The Renaissance of Combinatorics: Advances – Algorithms – Applications

Center of Combinatorics, Nankai University, Tianjin

A Symbolic Summation Toolbox to Evaluate 3–loop Feynman Integrals

Carsten Schneider
RISC, J. Kepler University Linz, Austria

16. August 2010

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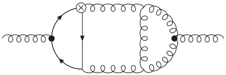
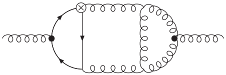
Evaluation of Feynman Integrals



Feynman diagrams

Evaluation of Feynman Integrals

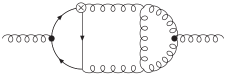


Feynman diagrams

$$\longrightarrow \quad \int \Phi(N,\epsilon,x)dx$$

Feynman integrals

Evaluation of Feynman Integrals



Feynman diagrams

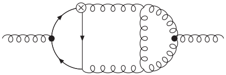
$$\int \Phi(N, \epsilon, x)dx$$

Feynman integrals

Reduction

multi-sums with
integer parameter N

Evaluation of Feynman Integrals



Feynman diagrams

$$\int \Phi(N, \epsilon, x)dx$$

Feynman integrals

Reduction

multi-sums with
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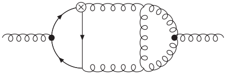
Sigma

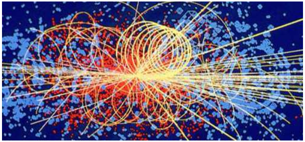
sum expressions
being processable by physicists

Evaluation of Feynman Integrals



Feynman diagrams → $\int \Phi(N, \epsilon, x)dx$

Feynman integrals



Reduction ↓

multi-sums with integer parameter N

Sigma ←

sum expressions being processable by physicists

A warm up example:

$$\text{GIVEN } F(N)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\times$$
$$\times\underbrace{\left(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+N)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+N)\Gamma(k+j+2)}\right.
+\left.\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+N)\Gamma(k+j+2+\frac{\varepsilon}{2})}\right)}_{f(N,k,j)}.$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

A warm up example:

GIVEN $F(N)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\times$

$$\times\underbrace{\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+N)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+N)\Gamma(k+j+2)} + \frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+N)\Gamma(k+j+2+\frac{\varepsilon}{2})}}_{f(N,k,j)}\Bigg).$$

FIND the first coefficients of the ϵ-expansion

$$F(N)=F_0(N)+\varepsilon F_1(N)+\dots$$

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A warm up example:

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+\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+N)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg)}_{f(N,k,j)}.$$

Step 1: Compute the first coefficients of the ϵ-expansion

$$f(N,k,j)=\qquad f_0(N,k,j)+\qquad \varepsilon f_1(N,k,j)+\dots$$

Arose in the context of

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A warm up example:

$$\text{GIVEN } F(N) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\times$$

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+\left.\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+N)\Gamma(k+j+2+\frac{\varepsilon}{2})}\right)}_{f(N,k,j)}.$$

Step 2: Simplify the sums in

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f(N,k,j) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j) + \varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_1(N,k,j) + \dots$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

Simplify the constant term:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(N+1)!}{(k+1)(N+1)(k+N+1)!}\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}}^{f(N,k,j)}$$
$$\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

where

$$S_1(N)=\sum_{i=1}^{N}\frac{1}{i}(=H_N)$$

Simplify the constant term:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(N+1)!}{(k+1)(N+1)(k+N+1)!}\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}}^{f(N,k,j)}$$

$$\sum_{j=0}^{a} f(N,k,j) =$$ ▸ Sigma

Simplify the constant term:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(N+1)!}{(k+1)(N+1)(k+N+1)!}\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}}^{f(N,k,j)}$$
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$$\sum_{j=0}^{a} f(N,k,j) =$$ Sigma

FIND $g(j)$:

$$\boxed{f(N,k,j)=g(j+1)-g(j)}$$

Simplify the constant term:

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$$\sum_{j=0}^{a} f(N,k,j) =$$ Sigma

FIND $g(j)$:

$$\boxed{f(N,k,j)=g(j+1)-g(j)}$$

Sigma (based on a refined version of M. Karr's difference fields (1981)) computes

$$g(j)=\frac{(j+k+1)(j+N+1)j!k!(j+k+N)!\Big(S_1(j)-S_1(j+k)-S_1(j+N)+S_1(j+k+N)\Big)}{kN(j+k+1)!(j+N+1)!(k+N+1)!}$$

