Symbolic Computation and its Applications, Maribor

Symbolic Summation and the Evaluation of 3–loop Feynman Integrals

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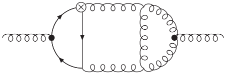
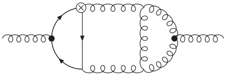
Evaluation of Feynman Integrals



Feynman diagrams

Evaluation of Feynman Integrals

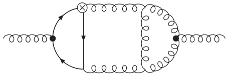


Feynman diagrams

$$\longrightarrow \qquad \int \Phi(N,\epsilon,x)dx$$

Feynman integrals

Evaluation of Feynman Integrals



Feynman diagrams

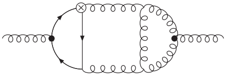
$$\int \Phi(N,\epsilon,x)dx$$

Feynman integrals

$$\Big\downarrow \text{Reduction}$$

multi-sums with upper bound N

Evaluation of Feynman Integrals



Feynman diagrams

$$\int \Phi(N, \epsilon, x)dx$$

Feynman integrals

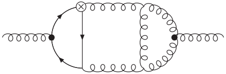
Reduction

multi-sums with
upper bound N

Sigma

sum expressions
being processable by physicists

Evaluation of Feynman Integrals

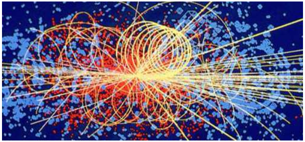


Feynman diagrams

$$\int \Phi(N,\epsilon,x)dx$$

Feynman integrals

Reduction



multi-sums with upper bound N

Sigma

sum expressions
being processable by physicists

A warm up example: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(N+1)!}{(k+1)(N+1)(k+N+1)!}\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}}^{f(N,k,j)}$$
$$\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

where

$$S_1(N)=\sum_{i=1}^{N}\frac{1}{i}(=H_N)$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

A warm up example: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(N+1)!}{(k+1)(N+1)(k+N+1)!}\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}}^{f(N,k,j)}$$
$$\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

$$\sum_{j=0}^{a}f(N,k,j)=$$ Sigma

A warm up example: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(N+1)!}{(k+1)(N+1)(k+N+1)!}\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}}^{f(N,k,j)}$$

$$\sum_{j=0}^{a} f(N,k,j) =$$ Sigma

FIND $g(j)$:

$$\boxed{f(N,k,j)=g(j+1)-g(j)}$$

A warm up example: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(N+1)!}{(k+1)(N+1)(k+N+1)!}\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}}^{f(N,k,j)}$$
$$\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

$$\sum_{j=0}^{a} f(N,k,j) =$$ Sigma

FIND $g(j)$:

$$\boxed{f(N,k,j)=g(j+1)-g(j)}$$

Sigma computes

$$g(j)=\frac{(j+k+1)(j+N+1)j!k!(j+k+N)!\Big(S_1(j)-S_1(j+k)-S_1(j+N)+S_1(j+k+N)\Big)}{kN(j+k+1)!(j+N+1)!(k+N+1)!}$$

A warm up example: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(N+1)!}{(k+1)(N+1)(k+N+1)!}\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}}^{f(N,k,j)}$$
$$\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

$$\sum_{j=0}^{a}f(N,k,j)=$$ Sigma

FIND $g(j)$:

$$\boxed{f(N,k,j)=g(j+1)-g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\boxed{\sum_{j=0}^{a}f(N,k,j)=g(a+1)-g(0)}$$

$$f(j) = \frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!} \times$$
$$\times \frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)} - S_1(j) + S_1(j+k) + S_1(j+N) - S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

with shift in j:

A difference field for the **summand:** A rational function field

$$\mathbb{F}$$

and a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

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, ,

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$$f(j)=\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}\times$$
$$\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

with shift in j: Elements from $\mathbb{Q}(N,k)$ are constants

A difference field for the **summand:** A rational function field

$$\mathbb{F}{:=}\mathbb{Q}(N,k)$$

and a field automorphism $\sigma:\mathbb{F}\to\mathbb{F}$ defined by

$$\sigma(\mathbf{c})=\mathbf{c}\quad\forall\mathbf{c}\in\mathbb{Q}(\mathbf{N},\mathbf{k}),$$

