\forall g \in \mathbb{Q}(k)(h): \quad \sigma(g) - g = h \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[h].$$

FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

Denominator bound: COMPUTE a polynomial $0 \neq d \in \mathbb{Q}(k)[h]$:

$$d = 1$$

$$\forall g \in \mathbb{Q}(k)(h): \quad \sigma(g) - g = h \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[h].$$

FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

Degree bound: COMPUTE $b \geq 0$:

$$b = 2$$

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq b.$$

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

Denominator bound: COMPUTE a polynomial $0 \neq d \in \mathbb{Q}(k)[h]$:

$$\forall g \in \mathbb{Q}(k)(h): \quad \sigma(g) - g = h \quad \Rightarrow \quad g\, d \in \mathbb{Q}(k)[h].$$

$d = 1$

FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma\left(\frac{g'}{d}\right) - \frac{g'}{d} = h.$$

Degree bound: COMPUTE $b \geq 0$:

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq b.$$

$b = 2$

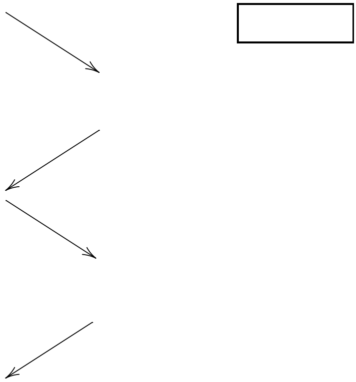
Polynomial Solution: FIND

$$g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h].$$

g=hk-k

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\sigma(g) - g = h$$



ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h+\tfrac{1}{k+1}\big)^2+\sigma(g_1 h+g_0)\Big] \\ -\big[g_2\, h^2+g_1 h+g_0\big]=h$$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\left[\sigma(g_2)\left(h+\tfrac{1}{k+1}\right)^2+\sigma(g_1h+g_0)\right]$$
$$-\left[g_2\,h^2+g_1h+g_0\right]=h$$

coeff. comp.

$$\boxed{\sigma(g_2)-g_2=0}$$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h+\tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\,h^2 + g_1 h + g_0\big] = h$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$$g_2 = c \in \mathbb{Q}$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\left[\sigma(g_2)\left(h+\tfrac{1}{k+1}\right)^2+\sigma(g_1 h+g_0)\right] - \left[g_2\, h^2+g_1 h+g_0\right] = h$$

coeff. comp.

$$\sigma(g_2)-g_2=0$$

$$g_2 = c \in \mathbb{Q}$$

$$\sigma(g_1\, h+g_0)-(g_1\, h+g_0) = h - c\left[\tfrac{2h(k+1)+1}{(k+1)^2}\right]$$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h+\tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\,h^2 + g_1 h + g_0\big] = h$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\,h + g_0) - (g_1\,h + g_0) = h - c\Big[\tfrac{2h(k+1)+1}{(k+1)^2}\Big]$$

coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}}$$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\left[\sigma(g_2)\left(h+\tfrac{1}{k+1}\right)^2 + \sigma(g_1 h + g_0)\right] - \left[g_2\,h^2 + g_1 h + g_0\right] = h$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\,h + g_0) - (g_1\,h + g_0) = h - c\left[\tfrac{2h(k+1)+1}{(k+1)^2}\right]$$

coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}}$$

$c = 0,\quad g_1 = k + d,\ d \in \mathbb{Q}$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\left[\sigma(g_2)\left(h+\tfrac{1}{k+1}\right)^2 + \sigma(g_1 h + g_0)\right] - \left[g_2\,h^2 + g_1 h + g_0\right] = h$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\,h + g_0) - (g_1\,h + g_0) = h - c\left[\tfrac{2h(k+1)+1}{(k+1)^2}\right]$$

coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}}$$

$c = 0$, $g_1 = k + d$, $d \in \mathbb{Q}$

$$\boxed{\sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}}$$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h+\tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\,h^2 + g_1 h + g_0\big] = h$$

coeff. comp.

