

Loops and Legs in QFT 2010

Modern Summation Methods and the Computation of 2- and 3-loop Feynman Diagrams

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27. April 2010

A warm up example: Simplify $f(N, k, j)$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} \right. \\ \left. + \frac{j!k!(j+k+N)!(-S_1(j) + S_1(j+k) + S_1(j+N) - S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!} \right)$$

where

$$S_1(N) = \sum_{i=1}^N \frac{1}{i}$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

A warm up example: Simplify $f(N, k, j)$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} \right. \\ \left. + \frac{j!k!(j+k+N)!(-S_1(j) + S_1(j+k) + S_1(j+N) - S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!} \right)$$

$$\sum_{j=0}^a f(N, k, j) = \text{▶ Sigma}$$

A warm up example: Simplify $f(N, k, j)$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} \right. \\ \left. + \frac{j!k!(j+k+N)!(-S_1(j) + S_1(j+k) + S_1(j+N) - S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!} \right)$$

$$\sum_{j=0}^a f(N, k, j) = \text{Sigma}$$

FIND $g(j)$:

$$f(N, k, j) = g(j+1) - g(j)$$

A warm up example: Simplify $f(N, k, j)$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} \right) + \frac{j!k!(j+k+N)!(-S_1(j) + S_1(j+k) + S_1(j+N) - S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!}$$

$$\sum_{j=0}^a f(N, k, j) = \text{Sigma}$$

FIND $g(j)$:

$$f(N, k, j) = g(j+1) - g(j)$$

Sigma (based on a refined version of M. Karr's difference fields (1981)) computes

$$g(j) = \frac{(j+k+1)(j+N+1)j!k!(j+k+N)!(S_1(j) - S_1(j+k) - S_1(j+N) + S_1(j+k+N))}{kN(j+k+1)!(j+N+1)!(k+N+1)!}$$

A warm up example: Simplify $f(N, k, j)$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} \right. \\ \left. + \frac{j!k!(j+k+N)!(-S_1(j) + S_1(j+k) + S_1(j+N) - S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!} \right)$$

$$\sum_{j=0}^a f(N, k, j) = \text{Sigma}$$

FIND $g(j)$:

$$f(N, k, j) = g(j+1) - g(j)$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^a f(N, k, j) = g(a+1) - g(0)$$

A warm up example: Simplify $f(N, k, j)$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} \right. \\ \left. + \frac{j!k!(j+k+N)!(-S_1(j) + S_1(j+k) + S_1(j+N) - S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!} \right)$$

$$\sum_{j=0}^a f(N, k, j) = \frac{(a+1)!(k-1)!(a+k+N+1)!(S_1(a) - S_1(a+k) - S_1(a+N) + S_1(a+k+N))}{N(a+k+1)!(a+N+1)!(k+N+1)!} \\ + \frac{S_1(k) + S_1(N) - S_1(k+N)}{kN(k+N+1)N!} + \frac{(2a+k+N+2)a!k!(a+k+N)!}{(a+k+1)(a+N+1)(a+k+1)!(a+N+1)!(k+N+1)!}$$

$a \rightarrow \infty$

A warm up example: Simplify $f(N, k, j)$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} \right. \\ \left. + \frac{j!k!(j+k+N)!(-S_1(j) + S_1(j+k) + S_1(j+N) - S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!} \right)$$

$$\sum_{j=0}^{\infty} f(N, k, j) = \frac{S_1(k) + S_1(N) - S_1(k+N)}{kN(k+N+1)N!}$$