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$$f(j) = \frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!} \times$$
$$\times \frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)} - S_1(j) + S_1(j+k) + S_1(j+N) - S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

with shift in j: $\mathcal{S}\, j = j+1$

A difference field for the **summand:** A rational function field

$$\mathbb{F}:=\mathbb{Q}(N,k)\ (j)$$

and a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(N,k),$$

$$\sigma(\mathbf{j}) = \mathbf{j}+\mathbf{1}, \qquad ,$$

$$, \qquad ,$$

$$,$$

$$f(j)=\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}\times$$
$$\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

with shift in j: $\mathcal{S}\,S_1(j)=S_1(j)+\frac{1}{j+1}$

A difference field for the **summand:** A rational function field

$$\mathbb{F}:=\mathbb{Q}(N,k)\ (j)\ (h_1)$$

and a field automorphism $\sigma:\mathbb{F}\to\mathbb{F}$ defined by

$$\sigma(c)=c\quad\forall c\in\mathbb{Q}(N,k),$$

$$\sigma(j)=j+1,$$

,

$$\sigma(\mathbf{h_1})=\mathbf{h_1}+\frac{\mathbf{1}}{\mathbf{j+1}},$$

,

,

$$f(j)=\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}\times$$
$$\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

with shift in j: $\mathcal{S}\,S_1(j+k)=S_1(j+k)+\frac{1}{j+k+1}$

A difference field for the **summand:** A rational function field

$$\mathbb{F}{:=}\mathbb{Q}(N,k)\ (j)\ (h_1)\ (h_2)$$

and a field automorphism $\sigma:\mathbb{F}\to\mathbb{F}$ defined by

$$\sigma(c)=c\quad\forall c\in\mathbb{Q}(N,k),$$

$$\sigma(j)=j+1,\qquad ,$$

$$\sigma(h_1)=h_1+\frac{1}{j+1},\qquad ,$$

$$\sigma(\mathbf{h_2})=\mathbf{h_2}+\frac{\mathbf{1}}{\mathbf{j+k+1}},$$

$$f(j)=\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}\times$$
$$\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

with shift in j: $\mathcal{S}\,S_1(j+N)=S_1(j+N)+\frac{1}{j+N+1}$

A difference field for the **summand:** A rational function field

$$\mathbb{F}:=\mathbb{Q}(N,k)\ (j)\ (h_1)\ (h_2)\ (h_3)$$

and a field automorphism $\sigma:\mathbb{F}\to\mathbb{F}$ defined by

$$\sigma(c)=c\quad\forall c\in\mathbb{Q}(N,k),$$

$$\sigma(j)=j+1,\qquad\qquad \sigma(\mathbf{h_3})=\mathbf{h_3}+\frac{\mathbf{1}}{\mathbf{j+N+1}},$$

$$\sigma(h_1)=h_1+\frac{1}{j+1},\qquad\qquad ,$$

$$\sigma(h_2)=h_2+\frac{1}{j+k+1},$$

$$f(j)=\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}\times$$
$$\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

with shift in j: $\mathcal{S}\,S_1(j+N+k)=S_1(j+N+k)+\frac{1}{j+N+k+1}$

A difference field for the **summand:** A rational function field

$$\mathbb{F}:=\mathbb{Q}(N,k)\;(j)\;(h_1)\;(h_2)\;(h_3)\;(h_4)$$

and a field automorphism $\sigma:\mathbb{F}\to\mathbb{F}$ defined by

$$\sigma(c)=c\quad\forall c\in\mathbb{Q}(N,k),$$

$$\sigma(j)=j+1, \qquad \sigma(h_3)=h_3+\frac{1}{j+N+1},$$

$$\sigma(h_1)=h_1+\frac{1}{j+1}, \qquad \sigma(\mathbf{h_4})=\mathbf{h_4}+\frac{\mathbf{1}}{\mathbf{j+N+k+1}},$$

$$\sigma(h_2)=h_2+\frac{1}{j+k+1},$$

$$f(j)=\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}\times$$
$$\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

with shift in j: $\mathcal{S}\,\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}=\frac{(j+2)(j+k+N+2)}{(j+k+2)(j+N+2)}\cdot\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}$