g=hk-k

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$$g_2 = c \in \mathbb{Q}$$

$$\sigma(g_1\,h + g_0) - (g_1\,h + g_0) = h - c\Big[\tfrac{2h(k+1)+1}{(k+1)^2}\Big]$$

coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}}$$

$c = 0,\quad g_1 = k + d,\ d \in \mathbb{Q}$

$$g_0 = -k,\ d = 0 \quad \leftarrow \quad \boxed{\sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}}$$

The general case (Karr's algorithm)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:
a rational function field $\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$ with an automorphism

The general case (Karr's algorithm)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:
a rational function field $\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$ with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

The general case (Karr's algorithm)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F},\sigma)$:
a rational function field $\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$ with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$
$$\sigma(t_1) = \alpha_1\, t_1 + \beta_1, \qquad \alpha_1 \in \mathbb{K}^*, \qquad \beta_1 \in \mathbb{K}$$

The general case (Karr's algorithm)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F},\sigma)$:
a rational function field $\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$ with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

$$\sigma(t_1) = \alpha_1\, t_1 + \beta_1, \qquad \alpha_1 \in \mathbb{K}^*, \qquad \beta_1 \in \mathbb{K}$$

$$\sigma(t_2) = \alpha_2\, t_2 + \beta_2, \qquad \alpha_2 \in \mathbb{K}(t_1)^*, \qquad \beta_2 \in \mathbb{K}(t_1)$$

The general case (Karr's algorithm)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F},\sigma)$:
a rational function field $\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$ with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= \alpha_1\,t_1 + \beta_1, && \alpha_1 \in \mathbb{K}^*, && \beta_1 \in \mathbb{K} \\
\sigma(t_2) &= \alpha_2\,t_2 + \beta_2, && \alpha_2 \in \mathbb{K}(t_1)^*, && \beta_2 \in \mathbb{K}(t_1) \\
\sigma(t_3) &= \alpha_3\,t_3 + \beta_3, && \alpha_3 \in \mathbb{K}(t_1,t_2)^*, && \beta_3 \in \mathbb{K}(t_1,t_2) \\
&\vdots
\end{aligned}$$

The general case (Karr's algorithm)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F},\sigma)$:
a rational function field $\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$ with an automorphism

$$
\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= \alpha_1\, t_1 + \beta_1, && \alpha_1 \in \mathbb{K}^*, && \beta_1 \in \mathbb{K} \\
\sigma(t_2) &= \alpha_2\, t_2 + \beta_2, && \alpha_2 \in \mathbb{K}(t_1)^*, && \beta_2 \in \mathbb{K}(t_1) \\
\sigma(t_3) &= \alpha_3\, t_3 + \beta_3, && \alpha_3 \in \mathbb{K}(t_1,t_2)^*, && \beta_3 \in \mathbb{K}(t_1,t_2) \\
&\vdots
\end{aligned}
$$

such that $\{c \in \mathbb{K}(t_1)\dots(t_e) | \sigma(c) = c\} = \mathbb{K}$.

The general case (Karr's algorithm)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F},\sigma)$:
a rational function field $\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$ with an automorphism

$$
\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= \alpha_1\, t_1 + \beta_1, && \alpha_1 \in \mathbb{K}^*, && \beta_1 \in \mathbb{K} \\
\sigma(t_2) &= \alpha_2\, t_2 + \beta_2, && \alpha_2 \in \mathbb{K}(t_1)^*, && \beta_2 \in \mathbb{K}(t_1) \\
\sigma(t_3) &= \alpha_3\, t_3 + \beta_3, && \alpha_3 \in \mathbb{K}(t_1,t_2)^*, && \beta_3 \in \mathbb{K}(t_1,t_2) \\
&\vdots
\end{aligned}
$$

such that $\{c \in \mathbb{K}(t_1)\dots(t_e) | \sigma(c) = c\} = \mathbb{K}$.

