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Rational general solutions of first order non-autonomous parametrizable ODEs☆

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ABSTRACT

In this paper we study non-autonomous algebraic ODEs $F(x, y, y') = 0$, where $F(x, y, z) \in \overline{\mathbb{Q}}[x, y, z]$, provided a proper rational parametrization $\mathcal{P}(s, t)$ of the corresponding algebraic surface $F(x, y, z) = 0$. We show the relation between a rational general solution of the non-autonomous differential equation $F(x, y, y') = 0$ and a rational general solution of its associated autonomous system with respect to $\mathcal{P}(s, t)$. The degrees of a rational solution $(s(x), t(x))$ of the associated system are studied by giving a degree bound for $t(x)$ in terms of the degree of $s(x)$ and the degree with respect to s of the first component of $\mathcal{P}(s, t)$. We also give a criterion for the existence of rational general solutions of the associated system provided a degree bound of its rational general solutions. The criterion is based on the vanishing of the differential pseudo remainder of Gao's differential polynomials with respect to the chain of differential polynomials derived from the associated system. We use this criterion to classify all autonomous linear systems of ODEs of order 1 having a rational general solution.

1. Introduction

In differential algebra, the solution set of a non-linear algebraic differential equation $F(x, y, y') = 0$, where $F(x, y, z) \in \overline{\mathbb{Q}}[x, y, z]$, can be decomposed into the components that are defined by their prime differential ideals. The general component is the one on which the separant of F does not vanish. In Hubert (1996), the general solutions of non-autonomous algebraic ordinary differential equations

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(ODEs) of the form $F(x, y, y') = 0$ are studied by giving a method to compute a basis of the prime differential ideal defining the general component.

The rational general solutions of autonomous algebraic ODEs of order 1, $F(y, y') = 0$, are well studied by R. Feng and X-S. Gao in recent papers Feng and Gao (2004, 2006). In these papers, the authors give an algorithm for explicitly computing a rational general solution of the autonomous algebraic ODE $F(y, y') = 0$. The method is based on a rational parametrization of the corresponding algebraic curve $F(y, z) = 0$.

In fact, $F(y, y')$ is supposed to be a non-zero irreducible differential polynomial of order 1 with coefficients in $\overline{\mathbb{Q}}$. One of the key observations in these papers is that a non-trivial rational solution of $F(y, y') = 0$ defines a proper rational parametrization of the corresponding algebraic curve $F(y, z) = 0$. Conversely, if a proper rational parametrization of the algebraic curve $F(y, z) = 0$ satisfies certain conditions, then a rational solution of $F(y, y') = 0$ can be derived from this parametrization. Moreover, from a non-trivial rational solution $y(x)$ of $F(y, y') = 0$, one can immediately create a rational general solution by shifting the variable x by an arbitrary constant c, namely $y(x + c)$ is a rational general solution of $F(y, y') = 0$. Therefore, the class of autonomous algebraic ODEs $F(y, y') = 0$ having a rational general solution can be viewed as a subclass of the class of rational algebraic curves. Moreover, the problem of computing a rational general solution is reduced to the problem of computing a non-trivial rational solution and hence computing a proper rational parametrization of a rational algebraic curve. This geometric approach has a great advantage because one can use the theory of rational algebraic curves (cf. for instance Walker, 1978; Sendra et al., 2008) to study the nature of rational solutions of autonomous algebraic ODEs of order 1. For instance, the degree of a non-trivial rational solution is exactly equal to the degree of y' in the algebraic differential equation $F(y, y') = 0$ (Feng and Gao, 2004, 2006; Sendra et al., 2008).

In this paper we study non-autonomous algebraic ODEs $F(x, y, y') = 0$ with unknown $y = y(x)$, where $F(x, y, z)$ is an irreducible polynomial in $\overline{\mathbb{Q}}[x, y, z]$, provided a proper rational parametrization $\mathcal{P}(s, t)$ of the corresponding algebraic surface $F(x, y, z) = 0$.

In Section 3 we use the parametrization $\mathcal{P}(s, t)$ to derive an autonomous system of ODEs in two differential indeterminates s, t of order 1 and of degree 1 in s' and t'. This system is called the associated system of the algebraic differential equation $F(x, y, y') = 0$ with respect to $\mathcal{P}(s, t)$. We show the relation between a rational general solution of the differential equation $F(x, y, y') = 0$ and a rational general solution of its associated system with respect to the given parametrization $\mathcal{P}(s, t)$.

In Section 4 we study the degrees of a rational solution $(s(x), t(x))$ of the associated system with respect to a parametrization $\mathcal{P}(s, t)$. We show that the degree of $t(x)$ can be bounded by the degree of $s(x)$ and the degree with respect to s of the first component of $\mathcal{P}(s, t)$.

In Section 5 we give a criterion for the existence of a rational general solution of the associated system provided a degree bound of its rational general solution. The criterion is based on the vanishing of the differential pseudo remainder of Gao's differential polynomials (Feng and Gao, 2006) with respect to a chain of differential polynomials coming from the associated system. Finally, we use this criterion to classify all autonomous linear systems of ODEs having a rational general solution.

2. Preliminaries

In this section we recall some basic notions in differential algebra such as order, initial, separant, ranking and Ritt's reduction in a ring of differential polynomials in two indeterminates. The more general definitions can be found in Ritt (1950) and Kolchin (1973).

Let $\overline{\mathbb{Q}}(x)$ be the differential field of rational functions over $\overline{\mathbb{Q}}$ with the usual derivation $\frac{\mathrm{d}}{\mathrm{d}x}$ which we also simply denote by $'$. Let s, t be two differential indeterminates over $\overline{\mathbb{Q}}(x)$. The i-th derivatives of s and t are denoted by s_i and t_i, respectively. The differential polynomial ring $\overline{\mathbb{Q}}(x)\{s, t\}$ is the ring consisting of all polynomials in s, t and all their derivatives of any order. In this paper we deal with both differential rings $\overline{\mathbb{Q}}(x)\{s, t\}$ and $\overline{\mathbb{Q}}(x)\{y\}$, where y is another differential indeterminate. We will define most notions for the case of two differential indeterminates. When we specify the situation to one indeterminate, these notions still apply.