A difference field for the **summand:** A rational function field

$$\mathbb{F}:=\mathbb{Q}(N,k)\;(j)\;(h_1)\;(h_2)\;(h_3)\;(h_4)\;(p)$$

and a field automorphism $\sigma:\mathbb{F}\to\mathbb{F}$ defined by

Karr's $\Pi\Sigma$-fields (1981)

$$\sigma(c)=c\quad\forall c\in\mathbb{Q}(N,k),$$

$$\sigma(j)=j+1, \qquad \sigma(h_3)=h_3+\frac{1}{j+N+1},$$

$$\sigma(h_1)=h_1+\frac{1}{j+1}, \qquad \sigma(h_4)=h_4+\frac{1}{j+N+k+1},$$

$$\sigma(h_2)=h_2+\frac{1}{j+k+1}, \qquad \sigma(\mathbf{p})=\frac{(\mathbf{j+2})(\mathbf{j+k+N+2})}{(\mathbf{j+k+2})(\mathbf{j+N+2})}\,\mathbf{p}.$$

$$f(j)=\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}\times$$
$$\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

FIND $g(j)$:

$$f(j)=g(j+1)-g(j)$$

$$\downarrow$$

GIVEN $f := p\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-h_1+h_2+h_3-h_4}{(j+1)(j+k+N+1)}$

FIND $g\in\mathbb{F}$:

$$\boxed{f=\sigma(g)-g}$$

$$f(j)=\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}\times$$
$$\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

FIND $g(j)$:

$$f(j)=g(j+1)-g(j)$$

$$\downarrow$$

GIVEN $f:=p\times\dfrac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-h_1+h_2+h_3-h_4}{(j+1)(j+k+N+1)}$

FIND $g\in\mathbb{F}$:

$$\boxed{f=\sigma(g)-g}$$

$\downarrow$ Sigma

$$g=p\times\frac{(h_1-h_2-h_3+h_4)(j+k+1)(j+N+1)}{(j+1)kN(j+k+N+1)}$$

$$f(j)=\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}\times$$
$$\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

FIND $g(j)$:

$$f(j)=g(j+1)-g(j)$$

$$\downarrow$$

GIVEN $f:=p\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-h_1+h_2+h_3-h_4}{(j+1)(j+k+N+1)}$

FIND $g\in\mathbb{F}$:

$$\boxed{f=\sigma(g)-g}$$

$$h_1\equiv S_1(j)$$
$$h_2\equiv S_1(j+k)$$
$$\vdots$$

$\downarrow$ Sigma

$$g=p\times\frac{(h_1-h_2-h_3+h_4)(j+k+1)(j+N+1)}{(j+1)kN(j+k+N+1)}$$

A warm up example: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(N+1)!}{(k+1)(N+1)(k+N+1)!}\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}}^{f(N,k,j)}$$
$$\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

$$\sum_{j=0}^{a}f(N,k,j)=\underbrace{\frac{(a+1)!(k-1)!(a+k+N+1)!(S_1(a)-S_1(a+k)-S_1(a+N)+S_1(a+k+N))}{N(a+k+1)!(a+N+1)!(k+N+1)!}+\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}+\frac{(2a+k+N+2)a!k!(a+k+N)!}{(a+k+1)(a+N+1)(a+k+1)!(a+N+1)!(k+N+1)!}}_{a\,\to\,\infty}$$

A warm up example: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(N+1)!}{(k+1)(N+1)(k+N+1)!}\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}}^{f(N,k,j)}$$
$$\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

$$\sum_{j=0}^{\infty}f(N,k,j)=\qquad\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

A warm up example: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(N+1)!}{(k+1)(N+1)(k+N+1)!}\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}}^{f(N,k,j)}$$
$$\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

$$\sum_{k=1}^{a}\sum_{j=0}^{\infty}f(N,k,j)=\sum_{k=1}^{a}\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