Simplify the constant term:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(N+1)!}{(k+1)(N+1)(k+N+1)!}\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}}^{f(N,k,j)}$$
$$\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

$$\sum_{j=0}^{a} f(N,k,j) =$$ Sigma

FIND $g(j)$:

$$\boxed{f(N,k,j)=g(j+1)-g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\boxed{\sum_{j=0}^{a} f(N,k,j)=g(a+1)-g(0)}$$

Simplify the constant term:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(N+1)!}{(k+1)(N+1)(k+N+1)!}\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}}^{f(N,k,j)}$$
$$\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

$$\sum_{j=0}^{a}f(N,k,j)=\underbrace{\frac{(a+1)!(k-1)!(a+k+N+1)!(S_1(a)-S_1(a+k)-S_1(a+N)+S_1(a+k+N))}{N(a+k+1)!(a+N+1)!(k+N+1)!}+\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}+\frac{(2a+k+N+2)a!k!(a+k+N)!}{(a+k+1)(a+N+1)(a+k+1)!(a+N+1)!(k+N+1)!}}_{a\,\to\,\infty}$$

Simplify the constant term:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(N+1)!}{(k+1)(N+1)(k+N+1)!}\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}}^{f(N,k,j)}$$

$$\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

$$\sum_{j=0}^{\infty}f(N,k,j)=\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

Simplify the constant term:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(N+1)!}{(k+1)(N+1)(k+N+1)!}\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}}^{f(N,k,j)}$$

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$$\sum_{k=1}^{a}\sum_{j=0}^{\infty}f(N,k,j)=\sum_{k=1}^{a}\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

Telescoping

GIVEN

$$\text{SUM}(N) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(N) - S_1(k+N)}{kN(k+N+1)N!}}_{=:\, f(N,k)}.$$

FIND $g(N,k)$:

$$\boxed{g(N,k+1) - g(N,k)} = \boxed{f(N,k)}$$

for all $0 \le k \le N$ and all $N \ge 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

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$$\mathsf{SUM}(N) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(N) - S_1(k+N)}{kN(k+N+1)N!}}_{=:\, f(N,k)}.$$

FIND $g(N,k)$ and $c_0(N), c_1(N)$:

$$\boxed{g(N,k+1) - g(N,k)} = \boxed{c_0(N) f(N,k) + c_1(N)\, f(N+1,k)}$$

for all $0 \leq k \leq N$ and all $N \geq 0$.

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$$\mathsf{SUM}(N) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(N) - S_1(k+N)}{kN(k+N+1)N!}}_{=: f(N,k)}.$$

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for all $0 \leq k \leq N$ and all $N \geq 0$.

Sigma *computes*: $c_0(N) = -N$, $c_1(N) = (N+1)(N+2)$ and

$$g(N,k) = \frac{kS_1(k) + (-N-1)S_1(N) - kS_1(k+N) - 2}{(k+N+1)N!(N+1)^2}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{SUM}(N) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(N) - S_1(k+N)}{kN(k+N+1)N!}}_{=:\, f(N,k)}.$$

FIND $g(N,k)$ and $c_0(N), c_1(N)$:

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for all $0 \leq k \leq N$ and all $N \geq 0$.

Summing this equation over k from 0 to a gives:

$$\boxed{g(N,a+1) - g(N,0)} = \boxed{c_0(N)\, \mathsf{SUM}(N) + c_1(N)\, \mathsf{SUM}(N+1)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{SUM}(N) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}}_{=:\, f(N,k)}.$$

FIND $g(N,k)$ and $c_0(N), c_1(N)$:

$$\boxed{g(N,k+1)-g(N,k)} = \boxed{c_0(N)f(N,k)+c_1(N)\,f(N+1,k)}$$

for all $0 \leq k \leq N$ and all $N \geq 0$.