A warm up example: Simplify $f(N, k, j)$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} \right. \\ \left. + \frac{j!k!(j+k+N)!(-S_1(j) + S_1(j+k) + S_1(j+N) - S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!} \right)$$

$$\sum_{k=1}^a \sum_{j=0}^{\infty} f(N, k, j) = \sum_{k=1}^a \frac{S_1(k) + S_1(N) - S_1(k+N)}{kN(k+N+1)N!}$$

$$= \text{Sigma}$$

A warm up example: Simplify $f(N, k, j)$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} \right) + \frac{j!k!(j+k+N)!(-S_1(j) + S_1(j+k) + S_1(j+N) - S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!}$$

$$\begin{aligned} \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} f(N, k, j) &= \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(N) - S_1(k+N)}{kN(k+N+1)N!} \\ &= \frac{S_1(N)^2 + S_2(N)}{2N(N+1)!} \end{aligned}$$

where

$$S_2(N) = \sum_{i=1}^N \frac{1}{i^2}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a **definite** sum

$$S(n) = \sum_{k=0}^n f(n, k);$$

$f(n, k)$: indefinite nested product-sum in k ;
 n : extra parameter

FIND a **recurrence** for $S(n)$

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2. Recurrence solving

GIVEN a recurrence

$$a_0(n), \dots, a_d(n), h(n):$$

indefinite nested product-sum expressions.

$$a_0(n)S(n) + \dots + a_d(n)S(n+d) = h(n);$$

FIND **all solutions** expressible by indefinite nested products and sums

(Nörlund 24, Abramov/Petkovšek 94, Hendriks/Singer 99/Sigma 01)

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NOTE: By construction, the solutions are highly nested.

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3. Indefinite summation

Simplify the solutions:

- ▶ The sums have **minimal nested depth**.
- ▶ **No algebraic relations** occur among the sums.

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a **definite** sum

$$S(n) = \sum_{k=0}^n f(n, k);$$

$f(n, k)$: indefinite nested product-sum in k ;
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FIND a **recurrence** for $S(n)$

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$$a_0(n)S(n) + \dots + a_d(n)S(n+d) = h(n);$$

FIND **all solutions** expressible by indefinite nested products and sums

(Nörlund 24, Abramov/Petkovšek 94, Hendriks/Singer 99/Sigma 01)

4. Find a “closed form”

$S(n)$ =combined solutions.

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$$\begin{aligned} \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} f(N, k, j) &= \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(N) - S_1(k+N)}{kN(k+N+1)N!} \\ &= \frac{S_1(N)^2 + S_2(N)}{2N(N+1)!} \end{aligned}$$

where

$$S_2(N) = \sum_{i=1}^N \frac{1}{i^2}$$

Automatic machinery

“Background”

- ▶ The following examples arise in the context of 2- and 3-loop massive single scale Feynman diagrams with operator insertion.
- ▶ These are related to the QCD anomalous dimensions and massive operator matrix elements.
- ▶ At 2-loop order all respective calculations are finished:

M. Buza, Y. Matiounine, J. Smith, R. Migneron, W.L. van Neerven, Nucl. Phys. **B472** (1996) 611;

I. Bierenbaum, J. Blümlein, S. Klein, Nucl. Phys. **B780** (2007) 40;

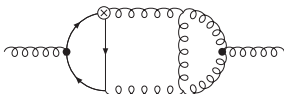
I. Bierenbaum, J. Blümlein, S. Klein, C. Schneider, Nucl.Phys. **B803** (2008)

and lead to representations in terms of harmonic sums.

Example 1: All N-Results for 3-Loop Ladder Graphs

Joint work with J. Ablinger, J. Blümlein, A. Hasselhuhn, S. Klein 2010

Consider, e.g., the diagram



Around 1000 sums have to be calculated

$$\sum_{j=0}^{N-2} \sum_{r=0}^{j+1} \sum_{s=0}^{N-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+N+r-2}{s} (-j+N-2)! r!}{(N-s)(s+1)(-j+N+r)!}$$