Let F be a differential polynomial in $\overline{\mathbb{Q}}(x)\{s, t\}$. The i-th derivative of F is denoted by $F^{(i)}$. We simply write s and t instead of s_0 and t_0, respectively, or simply write F' instead of $F^{(1)}$. The *order* of F in s, denoted by $\mathrm{ord}_s(F)$, is the highest n such that s_n occurs in F. We define $\mathrm{ord}_s(F)$ to be -1 if F does not involve any derivative of s. Analogously these notions are defined for t.

Definition 2.1. Let $F, G \in \overline{\mathbb{Q}}(x)\{s, t\}$. F is said to be *of higher rank than G in s* iff one of the following conditions holds:

(1) $\mathrm{ord}_s(F) > \mathrm{ord}_s(G)$;
(2) $\mathrm{ord}_s(F) = \mathrm{ord}_s(G) = n$ and $\deg_{s_n}(F) > \deg_{s_n}(G)$.

If F is of higher rank than G in s, then we also say G is *of lower rank than F in s*. Analogously these notions are defined for t.

Definition 2.2. Let $A = \{s_i \mid i \in \mathbb{N}\} \cup \{t_i \mid i \in \mathbb{N}\}$. The *ord-lex ranking* on A is the total order defined as follows:

$$\begin{cases} s_i < s_j & \text{if } i < j, \\ t_i < t_j & \text{if } i < j, \\ t_i < s_j & \text{if } i \leq j, \\ s_i < t_j & \text{if } i < j. \end{cases}$$

The ord-lex ranking is an orderly ranking. For a formal definition of orderly ranking we prefer to Kolchin (1973, page 75).

Definition 2.3. Let F be a differential polynomial in $\overline{\mathbb{Q}}(x)\{s, t\}$. The *leader* of F is the highest derivative occurring in F with respect to the ord-lex ranking on the set of derivatives $\{s_i \mid i \in \mathbb{N}\} \cup \{t_i \mid i \in \mathbb{N}\}$. The *initial* of F is the leading coefficient of F with respect to its leader. The *separant* of F is the partial derivative of F with respect to its leader.

Observe that the separant of F is also the initial of any proper derivative $F^{(i)}$ of F.

Definition 2.4. Let F and G be differential polynomials in $\overline{\mathbb{Q}}(x)\{s, t\}$. G is said to be *reduced* with respect to F iff G is of lower rank than F in the indeterminate defining the leader of F.

Remark 2.5. With the notion of leader, we can view a differential polynomial F in $\overline{\mathbb{Q}}(x)\{s, t\}$ as a univariate polynomial in the derivative defining its leader. Then both the initial and the separant of F are reduced with respect to F.

Definition 2.6. Let $F \in \overline{\mathbb{Q}}(x)\{s, t\}$. By Ritt's reduction, for any $G \in \overline{\mathbb{Q}}(x)\{s, t\}$ there exists a unique representation

$$S^m I^n G = \sum_{i \geq 0} Q_i F^{(i)} + R,$$

where S is the separant of F, I is the initial of F, $Q_i \in \overline{\mathbb{Q}}(x)\{s, t\}$, $F^{(i)}$ are the i-th derivatives of F, $m, n \in \mathbb{N}$ and $R \in \overline{\mathbb{Q}}(x)\{s, t\}$ is reduced with respect to F. The differential polynomial R is called the *differential pseudo remainder* of G with respect to F, denoted by

$$R = \mathrm{prem}(G, F).$$

The reduction of G with respect to F is *trivial* iff $R = G$. Otherwise, the reduction is *non-trivial.*

Now let F be an irreducible differential polynomial in $\overline{\mathbb{Q}}(x)\{y\}$ and let S be the separant of F. Then $\{F\}$, the radical differential ideal generated by F, can be decomposed as

$$\{F\} = (\{F\} : S) \cap \{F, S\},$$

where $\{F\} : S$ is the quotient of $\{F\}$ by the ideal $\langle S \rangle$ generated by S. Moreover, $\{F\} : S$ is a prime differential ideal in the differential ring $\overline{\mathbb{Q}}(x)\{y\}$ and G belongs to $\{F\} : S$ iff $\mathrm{prem}(G, F) = 0$ (Ritt, 1950, II, Section 13). The prime differential ideal $\{F\} : S$ defines the *general component* of $F(y) = 0$. The differential ideal $\{F, S\}$ defines the *singular solutions* of $F(y) = 0$.

Definition 2.7. Let I be a non-trivial prime differential ideal in $\overline{\mathbb{Q}}(x)\{y\}$. A zero η, in a differential field extension of $\overline{\mathbb{Q}}(x)$, of I is called a *generic zero* of I iff for any differential polynomial $P \in \overline{\mathbb{Q}}(x)\{y\}$, P vanishes on η if and only if P belongs to I.

By Ritt (Ritt, 1950, II, Section 6), there exists a differential field extension of $\overline{\mathbb{Q}}(x)$ in which the prime differential ideal $\{F\} : S$ has a generic zero.

Definition 2.8. Let F be an irreducible differential polynomial in $\overline{\mathbb{Q}}(x)\{y\}$. A generic zero of the prime differential ideal $\{F\} : S$ is called a *general solution* of the differential equation $F(y) = 0$. A *rational general solution of* the differential equation $F(y) = 0$ is defined as a general solution of $F(y) = 0$ of the form

$$y = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0},$$

where a_i, b_j are in some field of constants $\mathbb{K}$ and $b_m \neq 0$.

Definition 2.9. Let F be an irreducible differential polynomial in $\overline{\mathbb{Q}}(x)\{y\}$. Let $\mathcal{U}$ be a differential field extension of $\overline{\mathbb{Q}}(x)$ in which a general solution of the differential equation $F(y) = 0$ is contained. The constant field $\mathbb{K}$ of $\mathcal{U}$ is called a *field of constants of solutions* of $F(y) = 0$.

3. The associated system of the parametrizable equation $F(x, y, y') = 0$

In this section we consider an ODE of order 1,

$$F(x, y, y') = 0, \tag{1}$$

such that $F(x, y, z)$ is an irreducible polynomial in $\overline{\mathbb{Q}}[x, y, z]$. If F is not involving x, then the differential equation (1) is called *autonomous*. If F is possibly involving x, then the differential equation (1) is called *non-autonomous*. We will view the autonomous case as a special case of the non-autonomous one.