Summing this equation over k from 0 to a gives: Sigma

$$\boxed{g(N,a+1)-g(N,0)} = \boxed{c_0(N)\,\mathsf{SUM}(N)+c_1(N)\,\mathsf{SUM}(N+1)}$$

$$\frac{(a+1)(S_1(a)+S_1(N)-S_1(a+N))}{(N+1)^2(a+N+2)N!} + \frac{a(a+1)}{(N+1)^3(a+N+1)(a+N+2)N!} = -N\mathsf{SUM}(N)+(1+N)(2+N)\mathsf{SUM}(N+1)$$

Simplify the constant term:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(N+1)!}{(k+1)(N+1)(k+N+1)!}\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}}^{f(N,k,j)}$$

$$\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}f(N,k,j)=\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

$$=\frac{S_1(N)^2+S_2(N)}{2N(N+1)!}$$

where

$$S_2(N)=\sum_{i=1}^{N}\frac{1}{i^2}$$

1. Creative telescoping (for the original version for hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$S(n)=\sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $S(n)$

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n: extra parameter

FIND a recurrence for $S(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)S(n) + \cdots + a_d(n)S(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products and sums
(Abramov/Bronstein/Petkovšek/Schneider, in preparation)

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FIND all solutions expressible by indefinite nested products and sums
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NOTE: By construction, the solutions are highly nested.

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FIND all solutions expressible by indefinite nested products and sums
(Abramov/Bronstein/Petkovšek/Schneider, in preparation)

3. Indefinite summation (by Sigma's refined summation theory of $\Pi\Sigma^*$-fields)

Simplify the solutions:

- The sums have minimal nested depth.
- No algebraic relations occur among the sums.

1. Creative telescoping (for the original version for hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$S(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $S(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)S(n) + \cdots + a_d(n)S(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products and sums
(Abramov/Bronstein/Petkovšek/Schneider, in preparation)

4. Find a "closed form"

S(n)=combined solutions.

Simplify the constant term:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(N+1)!}{(k+1)(N+1)(k+N+1)!}\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}}^{f(N,k,j)}$$
$$\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}f(N,k,j)=\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$
$$=\frac{S_1(N)^2+S_2(N)}{2N(N+1)!}$$

where

$$S_2(N)=\sum_{i=1}^{N}\frac{1}{i^2}$$

Automatic machinery

Excursion: When Z's algorithm fails

Given $f(n,k) = (-1)^k \binom{n}{k}\binom{5\,k}{n}$ (from Paule, Schorn 1995)

Find $g(n,k)$ s.t.

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{f(n,k)}$$

Excursion: When Z's algorithm fails

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Find $g(n,k)$, $c_0(n), c_1(n)$ s.t.

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Find $g(n,k)$, $c_0(n), c_1(n), c_2(n)$ s.t.

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + \dots + c_2(n) f(n+2,k)}$$

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$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + \dots + c_3(n) f(n+3,k)}$$

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Given $f(n,k) = (-1)^k \binom{n}{k}\binom{5\,k}{n}$ (from Paule, Schorn 1995)

Find $g(n,k)$ $, c_0(n), c_1(n), c_2(n), c_3(n), c_4(n)$ s.t.

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n)f(n,k) + \dots + c_4(n)f(n+4,k)}$$

Excursion: When Z's algorithm fails

Given $f(n,k) = (-1)^k \binom{n}{k}\binom{5\,k}{n}$ (from Paule, Schorn 1995)

Find $g(n,k)$, $c_0(n), c_1(n), c_2(n), c_3(n), c_4(n), c_5(n)$ s.t.

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + \dots + c_5(n) f(n+5,k)}$$

☺ A solution implies a linear relation of the following sums:

$$\boxed{g(n,a+1) - g(n,0)} = \boxed{c_0(n)\sum_{k=0}^{a} f(n,k) + \dots + c_5(n)\sum_{k=0}^{a} f(n+5,k)}$$

Excursion: When Z's algorithm fails

Given $f(n,k) = (-1)^k \binom{n}{k}\binom{5k}{n}$ (from Paule, Schorn 1995)

Find $g(n,k)$, $c_0(n), c_1(n), c_2(n), c_3(n), c_4(n)$

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n)f(n,k) + \cdots + c_4(n)f(n+4,k)}$$

☹ No solution

Excursion: When Z's algorithm fails

Given $f(n,k) = (-1)^k \binom{n}{k}\binom{5\,k}{n}$ (from Paule, Schorn 1995)

Find $g(n,k)$, $c_0(n), c_1(n), c_2(n), c_3(n), c_4(n)$

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + \cdots + c_4(n) f(n+4,k)}$$

No solution implies that the sequences

$$\langle f(n,a)\rangle_{a\geq 0},\quad \langle\sum_{k=0}^{a} f(n,k)\rangle_{a\geq 0},\quad \dots,\quad \langle\sum_{k=0}^{a} f(n+4,k)\rangle_{a\geq 0}$$

are algebraic independent over the field of rational sequences.