Simple sum

$$\sum_{j=0}^{N-2} \sum_{r=0}^{j+1} \sum_{s=0}^{N-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+N+r-2}{s} (-j+N-2)! r!}{(N-s)(s+1)(-j+N+r)!}$$

||

$$\sum_{j=0}^{N-2} \sum_{r=0}^{j+1} \boxed{\sum_{s=0}^{N-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+N+r-2}{s} (-j+N-2)! r!}{(N-s)(s+1)(-j+N+r)!}}$$

$$\sum_{j=0}^{N-2} \sum_{r=0}^{j+1} \sum_{s=0}^{N-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+N+r-2}{s} (-j+N-2)! r!}{(N-s)(s+1)(-j+N+r)!}$$

||

$$\sum_{j=0}^{N-2} \sum_{r=0}^{j+1} \left[\sum_{s=0}^{N-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+N+r-2}{s} (-j+N-2)! r!}{(N-s)(s+1)(-j+N+r)!} \right]$$

||

$$\left(\binom{j+1}{r} \left(\frac{(-1)^r (-j+N-2)! r!}{(N+1)(-j+N+r-1)(-j+N+r)!} + \frac{(-1)^{N+r} (j+1)! (-j+N-2)! (-j+N-1)_r r!}{(N-1)N(N+1)(-j+N+r)! (-j-1)_r (2-N)_j} \right) \right)$$

$$\sum_{j=0}^{N-2} \sum_{r=0}^{j+1} \sum_{s=0}^{N-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+N+r-2}{s} (-j+N-2)! r!}{(N-s)(s+1)(-j+N+r)!}$$

||

$$\sum_{j=0}^{N-2} \left(\sum_{r=0}^{j+1} \binom{j+1}{r} \left(\frac{(-1)^r (-j+N-2)! r!}{(N+1)(-j+N+r-1)(-j+N+r)!} + \frac{(-1)^{N+r} (j+1)! (-j+N-2)! (-j+N-1)_r r!}{(N-1)N(N+1)(-j+N+r)! (-j-1)_r (2-N)_j} \right) \right)$$

$$\sum_{j=0}^{N-2} \sum_{r=0}^{j+1} \sum_{s=0}^{N-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+N+r-2}{s} (-j+N-2)! r!}{(N-s)(s+1)(-j+N+r)!}$$

||

$$\sum_{j=0}^{N-2} \left(\sum_{r=0}^{j+1} \binom{j+1}{r} \left(\frac{(-1)^r (-j+N-2)! r!}{(N+1)(-j+N+r-1)(-j+N+r)!} + \frac{(-1)^{N+r} (j+1)! (-j+N-2)! (-j+N-1)_r r!}{(N-1)N(N+1)(-j+N+r)! (-j-1)_r (2-N)_j} \right) \right)$$

||

$$\left(\frac{N^2 - N + 1}{(N-1)^2 N^2 (N+1)(2-N)_j} + \frac{\sum_{i=1}^j \frac{(2-N)_i}{(-i+N-1)^2 (i+1)!}}{(N+1)(2-N)_j} + \frac{(-1)^{j+N} (-j-2)(-j+N-2)!}{(j-N+1)(N+1)^2 N!} \right) (j+1)! - \frac{1}{(N+1)^2 (-j+N-1)}$$

$$\sum_{j=0}^{N-2} \sum_{r=0}^{j+1} \sum_{s=0}^{N-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+N+r-2}{s} (-j+N-2)! r!}{(N-s)(s+1)(-j+N+r)!}$$

||

$$\sum_{j=0}^{N-2} \left(\left(\frac{N^2 - N + 1}{(N-1)^2 N^2 (N+1)(2-N)_j} + \frac{\sum_{i=1}^j \frac{(2-N)_i}{(-i+N-1)^2 (i+1)!}}{(N+1)(2-N)_j} + \frac{(-1)^{j+N} (-j-2)(-j+N-2)!}{(j-N+1)(N+1)^2 N!} \right) (j+1)! - \frac{1}{(N+1)^2 (-j+N-1)} \right)$$

$$\sum_{j=0}^{N-2} \sum_{r=0}^{j+1} \sum_{s=0}^{N-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+N+r-2}{s} (-j+N-2)! r!}{(N-s)(s+1)(-j+N+r)!}$$