GIVEN $f \in \mathbb{F}$;

FIND $g \in \mathbb{F}$ with

$$\sigma(g) - g = f.$$

The general case (Karr's algorithm)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F},\sigma)$:
a rational function field $\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$ with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= \alpha_1\, t_1 + \beta_1, && \alpha_1 \in \mathbb{K}^*, && \beta_1 \in \mathbb{K} \\
\sigma(t_2) &= \alpha_2\, t_2 + \beta_2, && \alpha_2 \in \mathbb{K}(t_1)^*, && \beta_2 \in \mathbb{K}(t_1) \\
\sigma(t_3) &= \alpha_3\, t_3 + \beta_3, && \alpha_3 \in \mathbb{K}(t_1,t_2)^*, && \beta_3 \in \mathbb{K}(t_1,t_2) \\
&\vdots
\end{aligned}$$

such that $\{c \in \mathbb{K}(t_1)\dots(t_e) | \sigma(c) = c\} = \mathbb{K}$.

GIVEN $f \in \mathbb{F}$; (For refined versions see below)

FIND $g \in \mathbb{F}$ with

$$\sigma(g) - g = f.$$

telescoping

- GIVEN

$$\sum_{k=0}^{n} f(k).$$

- FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

telescoping

- GIVEN

$$\sum_{k=0}^{n} f(k).$$

- FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

- THEN (with some mild extra conditions)

$$\sum_{k=0}^{n} f(k) = g(n+1) - g(0)$$

Refined telescoping

- GIVEN
$$\sum_{k=0}^{n} f(k).$$
- FIND $g(k)$ and $f^*(k)$:
$$\boxed{f(k) = g(k+1) - g(k) + f^*(k)}$$
  where $f^*(k)$ is simpler than $f(k)$.
- THEN (with some mild extra conditions)
$$\sum_{k=0}^{n} f(k) = g(n+1) - g(0) + \sum_{k=0}^{n} f^*(k).$$

Degree optimal w.r.t the top extension

$$\sum_{k=1}^{n} \frac{1}{k(k+1)^2(k+2)^3} =$$

$$\sum_{k=2}^{n} \frac{2 - kH_k + H_k^4 - kH_k^5}{H_k - kH_k^2} =$$

Sigma

$$\sum_{k=1}^{n} H_k^4 =$$

Degree optimal w.r.t the top extension

$$\sum_{k=1}^{n}\frac{1}{k(k+1)^2(k+2)^3} = -\frac{1}{4}\sum_{k=1}^{n}\frac{9k+2}{k^3} + \frac{n\left(69n^5+585n^4+1967n^3+3283n^2+2728n+904\right)}{16(n+1)^3(n+2)^3}$$

$$\sum_{k=2}^{n}\frac{2-kH_k+H_k^4-kH_k^5}{H_k-kH_k^2} = \sum_{k=2}^{n}\frac{k^2+H_k}{k^2H_k} + (n+1)H_n^3-(2n+1)\left(\tfrac{3}{2}H_n^2-3H_n+\tfrac{3}{2}\right)+\tfrac{1}{H_n}$$

$$\sum_{k=1}^{n}H_k^4 = -H_n^{(3)}-2H_n^{(2)}+2\sum_{k=1}^{n}\frac{H_k}{k^2} + (n+1)H_n^4-2(2n+1)H_n^3+6(2n+1)H_n^2-12(2n+1)H_n+24n.$$

Simpler w.r.t. the depth

$$\sum_{k=1}^{n} H_k^2 H_k^{(2)} =$$

$$\sum_{k=0}^{a}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^2 =$$

Sigma

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{H_j}{j^2}}{k^3} =$$

Simpler w.r.t. the depth

$$\sum_{k=1}^{n} H_k^2 H_k^{(2)} = \frac{1}{3} H_n^{(3)} - \frac{1}{3} H_n^3 + \left((n+1) H_n^{(2)} + 1\right) H_n^2 + (2n+1)(1 - H_n) H_n^{(2)} - 2H_n$$

$$\sum_{k=0}^{a} \left(\sum_{i=0}^{k} \binom{n}{i}\right)^2 = (n-a)\binom{n}{a} \sum_{i=0}^{a} \binom{n}{i} - \frac{n-2a-2}{2} \left(\sum_{i=0}^{a} \binom{n}{i}\right)^2 - \frac{n}{2} \sum_{i=1}^{a} \binom{n}{i}^2$$