For more details see: Parameterized telescoping proves algebraic independence of sums. To appear in Annals of Combinatorics.

Excursion: When Z's algorithm fails

Given $f(n,k) = (-1)^k \binom{n}{k}\binom{5\,k}{n}$ (from Paule, Schorn 1995)

Find $g(n,k)$, $c_0(n), c_1(n), c_2(n), c_3(n), c_4(n)$

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + \dots + c_4(n) f(n+4,k)}$$

☹ No solution implies that the sequences

$$\langle f(n,a)\rangle_{a\geq 0},\quad \langle \sum_{k=0}^{a} f(n,k)\rangle_{a\geq 0},\quad \dots,\quad \langle \sum_{k=0}^{a} f(n+4,k)\rangle_{a\geq 0}$$

are algebraic independent over the field of rational sequences.

Remark: For $a = n$ we have

$$S(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{5\,k}{n} = (-5)^n$$

which satisfies $S(n+1) + 5S(n) = 0$.

Excursion: When Z's algorithm fails

Given $f(n,k) = (-1)^k \binom{n}{k}\binom{5\,k}{n}$ (from Paule, Schorn 1995)

Find $g(n,k)$, $c_0(n), c_1(n), c_2(n), c_3(n), c_4(n)$

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n)f(n,k) + \dots + c_4(n)f(n+4,k)}$$

☺ No solution implies that the sequences

$$\langle f(n,a)\rangle_{a\geq 0},\quad \langle\sum_{k=0}^{a} f(n,k)\rangle_{a\geq 0},\quad \dots,\quad \langle\sum_{k=0}^{a} f(n+4,k)\rangle_{a\geq 0}$$

are algebraic independent over the field of rational sequences.

Remark: For $a = n$ we have

$$S(n) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{5\,k}{n} = (-5)^n$$

which satisfies $S(n+1) + 5S(n) = 0$.

Any algorithm which searches a summand recurrence will fail to find the optimal recurrence for $S(n)$.

Excursion: When Z's algorithm fails

Given $f(n,k)=(-1)^k\binom{n}{k}\binom{5\,k}{n}$ (from Paule, Schorn 1995)

Find $g(n,k)$, $c_0(n), c_1(n), c_2(n), c_3(n), c_4(n)$

$$\boxed{g(n,k+1)-g(n,k)}=\boxed{c_0(n)f(n,k)+\dots+c_4(n)f(n+4,k)}$$

No solution implies that the sequences

$$\langle f(n,a)\rangle_{a\geq 0},\quad \langle\sum_{k=0}^{a}f(n,k)\rangle_{a\geq 0},\quad \dots,\quad \langle\sum_{k=0}^{a}f(n+4,k)\rangle_{a\geq 0}$$

are algebraic independent over the field of rational sequences.

Example: By Abramov's criterion Z's algorithm fails for any order with

$$f(n,k)=\frac{1}{nk+1}(-1)^k\binom{n+1}{k}\binom{2n-2k-1}{n-1}$$

Thus the following sequences are algebraic independent:

$$\langle f(n,a)\rangle_{a\geq 0}\quad\text{and}\quad \{\langle\sum_{k=0}^{a}f(n+i,k)\rangle_{a\geq 0}\ |\ i\geq 0\}$$

Excursion: When Z's algorithm fails

Given $f(n,k) = (-1)^k \binom{n}{k}\binom{5\,k}{n}$ (from Paule, Schorn 1995)

Find $g(n,k)$, $c_0(n), c_1(n), c_2(n), c_3(n), c_4(n)$

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + \dots + c_4(n) f(n+4,k)}$$

No solution implies that the sequences

$$\langle f(n,a)\rangle_{a\geq 0}, \quad \langle \sum_{k=0}^{a} f(n,k)\rangle_{a\geq 0}, \quad \dots, \quad \langle \sum_{k=0}^{a} f(n+4,k)\rangle_{a\geq 0}$$

are algebraic independent over the field of rational sequences.

Example: By a slight generalization (parametrized telescoping) it follows that the harmonic numbers with its generalized versions

$$\{\langle \sum_{k=0}^{a} \frac{1}{k^i}\rangle_{a\geq 0} \;\mid\; i \geq 1\}$$

are algebraic independent.