Definition 3.1. Let F be an irreducible polynomial in $\overline{\mathbb{Q}}[x, y, z]$. Let $\mathbb{K}$ be a field of constants of solutions of $F(x, y, y') = 0$. By viewing x, y and z as independent variables, the algebraic equation $F(x, y, z) = 0$ defines an algebraic surface $\mathcal{S}$ over $\mathbb{K}$. $\mathcal{S}$ is called the *solution surface* of $F(x, y, y') = 0$.

A rational solution $y = f(x)$ of (1) is an element of $\mathbb{K}(x)$ such that

$$F(x, f(x), f'(x)) = 0. \tag{2}$$

If f is a rational solution of (1), then the parametric space curve

$$\mathcal{C} = \{(x, y, z) \mid x \in \mathbb{K}, y = f(x), z = f'(x)\}$$

lies on the solution surface $\mathcal{S}$.

Definition 3.2. Let $y = f(x)$ be a rational solution of $F(x, y, y') = 0$. The parametric space curve

$$\mathcal{C} = \{(x, y, z) \mid x \in \mathbb{K}, y = f(x), z = f'(x)\}$$

is called the *solution curve* of f.

Let $F(x, y, z)$ be an irreducible polynomial in $\overline{\mathbb{Q}}[x, y, z]$. The algebraic surface

$$F(x, y, z) = 0$$

is called a *rational surface* iff there exists a rational mapping

$$\mathcal{P}(s, t) = (\chi_1(s, t), \chi_2(s, t), \chi_3(s, t))$$

such that $F(\mathcal{P}(s, t)) = 0$, where χ_1, χ_2 and χ_3 are rational functions in s and t, at least two of them non-constant. In this case $\mathcal{P}(s, t)$ is called a *rational parametrization* of $F(x, y, z) = 0$. A rational parametrization $\mathcal{P}(s,t)$ of $F(x,y,z)=0$ is called *proper* iff it has an inverse, i.e., there is a rational mapping

$$\mathcal{Q}(x,y,z)=(\psi_1(x,y,z),\,\psi_2(x,y,z))$$

such that $(\mathcal{Q}\circ\mathcal{P})(s,t)=(s,t)$ for almost all s,t and

$$(\mathcal{P}\circ\mathcal{Q})(x,y,z)=(x,y,z)$$

for almost all x,y,z with $F(x,y,z)=0$.

In this paper we always consider the differential equation $F(x,y,y')=0$ where the polynomial $F(x,y,z)$ defines a rational surface. We also assume that

$$\mathcal{P}(s,t)=(\chi_1(s,t),\,\chi_2(s,t),\,\chi_3(s,t))$$

is a proper rational parametrization of $F(x,y,z)=0$, where

$$\chi_1(s,t),\,\chi_2(s,t),\,\chi_3(s,t)\in\overline{\mathbb{Q}}(s,t).$$

The inverse map of $\mathcal{P}$, denoted by $\mathcal{P}^{-1}$, is defined on the surface $\mathcal{S}$, except for finitely many curves or points on $\mathcal{S}$.

Definition 3.3. Let $f(x)$ be a rational solution of the equation $F(x,y,y')=0$. Let $\mathcal{S}$ be the solution surface of $F(x,y,y')=0$ and $\mathcal{C}$ be the solution curve of f. Let $\mathcal{P}$ be a proper rational parametrization of $F(x,y,z)=0$. The solution curve $\mathcal{C}$ *is parametrizable by* $\mathcal{P}$ iff $\mathcal{C}$ is almost contained in $\mathrm{im}(\mathcal{P})\cap\mathrm{dom}(\mathcal{P}^{-1})$, i.e., except for finitely many points on $\mathcal{C}$.

Proposition 3.4. *Let $F(x,y,z)=0$ be such that the solution surface of F is rational with a proper parametrization*

$$\mathcal{P}(s,t)=(\chi_1(s,t),\,\chi_2(s,t),\,\chi_3(s,t)).$$

The differential equation $F(x,y,y')=0$ has a rational solution whose solution curve is parametrizable by $\mathcal{P}$ if and only if the system

$$\begin{cases}\chi_1(s(x),t(x))=x\\ \chi_2(s(x),t(x))'=\chi_3(s(x),t(x))\end{cases}\tag{3}$$

has a rational solution $(s(x),t(x))$. In that case $y=\chi_2(s(x),t(x))$ is a rational solution of $F(x,y,y')=0$.

Proof. Assume that $y=f(x)$ is a rational solution of $F(x,y,y')=0$, which is parametrizable by $\mathcal{P}$. Let

$$(s(x),t(x))=\mathcal{P}^{-1}(x,f(x),f'(x)).$$

Then $s(x)$ and $t(x)$ are rational functions because $f(x)$ is a rational function and $\mathcal{P}^{-1}$ is a rational map. We have

$$\mathcal{P}(s(x),t(x))=\mathcal{P}(\mathcal{P}^{-1}(x,f(x),f'(x)))=(x,f(x),f'(x)).$$

In other words, $(s(x),t(x))$ is a rational solution of the system

$$\begin{cases}\chi_1(s(x),t(x))=x\\ \chi_2(s(x),t(x))=f(x)\\ \chi_3(s(x),t(x))=f'(x).\end{cases}\tag{4}$$

Therefore,

$$\begin{cases}\chi_1(s(x),t(x))=x\\ \chi_2(s(x),t(x))'=\chi_3(s(x),t(x)).\end{cases}$$

Conversely, if two rational functions $s=s(x)$ and $t=t(x)$ satisfy the system (3), then $y=\chi_2(s(x),t(x))$ is a rational solution of the differential equation $F(x,y,y')=0$ because $F(\mathcal{P}(s(x),t(x)))=0$. $\square$

Remark 3.5. The rational parametrizations of the solution surface of $F(x, y, z) = 0$ are not unique. Suppose that $\mathscr{P}_1(s, t)$ and $\mathscr{P}_2(s, t)$ are two different proper rational parametrizations of $F(x, y, z) = 0$. It may happen that a rational solution $y = f(x)$ of $F(x, y, y') = 0$ is parametrizable by $\mathscr{P}_1(s, t)$ but it is not parametrizable by $\mathscr{P}_2(s, t)$. This is the case when the solution curve of f is not almost contained in $\mathrm{im}(\mathscr{P}_2) \cap \mathrm{dom}(\mathscr{P}_2^{-1})$.