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{H_j}{j^2}}{k^3} =$$

$$= H_n^{(3)} \sum_{j=1}^{n} \frac{H_j}{j^2} - \sum_{j=1}^{n} \frac{H_j H_j^{(3)}}{j^2} + \sum_{j=1}^{n} \frac{H_j}{j^5}$$

Simpler w.r.t. the depth

$$\sum_{k=1}^{n} H_k^2 H_k^{(2)} = \tfrac{1}{3} H_n^{(3)} - \tfrac{1}{3} H_n^3 + \Big((n+1) H_n^{(2)} + 1\Big) H_n^2 + (2n+1)(1 - H_n) H_n^{(2)} - 2H_n$$

$$\sum_{k=0}^{a} \left(\sum_{i=0}^{k} \binom{n}{i} \right)^2 = (n-a) \binom{n}{a} \sum_{i=0}^{a} \binom{n}{i} - \frac{n-2a-2}{2} \left(\sum_{i=0}^{a} \binom{n}{i} \right)^2 - \frac{n}{2} \sum_{i=1}^{a} \binom{n}{i}^2$$

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{H_j}{j^2}}{k^3} = S(3,2,1,N) \qquad \text{(Harmonic sum)}$$

$$= H_n^{(3)} \sum_{j=1}^{n} \frac{H_j}{j^2} - \sum_{j=1}^{n} \frac{H_j H_j^{(3)}}{j^2} + \sum_{j=1}^{n} \frac{H_j}{j^5} \qquad \text{(Euler sums)}$$

Further examples

$$\sum_{k=1}^{n} H_k^3 = \tfrac{1}{2}\Big(2(n+1)H_n^3 - 3(2n+1)H_n^2 + 6(2n+1)H_n - 12n - 1 + H_n^{(2)}\Big)$$

$$\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j}\sum_{i=1}^{j}\frac{1}{i} = \frac{1}{6}\left[H_n^3 + 3H_n\,H_n^{(2)} + 2\,H_n^{(3)}\right]$$

$$\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\frac{H_j^{(2)}}{j^3}\right)^2 = -\,({H_n^{(2)}}^2 + H_n^{(4)})\sum_{j=1}^{n}\frac{H_j^{(2)}}{j^3} + (n+1)\left(\sum_{j=1}^{n}\frac{H_j^{(2)}}{j^3}\right)^2 + \sum_{j=1}^{n}\frac{{H_j^{(2)}}^3}{j^3} - \sum_{j=1}^{n}\frac{{H_j^{(2)}}^2}{j^5} + \sum_{j=1}^{n}\frac{H_j^{(2)}H_j^{(4)}}{j^3}.$$

telescoping

GIVEN

$$S(n)=\sum_{k=0}^{n}\overbrace{\binom{n}{k}H_n}^{=:\,f(n,k)}\quad.$$

FIND $g(n,k)$:

$$g(n,k+1)-g(n,k)=\qquad f(n,k)$$

no solution ☹

Creative telescoping

GIVEN

$$S(n) = \sum_{k=0}^{n} \overbrace{\binom{n}{k} H_n}^{=:\, f(n,k)} .$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$g(n,k+1) - g(n,k) = c_0(n) f(n,k) + c_1(n) f(n+1,k)$$

no solution ☹

Creative telescoping

GIVEN

$$S(n)=\sum_{k=0}^{n}\overbrace{\binom{n}{k}H_n}^{=:\,f(n,k)}\quad.$$

FIND $g(n,k)$ and $c_0(n),c_1(n),c_2(n)$:

$$g(n,k+1)-g(n,k)\;=\;c_0(n)f(n,k)+c_1(n)f(n+1,k)+c_2(n)f(n+2,k).$$

Sigma *computes*:

$$c_0(n):=4\,(1+n),\quad c_1(n):=-2\,(3+2\,n),\quad c_2(n):=2+n,$$

$$g(n,k):=\frac{(1+n)\,(-2+k-n+(2\,k-2\,k^2+k\,n)H_k)\,\binom{n}{k}}{(1-k+n)\,(2-k+n)}.$$