||

$$\sum_{j=0}^{N-2} \left(\left(\frac{N^2 - N + 1}{(N-1)^2 N^2 (N+1)(2-N)_j} + \frac{\sum_{i=1}^j \frac{(2-N)_i}{(-i+N-1)^2 (i+1)!}}{(N+1)(2-N)_j} + \frac{(-1)^{j+N} (-j-2)(-j+N-2)!}{(j-N+1)(N+1)^2 N!} \right) (j+1)! - \frac{1}{(N+1)^2 (-j+N-1)} \right)$$

||

$$\frac{-N^2 - N - 1}{N^2(N+1)^3} + \frac{(-1)^N (N^2 + N + 1)}{N^2(N+1)^3} - \frac{2S_{-2}(N)}{N+1} + \frac{S_1(N)}{(N+1)^2} + \frac{S_2(N)}{-N-1}$$

Note: $S_a(N) = \sum_{i=1}^N \frac{\text{sign}(a)^i}{i^{|a|}}$, $a \in \mathbb{Z} \setminus \{0\}$.

A typical sum

$$\sum_{j=0}^{N-2} \sum_{s=1}^{j+1} \sum_{r=0}^{N+s-j-2} \sum_{\sigma=0}^{\infty} \frac{-2(-1)^{s+r} \binom{j+1}{s} \binom{-j+N+s-2}{r} (N-j)!(s-1)!\sigma! S_1(r+2)}{(N-r)(r+1)(r+2)(-j+N+\sigma+1)(-j+N+\sigma+2)(-j+N+s+\sigma)!}$$

A typical sum

$$\sum_{j=0}^{N-2} \sum_{s=1}^{j+1} \sum_{r=0}^{N+s-j-2} \sum_{\sigma=0}^{\infty} \frac{-2(-1)^{s+r} \binom{j+1}{s} \binom{-j+N+s-2}{r} (N-j)!(s-1)!\sigma! S_1(r+2)}{(N-r)(r+1)(r+2)(-j+N+\sigma+1)(-j+N+\sigma+2)(-j+N+s+\sigma)!}$$

$$= \frac{(2N^2 + 6N + 5) S_{-2}(N)^2}{2(N+1)(N+2)} + S_{-2,-1,2}(N) + S_{-2,1,-2}(N)$$

$$+ \dots$$

where, e.g.,

$$S_{-2,1,-2}(N) = \sum_{i=1}^N \frac{(-1)^i \sum_{j=1}^i \frac{\sum_{k=1}^j (-1)^k}{k^2}}{i^2}$$

Vermaseren 98/Blümlein/Kurth 99

A typical sum

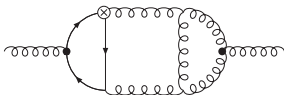
$$\begin{aligned}
& \sum_{j=0}^{N-2} \sum_{s=1}^{j+1} \sum_{r=0}^{N+s-j-2} \sum_{\sigma=0}^{\infty} \frac{-2(-1)^{s+r} \binom{j+1}{s} \binom{-j+N+s-2}{r} (N-j)!(s-1)!\sigma! S_1(r+2)}{(N-r)(r+1)(r+2)(-j+N+\sigma+1)(-j+N+\sigma+2)(-j+N+s+\sigma)!} \\
&= \frac{(2N^2 + 6N + 5) S_{-2}(N)^2}{2(N+1)(N+2)} + S_{-2,-1,2}(N) + S_{-2,1,-2}(N) \\
&+ \dots - S_{2,1,1,1}(-1, 2, \frac{1}{2}, -1; N) + S_{2,1,1,1}(1, \frac{1}{2}, 1, 2; N) \\
&+ \dots
\end{aligned}$$

where, e.g.,

145 S -sums occur

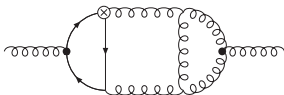
$$S_{2,1,1,1}(1, \frac{1}{2}, 1, 2; N) = \sum_{i=1}^N \frac{\sum_{j=1}^i \frac{\left(\frac{1}{2}\right)^j \sum_{k=1}^j \frac{\sum_{l=1}^k \frac{2^l}{l}}{k}}{j}}{i^2}$$