The system (3) can be expanded in more detail. Differentiating the first equation of (3) and expanding the last equation of (3), we obtain a linear system of equations in $s'(x)$ and $t'(x)$

$$\begin{cases} \dfrac{\partial \chi_1(s(x), t(x))}{\partial s} \cdot s'(x) + \dfrac{\partial \chi_1(s(x), t(x))}{\partial t} \cdot t'(x) = 1 \\ \\ \dfrac{\partial \chi_2(s(x), t(x))}{\partial s} \cdot s'(x) + \dfrac{\partial \chi_2(s(x), t(x))}{\partial t} \cdot t'(x) = \chi_3(s(x), t(x)). \end{cases} \tag{5}$$

If

$$\det \begin{pmatrix} \dfrac{\partial \chi_1(s(x), t(x))}{\partial s} & \dfrac{\partial \chi_1(s(x), t(x))}{\partial t} \\ \dfrac{\partial \chi_2(s(x), t(x))}{\partial s} & \dfrac{\partial \chi_2(s(x), t(x))}{\partial t} \end{pmatrix} \not\equiv 0, \tag{6}$$

then $(s(x), t(x))$ is a rational solution of the autonomous system of differential equations

$$\begin{cases} s' = \dfrac{f_1(s, t)}{g(s, t)} \\ t' = \dfrac{f_2(s, t)}{g(s, t)}, \end{cases} \tag{7}$$

where $f_1(s, t), f_2(s, t), g(s, t) \in \overline{\mathbb{Q}}(s, t)$ are defined by

$$\begin{aligned} f_1(s, t) &= \frac{\partial \chi_2(s, t)}{\partial t} - \chi_3(s, t) \cdot \frac{\partial \chi_1(s, t)}{\partial t}, \\ f_2(s, t) &= \chi_3(s, t) \cdot \frac{\partial \chi_1(s, t)}{\partial s} - \frac{\partial \chi_2(s, t)}{\partial s}, \\ g(s, t) &= \frac{\partial \chi_1(s, t)}{\partial s} \cdot \frac{\partial \chi_2(s, t)}{\partial t} - \frac{\partial \chi_1(s, t)}{\partial t} \cdot \frac{\partial \chi_2(s, t)}{\partial s}. \end{aligned} \tag{8}$$

If the determinant (6) is equal to 0, then $(s(x), t(x))$ is a solution of the system

$$\begin{cases} \bar{g}(s, t) = 0 \\ \bar{f}_1(s, t) = 0, \end{cases} \tag{9}$$

where $\bar{g}(s, t)$ and $\bar{f}_1(s, t)$ are the numerators of $g(s, t)$ and $f_1(s, t)$, respectively. Thus $(s(x), t(x))$ defines a curve if and only if $\gcd(\bar{g}(s, t), \bar{f}_1(s, t))$ is a non-constant polynomial in s, t. Otherwise, $(s(x), t(x))$ is just an intersection point of two algebraic curves $\bar{g}(s, t) = 0$ and $\bar{f}_1(s, t) = 0$, which does not satisfy the relation (3).

Definition 3.6. The autonomous system (7) is called the *associated system* of the differential equation $F(x, y, y') = 0$ with respect to $\mathscr{P}(s, t)$.

We would expect that a rational general solution of the system (7) completely determines a rational general solution of the differential equation $F(x, y, y') = 0$. At this point we define what we mean by a rational general solution of the system (7). For this purpose we need some preparation.

From now on, we consider $M_1, N_1, M_2, N_2 \in \overline{\mathbb{Q}}[s, t]$, $M_1 \neq 0, M_2 \neq 0$ and two special differential polynomials F_1 and F_2 in $\overline{\mathbb{Q}}(x)\{s, t\}$ defined as follows

$$F_1 := M_1 s' - N_1, \qquad F_2 := M_2 t' - N_2.$$

In this paper the ranking in the differential ring $\overline{\mathbb{Q}}(x)\{s,t\}$ is the ord-lex ranking defined in Definition 2.2. Note that the initial and separant of F_1 (respectively, of F_2) are the same. The differential ideal generated by F_1 and F_2 is denoted by $[F_1, F_2]$. Later we will take M_1, M_2, N_1, N_2 to be the polynomials in the denominators and the numerators of the right hand side of the system (7).

The set $A = \{F_1, F_2\}$ is an *autoreduced set relative to the ord-lex ranking* because F_1 is reduced with respect to F_2 and F_2 is reduced with respect to F_1. Let $G \in \overline{\mathbb{Q}}(x)\{s,t\}$. By Ritt's reduction (Kolchin, 1973, Proposition 1, page 79), we can reduce G with respect to the autoreduced set A, i.e., we have a unique representation

$$M_1^m M_2^n G = \sum_{i\geq 0} Q_{1i} F_1^{(i)} + \sum_{i\geq 0} Q_{2i} F_2^{(i)} + R,$$

where $Q_{1i}, Q_{2i} \in \overline{\mathbb{Q}}(x)\{s,t\}$, $m, n \in \mathbb{N}$, $F_1^{(i)}$ and $F_2^{(i)}$ are the i-th derivatives of F_1 and F_2 respectively, $R \in \overline{\mathbb{Q}}(x)[s,t]$ is reduced with respect to both F_1 and F_2. We call R the *differential pseudo remainder* of G with respect to the autoreduced set $\{F_1, F_2\}$ and we denote it by

$$R = \operatorname{prem}(G, F_1, F_2).$$

Remark 3.7. The differential pseudo remainder $R = \operatorname{prem}(G, F_1, F_2)$ is always a polynomial in $\overline{\mathbb{Q}}(x)[s,t]$ because F_1 and F_2 are of order 1 and of degree 1.

Lemma 3.8. *Let*

$$I = \{G \in \overline{\mathbb{Q}}(x)\{s,t\} \mid \operatorname{prem}(\mathrm{G}, \mathrm{F}_1, \mathrm{F}_2) = 0\}.$$

Then I is a prime differential ideal in $\overline{\mathbb{Q}}(x)\{s,t\}$.