Telescoping

GIVEN

$$\mathsf{SUM}(N) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}}_{=:\,f(N,k)}.$$

FIND $g(N,k)$:

$$\boxed{g(N,k+1)-g(N,k)} = \boxed{f(N,k)}$$

for all $0 \leq k \leq N$ and all $N \geq 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{SUM}(N) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}}_{=:\, f(N,k)}.$$

FIND $g(N,k)$ and $c_0(N), c_1(N)$:

$$\boxed{g(N,k+1)-g(N,k)} = \boxed{c_0(N)f(N,k)+c_1(N)\,f(N+1,k)}$$

for all $0 \leq k \leq N$ and all $N \geq 0$.

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{SUM}(N) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(N) - S_1(k+N)}{kN(k+N+1)N!}}_{=:\, f(N,k)}.$$

FIND $g(N,k)$ and $c_0(N), c_1(N)$:

$$\boxed{g(N,k+1) - g(N,k)} = \boxed{c_0(N) f(N,k) + c_1(N)\, f(N+1,k)}$$

for all $0 \leq k \leq N$ and all $N \geq 0$.

Sigma *computes*: $c_0(N) = -N$, $c_1(N) = (N+1)(N+2)$ and

$$g(N,k) = \frac{kS_1(k) + (-N-1)S_1(N) - kS_1(k+N) - 2}{(k+N+1)N!(N+1)^2}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{SUM}(N) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(N) - S_1(k+N)}{kN(k+N+1)N!}}_{=:\, f(N,k)}.$$

FIND $g(N,k)$ and $c_0(N), c_1(N)$:

$$\boxed{g(N,k+1) - g(N,k)} = \boxed{c_0(N) f(N,k) + c_1(N)\, f(N+1,k)}$$

for all $0 \leq k \leq N$ and all $N \geq 0$.

Summing this equation over k from 0 to a gives:

$$\boxed{g(N,a+1) - g(N,0)} = \boxed{c_0(N)\,\mathsf{SUM}(N) + c_1(N)\,\mathsf{SUM}(N+1)}$$

Zeilberger's creative telescoping paradigm

GIVEN
$$\mathsf{SUM}(N) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(N) - S_1(k+N)}{kN(k+N+1)N!}}_{=:\, f(N,k)}.$$

FIND $g(N,k)$ and $c_0(N), c_1(N)$:

$$\boxed{g(N,k+1) - g(N,k)} = \boxed{c_0(N) f(N,k) + c_1(N) f(N+1,k)}$$

for all $0 \leq k \leq N$ and all $N \geq 0$.

Summing this equation over k from 0 to a gives: Sigma

$$\boxed{g(N,a+1) - g(N,0)} = \boxed{c_0(N)\,\mathsf{SUM}(N) + c_1(N)\,\mathsf{SUM}(N+1)}$$

$$\frac{(a+1)(S_1(a)+S_1(N)-S_1(a+N))}{(N+1)^2(a+N+2)N!} + \frac{a(a+1)}{(N+1)^3(a+N+1)(a+N+2)N!} \quad || \quad -N\mathsf{SUM}(N) + (1+N)(2+N)\mathsf{SUM}(N+1)$$

A warm up example: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(N+1)!}{(k+1)(N+1)(k+N+1)!}\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}}^{f(N,k,j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}f(N,k,j)=\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

$$=\frac{S_1(N)^2+S_2(N)}{2N(N+1)!}$$

where

$$S_2(N)=\sum_{i=1}^{N}\frac{1}{i^2}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$S(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $S(n)$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$S(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $S(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)S(n) + \cdots + a_d(n)S(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products and sums
(Nörlund 24, Petkovšek 92, Abramov/Petkovšek 94, Hendriks/Singer 99/Sigma 01)

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$S(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $S(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)S(n) + \dots + a_d(n)S(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products and sums
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NOTE: By construction, the solutions are highly nested.