1. Creative telescoping (for the original version for hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$S(n)=\sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $S(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n),\dots,a_d(n),h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)S(n)+\dots+a_d(n)S(n+d)=h(n);$$

FIND all solutions expressible by indefinite nested products and sums
(Abramov/Bronstein/Petkovšek/Schneider, in preparation)

4. Find a "closed form"

S(n)=combined solutions.

"Background of our 3–loop computations"

- The following examples arise in the context of 2– and 3–loop massive single scale Feynman diagrams with operator insertion.
- These are related to the QCD anomalous dimensions and massive operator matrix elements.
- At 2-loop order all respective calculations are finished:

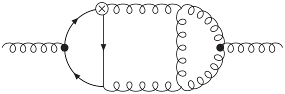
M. Buza, Y. Matiounine, J. Smith, R. Migneron, W.L. van Neerven, Nucl. Phys. **B472** (1996) 611;
I. Bierenbaum, J. Blümlein, S. Klein, Nucl. Phys. **B780** (2007) 40;
I. Bierenbaum, J. Blümlein, S. Klein, C. Schneider, Nucl.Phys. **B803** (2008)

and lead to representations in terms of harmonic sums.

Example 1: All N-Results for 3–Loop Ladder Graphs

Joint work with J. Ablinger (RISC), J. Blümlein (DESY),
A. Hasselhuhn (DESY), S. Klein (RWTH)

Consider, e.g., the diagram



(containing three massive fermion propagators)

$\Downarrow$

Around 1000 sums have to be calculated

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

Simple sum

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

$$||$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}}$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

$$||$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}}$$

$$||$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^r(-j+N-2)!r!}{(N+1)(-j+N+r-1)(-j+N+r)!}+\frac{(-1)^{N+r}(j+1)!(-j+N-2)!(-j+N-1)_r r!}{(N-1)N(N+1)(-j+N+r)!(-j-1)_r(2-N)_j}\Big)}$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

$$||$$

$$\sum_{j=0}^{N-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^{r}(-j+N-2)!r!}{(N+1)(-j+N+r-1)(-j+N+r)!}+\frac{(-1)^{N+r}(j+1)!(-j+N-2)!(-j+N-1)_r r!}{(N-1)N(N+1)(-j+N+r)!(-j-1)_r(2-N)_j}\Big)}$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

$$||$$

$$\sum_{j=0}^{N-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+N-2)!r!}{(N+1)(-j+N+r-1)(-j+N+r)!}+\frac{(-1)^{N+r}(j+1)!(-j+N-2)!(-j+N-1)_r r!}{(N-1)N(N+1)(-j+N+r)!(-j-1)_r(2-N)_j}\Big)}$$

$$||$$

$$\boxed{\Big(\frac{N^2-N+1}{(N-1)^2N^2(N+1)(2-N)_j}+\frac{\sum_{i=1}^{j}\frac{(2-N)_i}{(-i+N-1)^2(i+1)!}}{(N+1)(2-N)_j}+\frac{(-1)^{j+N}(-j-2)(-j+N-2)!}{(j-N+1)(N+1)^2N!}\Big)(j+1)!-\frac{1}{(N+1)^2(-j+N-1)}}$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

$$||$$

$$\sum_{j=0}^{N-2}\Bigg(\Big(\frac{N^2-N+1}{(N-1)^2N^2(N+1)(2-N)_j}+\frac{\displaystyle\sum_{i=1}^{j}\frac{(2-N)_i}{(-i+N-1)^2(i+1)!}}{(N+1)(2-N)_j}+\frac{(-1)^{j+N}(-j-2)(-j+N-2)!}{(j-N+1)(N+1)^2N!}\Big)(j+1)!-\frac{1}{(N+1)^2(-j+N-1)}\Bigg)$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

$$||$$

$$\sum_{j=0}^{N-2}\Bigg(\Big(\frac{N^2-N+1}{(N-1)^2N^2(N+1)(2-N)_j}+\frac{\sum_{i=1}^{j}\frac{(2-N)_i}{(-i+N-1)^2(i+1)!}}{(N+1)(2-N)_j}+\frac{(-1)^{j+N}(-j-2)(-j+N-2)!}{(j-N+1)(N+1)^2N!}\Big)(j+1)!-\frac{1}{(N+1)^2(-j+N-1)}\Bigg)$$

$$||$$

$$\frac{-N^2-N-1}{N^2(N+1)^3}+\frac{(-1)^N\left(N^2+N+1\right)}{N^2(N+1)^3}-\frac{2S_{-2}(N)}{N+1}+\frac{S_1(N)}{(N+1)^2}+\frac{S_2(N)}{-N-1}$$