Proof. Consider the set $H_A = \{M_1, M_2\}$. Denote $H_A^\infty = \{M_1^{m_1} M_2^{m_2} \mid m_1, m_2 \in \mathbb{N}\}$. Then

$$[F_1, F_2] : H_A^\infty := \{G \in \overline{\mathbb{Q}}(x)\{s,t\} \mid \exists J \in H_A^\infty, JG \in [F_1, F_2]\}$$

is a prime differential ideal (Ritt, 1950, V, Section 3, page 107). Let

$$I = \{G \in \overline{\mathbb{Q}}(x)\{s,t\} \mid \operatorname{prem}(G, F_1, F_2) = 0\}.$$

We prove that

$$I = [F_1, F_2] : H_A^\infty.$$

In fact it is clear that $I \subseteq [F_1, F_2] : H_A^\infty$. Let $G \in [F_1, F_2] : H_A^\infty$. Then there exists $J \in H_A^\infty$ such that $JG \in [F_1, F_2]$. On the other hand, let $R = \operatorname{prem}(G, F_1, F_2)$, we have

$$J_1 G - R \in [F_1, F_2]$$

for some $J_1 \in H_A^\infty$. It follows that $JR \in [F_1, F_2]$. Since $R, J \in \overline{\mathbb{Q}}(x)[s,t]$, this happens if and only if $JR = 0$. We must have $R = 0$ because $J \neq 0$. Therefore, $I = [F_1, F_2] : H_A^\infty$. Hence the lemma is proven. □

Definition 3.9. Let $M_1, N_1, M_2, N_2 \in \overline{\mathbb{Q}}[s,t]$, $M_1, M_2 \neq 0$. A rational solution $(s(x), t(x))$ of the autonomous system

$$\begin{cases} s' = \dfrac{N_1(s,t)}{M_1(s,t)} \\ t' = \dfrac{N_2(s,t)}{M_2(s,t)} \end{cases} \tag{10}$$

is called a *rational general solution* iff for any $G \in \overline{\mathbb{Q}}(x)\{s,t\}$ we have

$$G(s(x), t(x)) = 0 \Longleftrightarrow \operatorname{prem}(G, M_1 s' - N_1, M_2 t' - N_2) = 0.$$

Remark 3.10. A rational general solution of the system (10) is a generic zero of the prime differential ideal

$$I = \{G \in \overline{\mathbb{Q}}(x)\{s, t\} \mid \text{prem}(G, M_1 s' - N_1, M_2 t' - N_2) = 0\}.$$

Again there is a differential field extension $\mathcal{U}$ of $\overline{\mathbb{Q}}(x)$ in which the prime differential ideal I has a generic zero. We can take the smallest differential field extension of $\overline{\mathbb{Q}}(x)$ containing both general solutions of the differential equation $F(x, y, y') = 0$ and general solutions of the system (10) to be the common differential field. Therefore, we can assume that the associated system of the differential equation $F(x, y, y') = 0$ and the differential equation $F(x, y, y') = 0$ have the same field of constants of solutions, denoted by $\mathbb{K}$.

Lemma 3.11. *Let $(s(x), t(x))$ be a rational general solution of the system (10). Let G be a bivariate polynomial in $\overline{\mathbb{Q}}(x)[s, t]$. If $G(s(x), t(x)) = 0$, then $G = 0$ in $\overline{\mathbb{Q}}(x)[s, t]$.*

Proof. Since $G \in \overline{\mathbb{Q}}(x)[s, t]$, we have

$$\text{prem}(G, s'M_1(s, t) - N_1(s, t), t'M_2(s, t) - N_2(s, t)) = G.$$

By definition of general solutions, $G(s(x), t(x)) = 0$ implies $G = 0$ in $\overline{\mathbb{Q}}(x)[s, t]$. □

Lemma 3.12. *Let $\mathbb{K}$ be a field of constants of solutions of the system (10). Let*

$$s(x) = \frac{a_k x^k + a_{k-1}x^{k-1} + \cdots + a_0}{b_l x^l + b_{l-1}x^{l-1} + \cdots + b_0}$$

and

$$t(x) = \frac{c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0}{d_m x^m + d_{m-1}x^{m-1} + \cdots + d_0}$$

be a non-trivial rational solution of the system (10), where $a_i, b_i, c_i, d_i \in \mathbb{K}$, $b_l, d_m \neq 0$. If $(s(x), t(x))$ is a rational general solution of the system (10), then there exists a constant among the coefficients of $s(x)$ and $t(x)$, which is transcendental over $\overline{\mathbb{Q}}$.

Proof. Let

$$S = (b_l x^l + b_{l-1}x^{l-1} + \cdots + b_0)s - (a_k x^k + a_{k-1}x^{k-1} + \cdots + a_0)$$

and

$$T = (d_m x^m + d_{m-1}x^{m-1} + \cdots + d_0)t - (c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0).$$

Let $G = \text{res}_x(S, T)$ be the resultant of S and T with respect to x. Then G is a polynomial in s and t with the coefficients depending on a_i, b_i, c_i, d_i. If all a_i, b_i, c_i, d_i were in $\overline{\mathbb{Q}}$, then $G \in \overline{\mathbb{Q}}[s, t]$ and $G(s(x), t(x)) = 0$. Since $(s(x), t(x))$ is a rational general solution, it follows by Lemma 3.11 that $G = 0$. But G is the implicit equation of the rational curve with parametrization $(s(x), t(x))$; compare Chapter 4 Section 4.5 in Sendra et al. (2008). So $G \neq 0$, in contradiction. Therefore, there is a coefficient of $s(x)$ or $t(x)$ that does not belong to $\overline{\mathbb{Q}}$. Since $\overline{\mathbb{Q}}$ is an algebraically closed field, a constant which is not in $\overline{\mathbb{Q}}$ must be a transcendental element over $\overline{\mathbb{Q}}$. □

This lemma gives us a necessary condition for $(s(x), t(x))$ to be a rational general solution of the system (10). It requires that the rational general solution of the system has to contain at least one coefficient transcendental over the constant field of the system itself. Typically this transcendental coefficient appears as an arbitrary constant. Next we give a sufficient condition for a rational solution $(s(x), t(x))$ of the system (10) to be a rational general solution.