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3. Indefinite summation (by Sigma's refined summation theory of $\Pi\Sigma^*$-fields)

Simplify the solutions:

- The sums have minimal nested depth.
- No algebraic relations occur among the sums.

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4. Find a "closed form"

S(n)=combined solutions.

A warm up example: Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(N+1)!}{(k+1)(N+1)(k+N+1)!}\frac{(j+1)!(j+k+N+1)!}{(j+k+1)!(j+N+1)!}}^{f(N,k,j)}$$
$$\times\frac{\frac{2j+k+N+2}{(j+k+1)(j+N+1)}-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)}{(j+1)(j+k+N+1)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}f(N,k,j)=\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$
$$=\frac{S_1(N)^2+S_2(N)}{2N(N+1)!}$$

where

$$S_2(N)=\sum_{i=1}^{N}\frac{1}{i^2}$$

Automatic machinery

"Background of our 3–loop computations"

- The following examples arise in the context of 2– and 3–loop massive single scale Feynman diagrams with operator insertion.
- These are related to the QCD anomalous dimensions and massive operator matrix elements.
- At 2-loop order all respective calculations are finished:

M. Buza, Y. Matiounine, J. Smith, R. Migneron, W.L. van Neerven, Nucl. Phys. **B472** (1996) 611;

I. Bierenbaum, J. Blümlein, S. Klein, Nucl. Phys. **B780** (2007) 40;

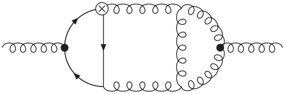
I. Bierenbaum, J. Blümlein, S. Klein, C. Schneider, Nucl.Phys. **B803** (2008)

and lead to representations in terms of harmonic sums.

Example 1: All N-Results for 3–Loop Ladder Graphs

Joint work with J. Ablinger (RISC), J. Blümlein (DESY), A. Hasselhuhn (DESY), S. Klein (RWTH)

Consider, e.g., the diagram



(containing three massive fermion propagators)

$$\Downarrow$$

Around 1000 sums have to be calculated

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

Simple sum

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

$$||$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}}$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

$$||$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}}$$

$$||$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^r(-j+N-2)!r!}{(N+1)(-j+N+r-1)(-j+N+r)!}+\frac{(-1)^{N+r}(j+1)!(-j+N-2)!(-j+N-1)_r r!}{(N-1)N(N+1)(-j+N+r)!(-j-1)_r(2-N)_j}\Big)}$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

$$||$$

$$\sum_{j=0}^{N-2}\boxed{\begin{array}{l}\displaystyle\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+N-2)!r!}{(N+1)(-j+N+r-1)(-j+N+r)!}+\\ \displaystyle\qquad\frac{(-1)^{N+r}(j+1)!(-j+N-2)!(-j+N-1)_r r!}{(N-1)N(N+1)(-j+N+r)!(-j-1)_r(2-N)_j}\Big)\end{array}}$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

$$\|$$

$$\sum_{j=0}^{N-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+N-2)!r!}{(N+1)(-j+N+r-1)(-j+N+r)!}+\frac{(-1)^{N+r}(j+1)!(-j+N-2)!(-j+N-1)_r r!}{(N-1)N(N+1)(-j+N+r)!(-j-1)_r(2-N)_j}\Big)}$$

$$\|$$

$$\boxed{\Big(\frac{N^2-N+1}{(N-1)^2N^2(N+1)(2-N)_j}+\frac{\sum_{i=1}^{j}\frac{(2-N)_i}{(-i+N-1)^2(i+1)!}}{(N+1)(2-N)_j}+\frac{(-1)^{j+N}(-j-2)(-j+N-2)!}{(j-N+1)(N+1)^2N!}\Big)(j+1)!-\frac{1}{(N+1)^2(-j+N-1)}}$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

$$||$$

$$\sum_{j=0}^{N-2}\Bigg(\Big(\frac{N^2-N+1}{(N-1)^2N^2(N+1)(2-N)_j}+\frac{\sum_{i=1}^{j}\frac{(2-N)_i}{(-i+N-1)^2(i+1)!}}{(N+1)(2-N)_j}+$$
$$\frac{(-1)^{j+N}(-j-2)(-j+N-2)!}{(j-N+1)(N+1)^2N!}\Big)(j+1)!-\frac{1}{(N+1)^2(-j+N-1)}\Bigg)$$