S. Moch, P. Uwer, S. Weinzierl 02



Sigma.m

Around 1000 sums are calculated containing in total 533 S -sums



Sigma.m

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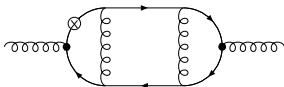


J. Ablinger's HarmonicSum.m

After elimination the following sums remain:

$$S_{-4}(N), S_{-3}(N), S_{-2}(N), S_1(N), S_2(N), S_3(N), S_4(N), S_{-3,1}(N), \\ S_{-2,1}(N), S_{2,-2}(N), S_{2,1}(N), S_{3,1}(N), S_{-2,1,1}(N), S_{2,1,1}(N)$$

Remark: For other diagrams like



non-trivial S -sums remain!
(see S. Klein's talk on Thursday)

For 3-loop ladder graphs we dealt with up to 6-fold sums:

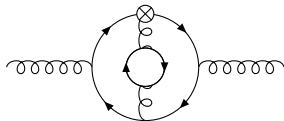
$$\begin{aligned}
 I = & C \frac{1}{(N+1)(N+2)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{l=2}^{N+2} \binom{N+2}{l} \sum_{j=2}^l \binom{l}{j} \left\{ \right. \\
 & \times \sum_{k=1}^j \binom{j}{k} \sum_{r=0}^{l-k} \binom{l-k}{r} (-1)^{l+j+k+r} B\left(k, m+1 + \frac{\varepsilon}{2}\right) \\
 & \times \Gamma \left[\begin{matrix} k+r+j+m+n + \frac{\varepsilon}{2} \\ m+1, n+1, k+r + \frac{\varepsilon}{2} \end{matrix} \right] \frac{B\left(k+m - \frac{\varepsilon}{2}, r+1+n - \frac{\varepsilon}{2}\right) B\left(r+l-1, n+1 + \frac{\varepsilon}{2}\right)}{(k+r+1+m+n-\varepsilon)(N+3-j)} \\
 & + \sum_{r=0}^{l-j} \binom{l-j}{r} (-1)^{l+j+r} B\left(j, m+1 + \frac{\varepsilon}{2}\right) \\
 & \times \Gamma \left[\begin{matrix} j+r+m+n + \frac{\varepsilon}{2} \\ m+1, n+1, j+r + \frac{\varepsilon}{2} \end{matrix} \right] \frac{B\left(j+m - \frac{\varepsilon}{2}, r+1+n - \frac{\varepsilon}{2}\right) B\left(r+l-1, n+1 - \frac{\varepsilon}{2}\right)}{(j+r+1+m+n-\varepsilon)(N+3-j)} \left. \right\}
 \end{aligned}$$

$$\begin{aligned}
 I = & \frac{C}{(N+1)(N+2)(N+3)} \left\{ \frac{1}{6} S_1^3 + \frac{N^2 + 12N + 16}{2(N+1)(N+2)} S_1^2 + \frac{4(2N+3)}{(N+1)^2(N+2)} S_1 \right. \\
 & + \frac{8(2N+3)}{(N+1)^3(N+2)} + 2 \left[-2^{N+3} + 3 - (-1)^N \right] \zeta_3 - (-1)^N S_{-3} + \left[\frac{3N^2 + 40N + 56}{2(N+1)(N+2)} - 2S_1 \right] S_2 \\
 & \left. - \frac{3N+17}{3} S_3 - 2(-1)^N S_{-2,1} - (N+3) S_{2,1} + 2^{N+4} S_{1,2} \left(\frac{1}{2}\right) + 2^{N+3} S_{1,1,1} \left(\frac{1}{2}\right) \right\} + O(\varepsilon),
 \end{aligned}$$

Example 2: 3-Loop All N-Results for the N_f Contributions

Joint work with J. Ablinger, J. Blümlein, F. Wißbrock, S. Klein 2010

E.g., for the diagram



around 700 sums are simplified (31 hours).