Note: $S_a(N)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}},\ a\in\mathbb{Z}\setminus\{0\}.$

A typical sum

$$\sum_{j=0}^{N-2}\sum_{s=1}^{j+1}\sum_{r=0}^{N+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+N+s-2}{r}(N-j)!(s-1)!\sigma!S_1(r+2)}{(N-r)(r+1)(r+2)(-j+N+\sigma+1)(-j+N+\sigma+2)(-j+N+s+\sigma)!}$$

A typical sum

$$\sum_{j=0}^{N-2}\sum_{s=1}^{j+1}\sum_{r=0}^{N+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+N+s-2}{r}(N-j)!(s-1)!\sigma!S_1(r+2)}{(N-r)(r+1)(r+2)(-j+N+\sigma+1)(-j+N+\sigma+2)(-j+N+s+\sigma)!}$$

$$=\frac{\left(2N^2+6N+5\right)S_{-2}(N)^2}{2(N+1)(N+2)}+S_{-2,-1,2}(N)+S_{-2,1,-2}(N)$$

$$+\dots$$

where, e.g.,

$$S_{-2,1,-2}(N)=\sum_{i=1}^{N}\frac{(-1)^i\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{(-1)^k}{k^2}}{j}}{i^2}$$

Vermaseren 98;Blümlein/Kurth 99

A typical sum

$$\sum_{j=0}^{N-2}\sum_{s=1}^{j+1}\sum_{r=0}^{N+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+N+s-2}{r}(N-j)!(s-1)!\sigma!S_1(r+2)}{(N-r)(r+1)(r+2)(-j+N+\sigma+1)(-j+N+\sigma+2)(-j+N+s+\sigma)!}$$

$$=\frac{\left(2N^2+6N+5\right)S_{-2}(N)^2}{2(N+1)(N+2)}+S_{-2,-1,2}(N)+S_{-2,1,-2}(N)$$

$$+\dots-S_{2,1,1,1}(-1,2,\frac{1}{2},-1;N)+S_{2,1,1,1}(1,\frac{1}{2},1,2;N)$$

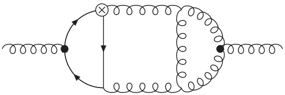
$$+\dots$$

where, e.g.,

145 S–sums occur

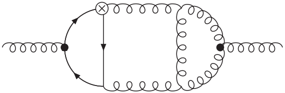
$$S_{2,1,1,1}(1,\frac{1}{2},1,2;N)=\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{\left(\frac{1}{2}\right)^j\sum_{k=1}^{j}\frac{\sum_{l=1}^{k}\frac{2^l}{l}}{k}}{j}}{i^2}$$

S. Moch, P. Uwer, S. Weinzierl 02



$\Downarrow$ Sigma.m

Around 1000 sums are calculated containing in total 533 S–sums



$\Downarrow$ Sigma.m

Around 1000 sums are calculated containing in total 533 S–sums

$\Downarrow$ J. Ablinger's HarmonicSum.m

After elimination the following sums remain:

$$S_{-4}(N), S_{-3}(N), S_{-2}(N), S_1(N), S_2(N), S_3(N), S_4(N), S_{-3,1}(N),$$
$$S_{-2,1}(N), S_{2,-2}(N), S_{2,1}(N), S_{3,1}(N), S_{-2,1,1}(N), S_{2,1,1}(N)$$

For 3–loop ladder graphs we dealt (so far) with up to 6–fold sums. E.g.,

$$\sum_{l=2}^{N}\sum_{j=2}^{l}\sum_{k=1}^{j}\sum_{r=0}^{l-k}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{2(-1)^{j+k+l+r}(k-1)!\binom{j}{k}\binom{l}{j}\binom{l-k}{r}\binom{N}{l}}{(N-1)N(k+m+n+r+2)(k+m)!}\frac{(k+m-1)!(N-j)!(l+r-2)!(n+r+1)!\,(k+m+n+r-1)!}{(-j+N+2)!(k+r-1)!(l+n+r-1)!\,(k+m+n+r+1)!}$$