$$\sum_{j=0}^{N-2}\sum_{r=0}^{j+1}\sum_{s=0}^{N-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+N+r-2}{s}(-j+N-2)!r!}{(N-s)(s+1)(-j+N+r)!}$$

$$||$$

$$\sum_{j=0}^{N-2}\Bigg(\Big(\frac{N^2-N+1}{(N-1)^2N^2(N+1)(2-N)_j}+\frac{\sum_{i=1}^{j}\frac{(2-N)_i}{(-i+N-1)^2(i+1)!}}{(N+1)(2-N)_j}+\frac{(-1)^{j+N}(-j-2)(-j+N-2)!}{(j-N+1)(N+1)^2N!}\Big)(j+1)!-\frac{1}{(N+1)^2(-j+N-1)}\Bigg)$$

$$||$$

$$\frac{-N^2-N-1}{N^2(N+1)^3}+\frac{(-1)^N\left(N^2+N+1\right)}{N^2(N+1)^3}-\frac{2S_{-2}(N)}{N+1}+\frac{S_1(N)}{(N+1)^2}+\frac{S_2(N)}{-N-1}$$

Note: $S_a(N)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}}$, $a\in\mathbb{Z}\setminus\{0\}$.

A typical sum

$$\sum_{j=0}^{N-2}\sum_{s=1}^{j+1}\sum_{r=0}^{N+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+N+s-2}{r}(N-j)!(s-1)!\sigma!S_1(r+2)}{(N-r)(r+1)(r+2)(-j+N+\sigma+1)(-j+N+\sigma+2)(-j+N+s+\sigma)!}$$

A typical sum

$$\sum_{j=0}^{N-2}\sum_{s=1}^{j+1}\sum_{r=0}^{N+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+N+s-2}{r}(N-j)!(s-1)!\sigma!S_1(r+2)}{(N-r)(r+1)(r+2)(-j+N+\sigma+1)(-j+N+\sigma+2)(-j+N+s+\sigma)!}$$

$$=\frac{\left(2N^2+6N+5\right)S_{-2}(N)^2}{2(N+1)(N+2)}+S_{-2,-1,2}(N)+S_{-2,1,-2}(N)$$

$$+\dots$$

where, e.g.,

$$S_{-2,1,-2}(N)=\sum_{i=1}^{N}\frac{(-1)^i\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{(-1)^k}{k^2}}{j}}{i^2}$$

Vermaseren 98/Blümlein/Kurth 99

A typical sum

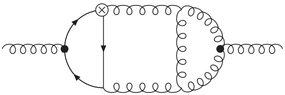
$$\sum_{j=0}^{N-2}\sum_{s=1}^{j+1}\sum_{r=0}^{N+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+N+s-2}{r}(N-j)!(s-1)!\sigma!S_1(r+2)}{(N-r)(r+1)(r+2)(-j+N+\sigma+1)(-j+N+\sigma+2)(-j+N+s+\sigma)!}$$
$$=\frac{\left(2N^2+6N+5\right)S_{-2}(N)^2}{2(N+1)(N+2)}+S_{-2,-1,2}(N)+S_{-2,1,-2}(N)$$
$$+\dots-S_{2,1,1,1}(-1,2,\frac{1}{2},-1;N)+S_{2,1,1,1}(1,\frac{1}{2},1,2;N)$$
$$+\dots$$

where, e.g.,

$\boxed{\text{145 } S\text{–sums occur}}$

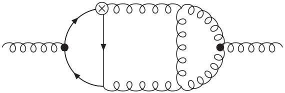
$$S_{2,1,1,1}(1,\frac{1}{2},1,2;N)=\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{\left(\frac{1}{2}\right)^j\sum_{k=1}^{j}\frac{\sum_{l=1}^{k}\frac{2^l}{l}}{k}}{j}}{i^2}$$