Simple example (6 seconds):

$$\begin{aligned}
& \sum_{j=1}^{N-2} \frac{j(j+1)(j+2)(N-j)((j-1)!)^2((-j+N-1)!)^2}{N(N+1)(-j+N-1)((N-1)!)^2} \\
&= \frac{(N+3)(6N^3+5N^2-14N+2)}{(N-1)^2 N(N+1)} \\
&+ \frac{(N!)^2(N+2)(N+3)}{(2N+1)!(N-1)} \left(-6 + \frac{3}{2} \sum_{i=1}^N \frac{(2i+1)!}{i(i!)^2} - 6 \sum_{i=1}^N \frac{(2i+1)!}{(i+1)(i!)^2} \right)
\end{aligned}$$

Not expressible in terms of harmonic sums or S -sums!

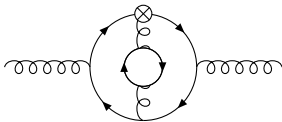
The final expression is given in terms of 700 indefinite nested sums and products. Typical examples are:

$$\sum_{i=1}^N \frac{(2i+1)! \sum_{j=1}^i \frac{1}{(2j+1)^2}}{i(i!)^2},$$

$$\sum_{i=1}^N \frac{(2i+1)! S_1(i) \sum_{j=1}^i \frac{(j+4)_{j-2}^2}{36(2j+1)!}}{(i+4)_{i-2}^2},$$

$$\sum_{i=1}^N \frac{(i!)^2 \sum_{j=1}^i \frac{(2j+1)! \sum_{k=1}^j \frac{1}{2k+3}}{j(j!)^2}}{(2i+1)!}$$

Sigma finds all algebraic relations among them. We get:



$$\begin{aligned}
 & - \frac{20S(1,N)^4}{27(N+1)(N+2)} + \frac{32(6N^3+61N^2-21N+24)S_1(N)^3}{81N^2(N+1)(N+2)} - \frac{16(48N^5+746N^4+2697N^3+2746N^2+1104N+240)S_1(N)^2}{81N^2(N+1)^2(N+2)^2} \\
 & + \frac{32(264N^7+4046N^6+21591N^5+52844N^4+74856N^3+66812N^2+30576N+2640)S_1(N)}{243N^2(N+1)^3(N+2)^3} \\
 & - \frac{4(48N^2+101N+96)S_2(N)^2}{9N(N+1)(N+2)} - \frac{32(363N^7+6758N^6+41285N^5+121235N^4+190235N^3+150758N^2+46964N+2904)}{243N(N+1)^4(N+2)^3} \\
 & + \left(-\frac{40S_1(N)^2}{9(N+1)(N+2)} + \frac{32(6N^3+61N^2-21N+24)S_1(N)}{27N^2(N+1)(N+2)} \right. \\
 & \left. - \frac{16(124N^5+198N^4-2387N^3-6162N^2-3632N-480)}{81N^2(N+1)^2(N+2)^2} \right) S_2(N) + \\
 & + \left(-\frac{32(9N^3-623N^2+894N+276)}{81N^2(N+1)(N+2)} - \frac{160S_1(N)}{27(N+1)(N+2)} \right) S_3(N) - \frac{8(56N^2+169N+112)S_4(N)}{9N(N+1)(N+2)} \\
 & + \left(\frac{64S_1(N)}{3(N+1)(N+2)} - \frac{128(N^3+9N^2-10N-6)}{9N^2(N+1)(N+2)} \right) S_{2,1}(N) + \frac{64S_{3,1}(N)}{3(N+1)(N+2)} + \frac{64(3N^2+7N+6)}{3N(N+1)(N+2)} S_{2,1,1}(N) \\
 & + \zeta_2 \left(\frac{8S_1(N)^2}{3(N+1)(N+2)} + \frac{16(3N^3-N^2+30N+12)S_1(N)}{9N^2(N+1)(N+2)} - \frac{16(3N^3+2N^2+17N+6)}{9N(N+1)^2(N+2)} - \frac{8(4N^2+9N+8)S_2(N)}{3N(N+1)(N+2)} \right) + \\
 & + \zeta_3 \left(\frac{448}{9(N+1)(N+2)} - \frac{448S_1(N)}{9(N+1)(N+2)} \right)
 \end{aligned}$$