Creative telescoping

GIVEN

$$S(n)=\sum_{k=0}^{n}\overbrace{\binom{n}{k}H_n}^{=:\,f(n,k)}\quad.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$g(n,k+1)-g(n,k)\;=\;c_0(n)f(n,k)+c_1(n)f(n+1,k)+c_2(n)f(n+2,k).$$

Summing this equation over k from 0 to n gives:

$$\begin{aligned}g(n,n+1)-g(n,0)=&c_0(n)\,S(n)+\\&c_1(n)\,\Big[S(n+1)-f(n+1,n+1)\Big]+\\&c_2(n)\Big[S(n+2)-f(n+2,n+1)-f(n+2,n+2)\Big]\end{aligned}$$

FIND a recurrence for

$$S(n) := \sum_{k=1}^{n} \binom{n}{k} H_k.$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(n)(k)(h)(b)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

$$\sigma(h) = h + \frac{1}{k+1}, \qquad \mathcal{S}\, H_k = H_k + \frac{1}{k+1},$$

$$\sigma(b) = \frac{n-k}{k+1}\, b, \qquad \mathcal{S} \binom{n}{k} = \frac{n-k}{k+1} \binom{n}{k}.$$

REPRESENT $f(n+i,k)$ in $\mathbb{F}$:

$$f(n,k) = \mathsf{H}_k \binom{n}{k} \quad \longleftrightarrow \quad h\,b =: f_0$$

REPRESENT $f(n+i,k)$ in $\mathbb{F}$:

$$f(n,k) = \mathsf{H}_k \binom{n}{k} \quad \longleftrightarrow \quad h\,b =: f_0$$

$$f(n+1,k) = \frac{(n+1)\,\mathsf{H}_k \binom{n}{k}}{n+1-k} \quad \longleftrightarrow \quad \frac{(n+1)\,h\,b}{n+1-k} =: f_1$$

$$f(n+2,k) = \frac{(n+1)\,(n+2)\,\mathsf{H}_k \binom{n}{k}}{(n+1-k)\,(n+2-k)} \quad \longleftrightarrow \quad \frac{(n+1)\,(n+2)\,h\,b}{(n+1-k)\,(n+2-k)} =: f_2.$$

REPRESENT $f(n+i,k)$ in $\mathbb{F}$:

$$f(n,k)=\mathsf{H}_k\binom{n}{k}\qquad\longleftrightarrow\qquad h\,b=:f_0$$

$$f(n+1,k)=\frac{(n+1)\,\mathsf{H}_k\binom{n}{k}}{n+1-k}\qquad\longleftrightarrow\qquad\frac{(n+1)\,h\,b}{n+1-k}=:f_1$$

$$f(n+2,k)=\frac{(n+1)\,(n+2)\,\mathsf{H}_k\binom{n}{k}}{(n+1-k)\,(n+2-k)}\qquad\longleftrightarrow\qquad\frac{(n+1)\,(n+2)\,h\,b}{(n+1-k)\,(n+2-k)}=:f_2.$$

FIND $c_0,c_1,c_2\in\mathbb{Q}(n)$ and $g\in\mathbb{F}$:

$$\boxed{\sigma(g)-g=c_0f_0+c_1f_1+c_2f_2.}$$

REPRESENT $f(n+i,k)$ in $\mathbb{F}$:

$$f(n,k) = \mathsf{H}_k \binom{n}{k} \quad \longleftrightarrow \quad h\,b =: f_0$$

$$f(n+1,k) = \frac{(n+1)\,\mathsf{H}_k \binom{n}{k}}{n+1-k} \quad \longleftrightarrow \quad \frac{(n+1)\,h\,b}{n+1-k} =: f_1$$

$$f(n+2,k) = \frac{(n+1)\,(n+2)\,\mathsf{H}_k \binom{n}{k}}{(n+1-k)\,(n+2-k)} \quad \longleftrightarrow \quad \frac{(n+1)\,(n+2)\,h\,b}{(n+1-k)\,(n+2-k)} =: f_2.$$