S. Moch, P. Uwer, S. Weinzierl 02



$\Downarrow$ Sigma.m

Around 1000 sums are calculated containing in total 533 S–sums



$\Downarrow$ Sigma.m

Around 1000 sums are calculated containing in total 533 S–sums

$\Downarrow$ J. Ablinger's HarmonicSum.m

After elimination the following sums remain:

$$S_{-4}(N), S_{-3}(N), S_{-2}(N), S_1(N), S_2(N), S_3(N), S_4(N), S_{-3,1}(N),$$
$$S_{-2,1}(N), S_{2,-2}(N), S_{2,1}(N), S_{3,1}(N), S_{-2,1,1}(N), S_{2,1,1}(N)$$

For 3–loop ladder graphs we dealt (so far) with up to 6–fold sums. E.g.,

$$\boxed{\begin{aligned}&\sum_{l=2}^{N}\sum_{j=2}^{l}\sum_{k=1}^{j}\sum_{r=0}^{l-k}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{2(-1)^{j+k+l+r}(k-1)!\binom{j}{k}\binom{l}{j}\binom{l-k}{r}\binom{N}{l}}{(N-1)N(k+m+n+r+2)(k+m)!}\\ &\quad\frac{(k+m-1)!(N-j)!(l+r-2)!(n+r+1)!\,(k+m+n+r-1)!}{(-j+N+2)!(k+r-1)!(l+n+r-1)!\,(k+m+n+r+1)!}\end{aligned}}$$

$$\begin{aligned}=\frac{1}{N(N+1)(N+2)}\Big(&2\big((3-2^{N+3})-(-1)^N\big)\zeta_3\\ &+\tfrac{1}{6}S_1(N)^3+\tfrac{4(2N+3)}{(N+1)^2(N+2)}S_1(N)+\tfrac{8(2N+3)}{(N+1)^3(N+2)}\\ &-\frac{-56-40N-3N^2+2S_1(N)+3NS_1(N)+N^2S_1(N)}{2(1+N)(2+N)}S_2(N)\\ &+\tfrac{\left(16+12N+N^2\right)}{2(1+N)\,(2+N)}S_1(N)^2+\tfrac{1}{3}(-3N-17)S_3(N)\\ &-(-1)^NS_{-3}(N)+(-N-3)S_{2,1}(N)-2(-1)^NS_{-2,1}(N)\\ &+2^{N+4}S_{1,2}\big(\tfrac{1}{2},1,N\big)+2^{N+3}S_{1,1,1}\big(\tfrac{1}{2},1,1,N\big)\Big)\end{aligned}$$

and ...

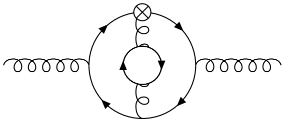
$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+N-3}\sum_{s=1}^{-l+N-q-3}\sum_{r=0}^{-l+N-q-s-3}$$
$$\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{N-1}{j+2}\binom{-j+N-3}{q}\binom{-l+N-q-3}{s}\binom{-l+N-q-s-3}{r}r!(-l+N-q-r-s-3)!(s-1)!}{(-l+N-q-2)!}$$
$$\Bigg(\frac{\frac{2\ (-1)^{-j+k-l+N-q-3}(2S_1(-j+N-1)-S_1(-j+N-2)}{-j+N-1}-\frac{(-1)^{-j+k-l+N-q-3}S_1(k)}{-j+N-1}}{(N-q-r-s-2)(q+s+1)}$$
$$-\frac{(-1)^{-j+k-l+N-q-3}(S_1(-l+N-q-2)+S_1(-l+N-q-r-s-3)-2S_1(r+s))}{(-j+N-1)(N-q-r-s-2)(q+s+1)}$$
$$+\frac{2(-1)^{-j+k-l+N-q-3}(S_1(s-1)-S_1(r+s))}{(-j+N-1)(N-q-r-s-2)(q+s+1)}\Bigg)$$

$=$ polynomial expression in terms of 49 harmonic sums and S–sums

Example 2: 3-Loop All N-Results for the N_f Contributions

Joint work with J. Ablinger (RISC), J. Blümlein (DESY),
F. Wißbrock (DESY), S. Klein (RWTH)

E.g., for the diagram



768 sums are simplified.