Lemma 3.13. *Let $(s(x), t(x))$ be a rational solution of the system (10). Let $H(s, t)$ be the defining polynomial of the rational algebraic curve defined by $(s(x), t(x))$. If there is an arbitrary constant in the set of coefficients of $H(s, t)$, then $(s(x), t(x))$ is a rational general solution of the system (10).*

Proof. Suppose that $(s(x), t(x))$ is a rational solution of the system (10). Let $G \in \overline{\mathbb{Q}}(x)\{s, t\}$ be a differential polynomial such that $G(s(x), t(x)) = 0$. Let

$$R = \operatorname{prem}(G, M_1 s' - N_1, M_2 t' - N_2).$$

Then $R \in \overline{\mathbb{Q}}(x)[s, t]$ and $R(s(x), t(x)) = 0$. Since the set of coefficients of $H(s, t)$ contains an arbitrary constant, $H(s, t)$ can be viewed as a family of polynomials in $\overline{\mathbb{Q}}[s, t]$ by specializing the arbitrary constants by algebraic numbers in $\overline{\mathbb{Q}}$. Moreover, $H(s, t)$ is a family of irreducible polynomials because $H(s, t) = 0$ is a family of rational algebraic curves (Sendra et al., 2008, Theorem 4.4). Therefore, the polynomial $R(s, t)$ must be a multiple of each individual irreducible polynomial of the infinite family $H(s, t)$. This happens if and only if $R = 0$. It follows that $(s(x), t(x))$ is a rational general solution of the system (10). □

Theorem 3.14. *Let $y = f(x)$ be a rational general solution of $F(x, y, y') = 0$. Suppose that the solution curve of f is parametrizable by $\mathcal{P}(s, t) = (\chi_1(s, t), \chi_2(s, t), \chi_3(s, t))$. Let*

$$(s(x), t(x)) = \mathcal{P}^{-1}(x, f(x), f'(x))$$

and

$$g(s, t) = \frac{\partial \chi_1(s, t)}{\partial s} \cdot \frac{\partial \chi_2(s, t)}{\partial t} - \frac{\partial \chi_1(s, t)}{\partial t} \cdot \frac{\partial \chi_2(s, t)}{\partial s}.$$

If $g(s(x), t(x)) \neq 0$, then $(s(x), t(x))$ is a rational general solution of the system (7).

Proof. Since $g(s(x), t(x)) \neq 0$, $(s(x), t(x))$ is a solution of (7). Suppose that $P \in \overline{\mathbb{Q}}(x)\{s, t\}$ is a differential polynomial such that $P(s(x), t(x)) = 0$. Let

$$R = \operatorname{prem}(P, s' M_1(s, t) - N_1(s, t), t' M_2(s, t) - N_2(s, t)),$$

where N_1, N_2, M_1 and M_2 are numerators and denominators of the right hand side of the system (7). Then $R \in \overline{\mathbb{Q}}(x)[s, t]$. We have to prove that $R = 0$. We know that

$$R(s(x), t(x)) = R(\mathcal{P}^{-1}(x, f(x), f'(x))) = 0.$$

Let us consider the rational function $R(\mathcal{P}^{-1}(x, y, z)) = \frac{U(x,y,z)}{V(x,y,z)}$. Then $U(x, y, y')$ is a differential polynomial satisfying the condition

$$U(x, f(x), f'(x)) = 0.$$

Since $f(x)$ is a rational general solution of $F(y) = 0$ and $U(x, y, y')$ vanishes on $y = f(x)$, the differential pseudo remainder of U with respect to F must be zero. On the other hand, both F and U are differential polynomials of order 1, we only divide U by F and not by any of its derivatives. Hence, we have the reduction

$$I^m U(x, y, y') = Q_0 F,$$

where I is the initial of F, $m \in \mathbb{N}$ and Q_0 is a differential polynomial of order 1 in $\overline{\mathbb{Q}}(x)\{y\}$. Therefore,

$$R(s, t) = R(\mathcal{P}^{-1}(\mathcal{P}(s, t))) = \frac{U(\mathcal{P}(s, t))}{V(\mathcal{P}(s, t))} = \frac{Q_0(\mathcal{P}(s, t))F(\mathcal{P}(s, t))}{I^m(\mathcal{P}(s, t))V(\mathcal{P}(s, t))} = 0$$

because $F(\mathcal{P}(s, t)) = 0$ and $I(\mathcal{P}(s, t)) \neq 0$. Thus $(s(x), t(x))$ is a rational general solution of (7). □

Now let us demonstrate how to construct a rational general solution of $F(x, y, y') = 0$ from a rational general solution of its associated system. Assume that $(s(x), t(x))$ is a rational general solution of the associated system (7). Substituting $s(x)$ and $t(x)$ into $\chi_1(s, t)$ and using the relation (5) we get

$$\chi_1(s(x), t(x)) = x + c$$

for some constant c. Hence

$$\chi_1(s(x - c), t(x - c)) = x.$$

It follows that $y = \chi_2(s(x-c), t(x-c))$ is a solution of the differential equation

$$F(x, y, y') = 0.$$

Moreover, we will prove that $y = \chi_2(s(x-c), t(x-c))$ is a rational general solution of $F(x, y, y') = 0$. The main theorem is the following.

Theorem 3.15. *Let $(s(x), t(x))$ be a rational general solution of the system (7). Let $c = \chi_1(s(x), t(x)) - x$. Then*

$$y = \chi_2(s(x-c), t(x-c))$$

is a rational general solution of $F(x, y, y') = 0$.

Proof. By the above discussion, it is clear that $y = \chi_2(s(x-c), t(x-c))$ is a rational solution of $F(x, y, y') = 0$. Let G be an arbitrary differential polynomial in $\overline{\mathbb{Q}}(x)\{y\}$ such that $G(y) = 0$. Let

$$R = \mathrm{prem}(G, F)$$

be the differential pseudo remainder of G with respect to F. It follows that

$$R(y) = 0.$$

We have to prove that $R = 0$. Assume that $R \neq 0$. Then

$$R(\chi_1(s,t), \chi_2(s,t), \chi_3(s,t)) = \frac{W(s,t)}{Z(s,t)} \in \overline{\mathbb{Q}}(s,t).$$

On the other hand,

$$R(\chi_1(s(x), t(x)), \chi_2(s(x), t(x)), \chi_3(s(x), t(x))) = 0.$$

It follows that $W(s(x), t(x)) = 0$. By Lemma 3.11 we must have $W(s,t) = 0$. Thus

$$R(\chi_1(s,t), \chi_2(s,t), \chi_3(s,t)) = 0.$$

Since F is irreducible and $\deg_{y'} R < \deg_{y'} F$, we have $R = 0$ in $\overline{\mathbb{Q}}[x, y, z]$. Therefore, y is a rational general solution of $F(x, y, y') = 0$. $\square$