$$\begin{aligned}
=\frac{1}{N(N+1)(N+2)}\Big(2\big((3-2^{N+3}\big)-(-1)^{N}\big)\zeta_3 \\
+\tfrac{1}{6}S_1(N)^3+\tfrac{4(2N+3)}{(N+1)^2(N+2)}S_1(N)+\tfrac{8(2N+3)}{(N+1)^3(N+2)} \\
-\frac{-56-40N-3N^2+2S_1(N)+3NS_1(N)+N^2S_1(N)}{2(1+N)(2+N)}S_2(N) \\
+\tfrac{\left(16+12N+N^2\right)}{2(1+N)\,(2+N)}S_1(N)^2+\tfrac{1}{3}(-3N-17)S_3(N) \\
-(-1)^{N}S_{-3}(N)+(-N-3)S_{2,1}(N)-2(-1)^{N}S_{-2,1}(N) \\
+2^{N+4}S_{1,2}\big(\tfrac{1}{2},1,N\big)+2^{N+3}S_{1,1,1}\big(\tfrac{1}{2},1,1,N\big)\Big)
\end{aligned}$$

and ...

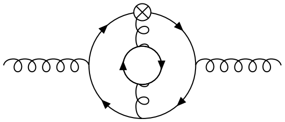
$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+N-3}\sum_{s=1}^{-l+N-q-3}\sum_{r=0}^{-l+N-q-s-3}$$
$$\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{N-1}{j+2}\binom{-j+N-3}{q}\binom{-l+N-q-3}{s}\binom{-l+N-q-s-3}{r}r!(-l+N-q-r-s-3)!(s-1)!}{(-l+N-q-2)!}$$
$$\Bigg(\frac{\frac{2\ (-1)^{-j+k-l+N-q-3}(2S_1(-j+N-1)-S_1(-j+N-2)}{-j+N-1}-\frac{(-1)^{-j+k-l+N-q-3}S_1(k)}{-j+N-1}}{(N-q-r-s-2)(q+s+1)}$$
$$-\frac{(-1)^{-j+k-l+N-q-3}(S_1(-l+N-q-2)+S_1(-l+N-q-r-s-3)-2S_1(r+s))}{(-j+N-1)(N-q-r-s-2)(q+s+1)}$$
$$+\frac{2(-1)^{-j+k-l+N-q-3}(S_1(s-1)-S_1(r+s))}{(-j+N-1)(N-q-r-s-2)(q+s+1)}\Bigg)$$

$=$ polynomial expression in terms of 49 harmonic sums and S–sums

Example 2: 3-Loop All N-Results for the N_f Contributions

Joint work with J. Ablinger (RISC), J. Blümlein (DESY),
F. Wißbrock (DESY), S. Klein (RWTH)

E.g., for the diagram



768 sums are simplified.

Simple example:

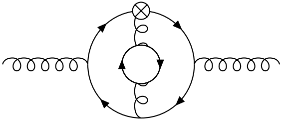
$$
\begin{aligned}
\sum_{j=1}^{N-2} & \frac{j(j+1)(j+2)(N-j)(j-1)!^2(-j+N-1)!^2}{-j+N-1} \\
&= \frac{\big(-N^3-5N^2-4N+6\big)(N!)^2}{(N-1)^2N^2} \\
&+ \frac{3}{2}\frac{(N!)^2\big(N^3+6N^2+11N+6\big)}{(N-1)N(2N+1)\binom{2N}{N}}\sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}.
\end{aligned}
$$

Not expressible in terms of harmonic sums or S-sums!

The final expression is given in terms of 703 indefinite nested sums and products. Typical examples are:

$$\sum_{i=1}^{N}\frac{\displaystyle\sum_{j=1}^{i}\frac{\binom{2j}{j}}{j}}{\binom{2\,i}{i}},\quad \sum_{i=1}^{N}\frac{S_1(i)\displaystyle\sum_{j=1}^{i}\frac{\binom{2j}{j}}{j}}{\binom{2i}{i}};$$