Simple example:

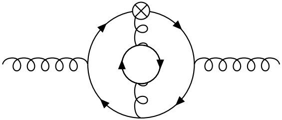
$$\sum_{j=1}^{N-2} \frac{j(j+1)(j+2)(N-j)(j-1)!^2(-j+N-1)!^2}{-j+N-1}$$
$$= \frac{\big(-N^3-5N^2-4N+6\big)(N!)^2}{(N-1)^2N^2}$$
$$+\frac{3}{2}\frac{(N!)^2\big(N^3+6N^2+11N+6\big)}{(N-1)N(2N+1)\binom{2N}{N}}\sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}.$$

Not expressible in terms of harmonic sums or S-sums!

The final expression is given in terms of 703 indefinite nested sums and products. Typical examples are:

$$\sum_{i=1}^{N} \frac{\sum_{j=1}^{i} \frac{\binom{2j}{j}}{j}}{\binom{2\,i}{i}}, \quad \sum_{i=1}^{N} \frac{S_1(i) \sum_{j=1}^{i} \frac{\binom{2j}{j}}{j}}{\binom{2i}{i}};$$

Sigma finds all algebraic relations among them. We get:



$$\Downarrow$$

$$
\begin{aligned}
&-\frac{20S(1,N)^4}{27(N+1)(N+2)}+\frac{32\left(6N^3+61N^2-21N+24\right)S_1(N)^3}{81N^2(N+1)(N+2)}-\frac{16\left(48N^5+746N^4+2697N^3+2746N^2+1104N+240\right)S_1(N)^2}{81N^2(N+1)^2(N+2)^2}\\
&+\frac{32\left(264N^7+4046N^6+21591N^5+52844N^4+74856N^3+66812N^2+30576N+2640\right)S_1(N)}{243N^2(N+1)^3(N+2)^3}\\
&-\frac{4\left(48N^2+101N+96\right)S_2(N)^2}{9N(N+1)(N+2)}-\frac{32\left(363N^7+6758N^6+41285N^5+121235N^4+190235N^3+150758N^2+46964N+2904\right)}{243N(N+1)^4(N+2)^3}\\
&+\Bigg(-\frac{40S_1(N)^2}{9(N+1)(N+2)}+\frac{32\left(6N^3+61N^2-21N+24\right)S_1(N)}{27N^2(N+1)(N+2)}\\
&-\frac{16\left(124N^5+198N^4-2387N^3-6162N^2-3632N-480\right)}{81N^2(N+1)^2(N+2)^2}\Bigg)S_2(N)+\\
&+\left(-\frac{32\left(9N^3-623N^2+894N+276\right)}{81N^2(N+1)(N+2)}-\frac{160S_1(N)}{27(N+1)(N+2)}\right)S_3(N)-\frac{8\left(56N^2+169N+112\right)S_4(N)}{9N(N+1)(N+2)}\\
&+\left(\frac{64S_1(N)}{3(N+1)(N+2)}-\frac{128\left(N^3+9N^2-10N-6\right)}{9N^2(N+1)(N+2)}\right)S_{2,1}(N)+\frac{64S_{3,1}(N)}{3(N+1)(N+2)}+\frac{64\left(3N^2+7N+6\right)}{3N(N+1)(N+2)}S_{2,1,1}(N)\\
&+\zeta_2\left(\frac{8S_1(N)^2}{3(N+1)(N+2)}+\frac{16\left(3N^3-N^2+30N+12\right)S_1(N)}{9N^2(N+1)(N+2)}-\frac{16\left(3N^3+2N^2+17N+6\right)}{9N(N+1)^2(N+2)}-\frac{8\left(4N^2+9N+8\right)S_2(N)}{3N(N+1)(N+2)}\right)+\\
&+\zeta_3\left(\frac{448}{9(N+1)(N+2)}-\frac{448S_1(N)}{9(N+1)(N+2)}\right)
\end{aligned}
$$