4. A degree bound for rational solutions of the associated system

We have studied the algebraic ODE of order 1, $F(x, y, y') = 0$, provided a proper rational parametrization

$$\mathcal{P}(s,t) = (\chi_1(s,t), \chi_2(s,t), \chi_3(s,t))$$

of the solution surface $F(x, y, z) = 0$. We know that every rational solution $(s(x), t(x))$ of the associated system of $F(x, y, y') = 0$ w.r.t $\mathcal{P}(s,t)$ satisfies the condition

$$\chi_1(s(x), t(x)) = x.$$

From this condition we can deduce that the degree of $t(x)$ is bounded in terms of the degree of $s(x)$ and the degree of $\chi_1(s,t)$ with respect to s.

Theorem 4.1. *Let*

$$\chi_1(s,t) = \frac{a_n(t)s^n + a_{n-1}(t)s^{n-1} + \cdots + a_0(t)}{b_m(t)s^m + b_{m-1}(t)s^{m-1} + \cdots + b_0(t)} \in \mathbb{K}(s,t)$$

be such that $\chi_1(s,t) \notin \mathbb{K}(s)$ and $\chi_1(s,t) \notin \mathbb{K}(t)$, where $m, n \in \mathbb{N}$ and $b_m(t) \neq 0$. Suppose that $s(x)$ and $t(x)$ are rational functions in $\mathbb{K}(x)$, which satisfy the condition

$$\chi_1(s(x), t(x)) = x.$$

Let $\delta = \deg s(x)$. Then

$$\deg t(x) \leq 1 + \delta \max\{m, n\}.$$

Proof. We have

$$\chi_1(s(x), t(x)) = x \iff \frac{a_n(t(x))s(x)^n + a_{n-1}(t(x))s(x)^{n-1} + \cdots + a_0(t(x))}{b_m(t(x))s(x)^m + b_{m-1}(t)s(x)^{m-1} + \cdots + b_0(t(x))} = x.$$

We know that for any rational function $t \in \mathbb{K}(x)$, x is algebraic over $\mathbb{K}(t)$ and

$$\deg t(x) = [\mathbb{K}(x) : \mathbb{K}(t)].$$

Therefore, in order to find a degree bound for t, it is enough to find an algebraic equation for x over $\mathbb{K}(t)$. Let $s(x) = \frac{P}{Q}$, where $P, Q \in \mathbb{K}[x]$, $Q \neq 0$. Let

$$\delta = \deg s(x) = \max\{\deg P, \deg Q\}, \quad l = \deg Q.$$

We have

$$\begin{aligned} x &= \frac{Q^m}{Q^n} \cdot \frac{(a_n(t)P^n + \cdots + a_0(t)Q^n)}{(b_m(t)P^m + \cdots + b_0(t)Q^m)} \\ &= Q^{m-n} \cdot \frac{(a_n(t)P^n + \cdots + a_0(t)Q^n)}{(b_m(t)P^m + \cdots + b_0(t)Q^m)}. \end{aligned}$$

This equation derives a non-zero algebraic equation of x over $\mathbb{K}(t)$ because $\chi_1(s, t) \notin \mathbb{K}(s)$ and $\chi_1(s, t) \notin \mathbb{K}(t)$. We can compute the degree of x in the above equation regarding $l \leq \delta$.

If $n \geq m$, then

$$\deg t(x) \leq \max\{1 + m\delta + l(n - m), n\delta\} \leq 1 + n\delta.$$

If $n < m$, then

$$\deg t(x) \leq \max\{1 + m\delta, n\delta + l(m - n)\} \leq 1 + m\delta.$$

Therefore, $\deg t(x) \leq 1 + \delta \max\{m, n\}$. □

Of course, the degree of $s(x)$ can also be bounded in the same way by the degree of $t(x)$ and the degree of the first component of $\mathcal{P}(s, t)$ with respect to t.

5. A criterion for existence of rational general solutions of the associated system

In this section we derive a criterion for the existence of a rational general solution of the associated system of the equation $F(x, y, y') = 0$. The following lemma can be found in Feng and Gao (2006).

Lemma 5.1. *Let $n, m \in \mathbb{N}$. There exists a differential polynomial $D_{n,m}(y)$ such that every rational function*

$$y = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}$$

is a solution of $D_{n,m}(y)$, where a_i, b_j are constants in $\mathbb{K}$. Moreover, the differential polynomial $D_{n,m}(y)$ has only rational solutions.

Definition 5.2. The differential polynomial in Lemma 5.1 is given by

$$D_{n,m}(y) = \begin{vmatrix} \binom{n+1}{0} y^{(n+1)} & \binom{n+1}{1} y^{(n)} & \cdots & \binom{n+1}{m} y^{(n+1-m)} \\ \binom{n+2}{0} y^{(n+2)} & \binom{n+2}{1} y^{(n+1)} & \cdots & \binom{n+2}{m} y^{(n+2-m)} \\ \vdots & \vdots & \cdots & \vdots \\ \binom{n+1+m}{0} y^{(n+1+m)} & \binom{n+1+m}{1} y^{(n+m)} & \cdots & \binom{n+1+m}{m} y^{(n+1)} \end{vmatrix}.$$

We call $D_{n,m}(y)$ *Gao's differential polynomials.*

Using Gao's differential polynomials we have the following criterion.