Sigma finds all algebraic relations among them. We get:



$$\Downarrow$$

$$
\begin{aligned}
&-\frac{20S(1,N)^4}{27(N+1)(N+2)}+\frac{32\left(6N^3+61N^2-21N+24\right)S_1(N)^3}{81N^2(N+1)(N+2)}-\frac{16\left(48N^5+746N^4+2697N^3+2746N^2+1104N+240\right)S_1(N)^2}{81N^2(N+1)^2(N+2)^2}\\
&+\frac{32\left(264N^7+4046N^6+21591N^5+52844N^4+74856N^3+66812N^2+30576N+2640\right)S_1(N)}{243N^2(N+1)^3(N+2)^3}\\
&-\frac{4\left(48N^2+101N+96\right)S_2(N)^2}{9N(N+1)(N+2)}-\frac{32\left(363N^7+6758N^6+41285N^5+121235N^4+190235N^3+150758N^2+46964N+2904\right)}{243N(N+1)^4(N+2)^3}\\
&+\Bigg(-\frac{40S_1(N)^2}{9(N+1)(N+2)}+\frac{32\left(6N^3+61N^2-21N+24\right)S_1(N)}{27N^2(N+1)(N+2)}\\
&-\frac{16\left(124N^5+198N^4-2387N^3-6162N^2-3632N-480\right)}{81N^2(N+1)^2(N+2)^2}\Bigg)S_2(N)+\\
&+\left(-\frac{32\left(9N^3-623N^2+894N+276\right)}{81N^2(N+1)(N+2)}-\frac{160S_1(N)}{27(N+1)(N+2)}\right)S_3(N)-\frac{8\left(56N^2+169N+112\right)S_4(N)}{9N(N+1)(N+2)}\\
&+\left(\frac{64S_1(N)}{3(N+1)(N+2)}-\frac{128\left(N^3+9N^2-10N-6\right)}{9N^2(N+1)(N+2)}\right)S_{2,1}(N)+\frac{64S_{3,1}(N)}{3(N+1)(N+2)}+\frac{64\left(3N^2+7N+6\right)}{3N(N+1)(N+2)}S_{2,1,1}(N)\\
&+\zeta_2\left(\frac{8S_1(N)^2}{3(N+1)(N+2)}+\frac{16\left(3N^3-N^2+30N+12\right)S_1(N)}{9N^2(N+1)(N+2)}-\frac{16\left(3N^3+2N^2+17N+6\right)}{9N(N+1)^2(N+2)}-\frac{8\left(4N^2+9N+8\right)S_2(N)}{3N(N+1)(N+2)}\right)+\\
&+\zeta_3\left(\frac{448}{9(N+1)(N+2)}-\frac{448S_1(N)}{9(N+1)(N+2)}\right)
\end{aligned}
$$

The beginning of our story: A challenging email (7/2004)

```
From: Doron Zeilberger
To: Robin Pemantle, Herbert Wilf
CC:Carsten Schneider


Robin and Herb,

I am willing to bet that Carsten Schneider's SIGMA package
for handling sums with harmonic numbers (among others)
can do it in a jiffy. I am Cc-ing this to Carsten.

Carsten: please do it, and Cc- the  answer to me.
             -Doron
```

The problem

```
From: Robin Pemantle [University of Pennsylvania]
To: herb wilf; doron zeilberger

Herb, Doron,

I have a sum that, when I evaluate numerically, looks suspiciously
like it comes out to exactly 1.
Is there a way I can automatically decide this?
The sum may be written in many ways, but one is:
```

$$\sum_{j,k=1}^{\infty} \frac{S_1(j)(S_1(k+1)-1)}{jk(k+1)(j+k)}; \qquad S_1(j) := \sum_{i=1}^{j} \frac{1}{i}.$$

After one week of (hard) work Sigma found/proved:

$$\sum_{j,k=1}^{\infty} \frac{S_1(j)(S_1(k+1)-1)}{jk(k+1)(j+k)} = -4\zeta(2) - 2\zeta(3) + 4\zeta(2)\zeta(3) + 2\zeta(5)$$

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$$= 0.999222\dots$$

After one week of (hard) work Sigma found/proved:

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$$=0.999222\ldots$$

Doron's reply: `Wow, you (and your computer!) are wizhes!`

```
I suggest that Carsten and Robin write a short Monthly paper,
that will serve, among other things, as a cautionary tale not to
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cheat and use some human-previously-proved identities, and hence
the proof is not fully rigorous (from my point of view, since it
uses human mathematics).
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Six years later: Now it is possible in a jiffy and without cheating!

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cheat and use some human-previously-proved identities, and hence
the proof is not fully rigorous (from my point of view, since it
uses human mathematics).
  Anyway, even though the bet was one sided, I still feel that
Robin and/or Herb owe me a free lunch (and they owe Carsten,
and his computer, a free dinner).


            Best wishes
                    Doron
```

Dear Doron,

happy birthday

and many free lunches and dinners in your future!
(if possible Sigma and I try to help)