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 - ▶ Algebraic relations of harmonic sums (up to weight 8)
(covering also differentiation/half integer relations)

Weight	Number of							
	<i>All</i>	N_A	N_D	N_H	N_{AD}	N_{AH}	N_{DH}	N_{ADH}
1	2	2	2	1	2	1	1	1
2	6	3	4	4	1	2	3	1
3	18	8	12	14	5	6	10	4
4	54	18	36	46	10	15	32	9
5	162	48	108	146	30	42	100	27
6	486	116	324	454	68	107	308	65
7	1458	312	972	1394	196	294	940	187
8	4374	810	2916	4246	498	780	2852	486

Algebraic relations for S -sums: up to weight 6 (so far)

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 - ▶ asymptotic expansion of harmonic sums

$$\begin{aligned}
 S(3, 2, 1, n) = & \frac{143}{210} \zeta_2^3 + 3 \zeta_3^2 + \left(\frac{-1}{n^2} + \frac{1}{n^3} - \frac{1}{2n^4} + \frac{1}{6n^6} - \frac{1}{6n^8} + \frac{3}{10n^{10}} - \frac{5}{6n^{12}} + \frac{691}{210n^{14}} - \frac{35}{2n^{16}} + \frac{3617}{30n^{18}} - \frac{43867}{42n^{20}} \right) \zeta_3 \\
 & + \left(\frac{1}{3n^3} - \frac{5}{8n^4} + \frac{37}{60n^5} - \frac{7}{24n^6} - \frac{37}{420n^7} + \frac{13}{80n^8} + \frac{118}{945n^9} - \frac{11}{42n^{10}} - \frac{785}{2772n^{11}} + \frac{169}{240n^{12}} \right. \\
 & + \left. \frac{42376}{45045n^{13}} - \frac{725}{264n^{14}} - \frac{1664693}{386100n^{15}} + \frac{105723}{7280n^{16}} + \frac{19976612}{765765n^{17}} - \frac{399}{4n^{18}} - \frac{58623353743}{290990700n^{19}} + \frac{3519341}{4080n^{20}} \right) (\log(n) + \gamma) \\
 & + \frac{4}{9n^3} - \frac{15}{32n^4} + \frac{107}{1800n^5} + \frac{17}{48n^6} - \frac{3341}{9800n^7} - \frac{583}{9600n^8} + \frac{2464639}{9525600n^9} + \frac{24449}{141120n^{10}} - \frac{20380229}{42688800n^{11}} \\
 & - \frac{10399}{17280n^{12}} + \frac{60101665187}{43286443200n^{13}} + \frac{643811}{232320n^{14}} - \frac{3703037408669}{649296648000n^{15}} - \frac{37263089359}{2248646400n^{16}} \\
 & + \frac{585708761937371}{18764673127200n^{17}} + \frac{77481991}{617760n^{18}} - \frac{14828831197152090581}{67740469989192000n^{19}} - \frac{102648938023}{87393600n^{20}} + O\left(\frac{1}{n^{21}}\right)
 \end{aligned}$$

Needed for limits (see first example) and for analytic continuation of harmonic sums

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- ▶ Alternative summation methods are in preparation (J. Blümlein, S. Klein, C.S., F. Stan; 2010)