FIND $c_0, c_1, c_2 \in \mathbb{Q}(n)$ and $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = c_0 f_0 + c_1 f_1 + c_2 f_2.}$$

Sigma computes

$$c_0 := 4\,(1+n), \quad c_1 := -2\,(3+2\,n), \quad c_2 := 2+n,$$

$$g := \frac{(1+n)\,(-2+k-n+(2\,k-2\,k^2+k\,n)h)\,b}{(1-k+n)\,(2-k+n)}.$$

REPRESENT $f(n+i,k)$ in $\mathbb{F}$:

$$f(n,k)=\mathsf{H}_k\binom{n}{k}\quad\longleftrightarrow\quad h\,b=:f_0$$

$$f(n+1,k)=\frac{(n+1)\,\mathsf{H}_k\binom{n}{k}}{n+1-k}\quad\longleftrightarrow\quad\frac{(n+1)\,h\,b}{n+1-k}=:f_1$$

$$f(n+2,k)=\frac{(n+1)\,(n+2)\,\mathsf{H}_k\binom{n}{k}}{(n+1-k)\,(n+2-k)}\quad\longleftrightarrow\quad\frac{(n+1)\,(n+2)\,h\,b}{(n+1-k)\,(n+2-k)}=:f_2.$$

This gives

$$\boxed{g(n,k+1)-g(n,k)=c_0(n)f(n,k)+c_1(n)f(n+1,k)+c_2(n)f(n+2,k)}$$

with

$$c_0(n):=4\,(1+n),\quad c_1(n):=-2\,(3+2\,n),\quad c_2(n):=2+n,$$

$$g(n,k):=\frac{(1+n)\,(-2+k-n+(2\,k-2\,k^2+k\,n)H_k)\binom{n}{k}}{(1-k+n)\,(2-k+n)}.$$

REPRESENT $f(n+i,k)$ in $\mathbb{F}$:

$$f(n,k) = \mathsf{H}_k \binom{n}{k} \quad \longleftrightarrow \quad h\,b =: f_0$$

$$f(n+1,k) = \frac{(n+1)\,\mathsf{H}_k \binom{n}{k}}{n+1-k} \quad \longleftrightarrow \quad \frac{(n+1)\,h\,b}{n+1-k} =: f_1$$

$$f(n+2,k) = \frac{(n+1)\,(n+2)\,\mathsf{H}_k \binom{n}{k}}{(n+1-k)\,(n+2-k)} \quad \longleftrightarrow \quad \frac{(n+1)\,(n+2)\,h\,b}{(n+1-k)\,(n+2-k)} =: f_2.$$

Summing over k from 0 to n gives

$$\boxed{1 = 4(1+n)S(n) - 2(3+2n)S(n+1) + (2+n)S(n+2)}$$

for

$$S(n) = \sum_{k=0}^{n} \binom{n}{k} H_k.$$

Sigma

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$S(n)=\sum_{k=0}^{n}f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $S(n)$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$S(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $S(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$: indefinite nested product-sum expressions.

$$a_0(n)S(n) + \dots + a_d(n)S(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products and sums
(Nörlund 24, Petkovšek 92, Abramov/Petkovšek 94, Hendriks/Singer 99/Sigma 01)

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$S(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $S(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)S(n) + \dots + a_d(n)S(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products and sums
(Nörlund 24, Petkovšek 92, Abramov/Petkovšek 94, Hendriks/Singer 99/Sigma 01)

NOTE: By construction, the solutions are highly nested.

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$S(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $S(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)S(n) + \dots + a_d(n)S(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products and sums
(Nörlund 24, Petkovšek 92, Abramov/Petkovšek 94, Hendriks/Singer 99/Sigma 01)

3. Indefinite summation (by Sigma's refined summation theory of $\Pi\Sigma^*$-fields)

Simplify the solutions:

- The sums have minimal nested depth.
- No algebraic relations occur among the sums.

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$S(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $S(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)S(n) + \dots + a_d(n)S(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products and sums
(Nörlund 24, Petkovšek 92, Abramov/Petkovšek 94, Hendriks/Singer 99/Sigma 01)

4. Find a "closed form"

S(n)=combined solutions.

Real life problem: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!}}^{f(N,k,j)}$$
$$+\frac{j!k!(j+k+N)!\left(-H_j+H_{j+k}+H_{j+N}-H_{j+k+N}\right)}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)$$

where

$$H_N=\sum_{i=1}^{N}\frac{1}{i}$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

Real life problem: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\left(-H_j+H_{j+k}+H_{j+N}-H_{j+k+N}\right)}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{j=0}^{a} f(N,k,j) =$$ Sigma

Real life problem: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!}+\frac{j!k!(j+k+N)!\left(-H_j+H_{j+k}+H_{j+N}-H_{j+k+N}\right)}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{j=0}^{a} f(N,k,j) =$$ ▸ Sigma

FIND $g(j)$:

$$\boxed{f(N,k,j)=g(j+1)-g(j)}$$

Real life problem: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!}+\frac{j!k!(j+k+N)!\,(-H_j+H_{j+k}+H_{j+N}-H_{j+k+N})}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{j=0}^{a} f(N,k,j) =$$ ▸ Sigma

FIND $g(j)$:

$$\boxed{f(N,k,j)=g(j+1)-g(j)}$$

Sigma computes

$$g(j)=\frac{(j+k+1)(j+N+1)j!k!(j+k+N)!\Big(H_j-Hj+k-Hj+N+H_{j+k+N}\Big)}{kN(j+k+1)!(j+N+1)!(k+N+1)!}$$

Real life problem: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!}+\frac{j!k!(j+k+N)!\left(-H_j+H_{j+k}+H_{j+N}-H_{j+k+N}\right)}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{j=0}^{a} f(N,k,j) =$$ ▸ Sigma

FIND $g(j)$:

$$\boxed{f(N,k,j)=g(j+1)-g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\boxed{\sum_{j=0}^{a} f(N,k,j)=g(a+1)-g(0)}$$

Real life problem: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!}+\frac{j!k!(j+k+N)!\,(-H_j+H_{j+k}+H_{j+N}-H_{j+k+N})}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{j=0}^{a} f(N,k,j) = \underbrace{\frac{(a+1)!(k-1)!(a+k+N+1)!(H_a-H_{a+k}-H_{a+N}+H_{a+k+N})}{N(a+k+1)!(a+N+1)!(k+N+1)!}+\frac{H_k+H_N-H_{k+N}}{kN(k+N+1)N!}+\frac{(2a+k+N+2)a!k!(a+k+N)!}{(a+k+1)(a+N+1)(a+k+1)!(a+N+1)!(k+N+1)!}}_{a\,\to\,\infty}$$

Real life problem: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\left(-H_j+H_{j+k}+H_{j+N}-H_{j+k+N}\right)}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{j=0}^{\infty} f(N,k,j) = \frac{H_k+H_N-H_{k+N}}{kN(k+N+1)N!}$$

Real life problem: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\left(-H_j+H_{j+k}+H_{j+N}-H_{j+k+N}\right)}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{k=1}^{a}\sum_{j=0}^{\infty}f(N,k,j)=\sum_{k=1}^{a}\frac{H_k+H_N-H_{k+N}}{kN(k+N+1)N!}$$

$$= \text{Sigma}$$

Real life problem: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\,(-H_j+H_{j+k}+H_{j+N}-H_{j+k+N})}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(N,k,j) = \sum_{k=1}^{\infty}\frac{H_k+H_N-H_{k+N}}{kN(k+N+1)N!} = \frac{H_N^2+H_N^{(2)}}{2N(N+1)!}$$

where

$$H_N^{(2)} = \sum_{i=1}^{N}\frac{1}{i^2}$$