Theorem 5.3. *Let $M_1, N_1, M_2, N_2 \in \overline{\mathbb{Q}}[s,t]$, $M_1, M_2 \neq 0$. The autonomous system*

$$\begin{cases} s' = \dfrac{N_1(s,t)}{M_1(s,t)} \\ t' = \dfrac{N_2(s,t)}{M_2(s,t)} \end{cases} \tag{11}$$

has a rational general solution $(s(x), t(x))$ with $\deg s(x) \le n$ and $\deg t(x) \le m$ if and only if

$$\begin{cases} \mathrm{prem}(\mathrm{D_{n,n}(s), M_1 s' - N_1, M_2 t' - N_2}) = 0 \\ \mathrm{prem}(\mathrm{D_{m,m}(t), M_1 s' - N_1, M_2 t' - N_2}) = 0. \end{cases} \tag{12}$$

Proof. Suppose that the system (11) has a rational general solution $(s(x), t(x))$ with $\deg s(x) \le n$ and $\deg t(x) \le m$. Then $(s(x), t(x))$ is a solution of $D_{n,n}(s)$ and $D_{m,m}(t)$ as well. By definition of rational general solutions of the system (11) we have

$$\begin{cases} \mathrm{prem}(\mathrm{D_{n,n}(s), M_1 s' - N_1, M_2 t' - N_2}) = 0 \\ \mathrm{prem}(\mathrm{D_{m,m}(t), M_1 s' - N_1, M_2 t' - N_2}) = 0. \end{cases}$$

Conversely, if these two conditions hold, then $D_{n,n}(s)$ and $D_{m,m}(t)$ belong to the set I defined as follows

$$I := \{G \in \overline{\mathbb{Q}}(x)\{s,t\} \mid \mathrm{prem}(G, M_1 s' - N_1, M_2 t' - N_2) = 0\}.$$

Since I is a prime differential ideal, I has a generic zero. This generic zero is a zero of $D_{n,n}(s)$ and $D_{m,m}(t)$. By Lemma 5.1, these two differential polynomials have only rational solutions. Therefore, the generic zero of I must be rational. □

Remark 5.4. If we know a degree bound of the rational solutions of the system (11), then Theorem 5.3 gives us a criterion for the existence of a rational general solution of the system (11).

6. The linear system of autonomous ODEs with rational general solutions

We have seen that the associated system of the differential equation $F(x, y, y') = 0$ is an autonomous system. In this section we consider the linear system of autonomous differential equations

$$\begin{cases} s' = as + bt + e \\ t' = cs + dt + h \end{cases} \tag{13}$$

where a, b, c, d, e, h are constants in $\overline{\mathbb{Q}}$.

Before studying rational solutions of the system (13) we need to introduce the notation of order of an irreducible polynomial in a rational function.

Definition 6.1. Let $\mathbb{K}$ be a field. Let $s \in \mathbb{K}(x)$ be a rational function in x. Suppose that s has a complete decomposition as follows

$$s = \frac{A}{p_1^{\alpha_1} \cdots p_n^{\alpha_n}}$$

where $A \in \mathbb{K}[x]$ and p_i are distinct irreducible polynomials over $\overline{\mathbb{K}}$ and $\gcd(A, p_i) = 1$ for all $i = 1, \ldots, n$. The power α_i in this representation of s is called the *order* of s with respect to p_i, denoted by $\mathrm{ord}_{p_i}(s)$. By convention, if an irreducible polynomial p does not effectively appear in the denominator of s, then we define $\mathrm{ord}_p(s) = 0$.

Lemma 6.2. *Every rational solution of the linear system (13) is a polynomial solution.*

Proof. Suppose that $(s(x), t(x))$ is a rational solution of the linear system (13). If $s(x)$ or $t(x)$ is not a polynomial, then we can assume without loss of generality that

$$s = \frac{A}{p_1^{\alpha_1} \cdots p_n^{\alpha_n}},$$

where $A \in \mathbb{K}[x]$, p_i are irreducible polynomials over $\overline{\mathbb{K}}$, $\gcd(A, p_i) = 1$ and $\alpha_i > 0$ for all $i = 1, \ldots, n$. Let

$$\beta_i = \mathrm{ord}_{p_i}(t) \geq 0 \quad \forall i = 1, \ldots, n.$$

Since $\alpha_i > 0$, computing the derivative of $s(x)$ we have

$$\mathrm{ord}_{p_i}(s') = \alpha_i + 1 \quad \forall i = 1, \ldots, n.$$

On the other hand,

$$\mathrm{ord}_{p_i}(as + bt + e) \leq \max\{\alpha_i, \beta_i\}, \qquad \mathrm{ord}_{p_i}(cs + dt + h) \leq \max\{\alpha_i, \beta_i\}.$$

Let us compare the orders with respect to p_i of the left and the right hand sides of the linear system (13). There are two cases as follows.

- Either $\alpha_i \geq \beta_i$, then $\mathrm{ord}_{p_i}(as + bt + e) \leq \alpha_i < \mathrm{ord}_{p_i}(s')$, which is impossible;
- or $0 < \alpha_i < \beta_i$, then $\mathrm{ord}_{p_i}(cs + dt + h) \leq \beta_i < \mathrm{ord}_{p_i}(t')$, which is also impossible.

Therefore, $\alpha_i = 0$ for all $i = 1, \ldots, n$. Thus s is a polynomial. Replacing the role of s and t we also prove that t is a polynomial. Therefore, $(s(x), t(x))$ is a polynomial solution. $\square$

Theorem 6.3. *Every rational general solution of the linear system* (13) *is a couple of polynomials of degree at most 2.*

Proof. By Lemma 6.2, every rational solution of the linear system (13) is a polynomial solution. In this case the Gao's differential polynomials for checking polynomial general solutions of the system are of simple forms $s^{(n+1)}$ and $t^{(n+1)}$ for some n. We can write the linear system in the matrix form

$$\begin{pmatrix} s' \\ t' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} + \begin{pmatrix} e \\ h \end{pmatrix}.$$

Hence

$$\begin{pmatrix} s'' \\ t'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 \begin{pmatrix} s \\ t \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e \\ h \end{pmatrix},$$

$$\vdots$$

$$\begin{pmatrix} s^{(n+1)} \\ t^{(n+1)} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{n+1} \begin{pmatrix} s \\ t \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}^n \begin{pmatrix} e \\ h \end{pmatrix},$$

for $n \in \mathbb{N}$. By Theorem 5.3, the system (13) has a polynomial general solution of degree at most n if and only if

$$\begin{cases} \mathrm{prem}(s^{(n+1)}, s' - as - bt - e, t' - cs - dt - h) = 0 \\ \mathrm{prem}(t^{(n+1)}, s' - as - bt - e, t' - cs - dt - h) = 0, \end{cases}$$

or equivalently when

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{n+1} = 0 \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^n \begin{pmatrix} e \\ h \end{pmatrix} = 0.$$

These relations hold for $n \geq 2$ if and only if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = 0.$$

Therefore, the conclusion of the theorem follows immediately. □

The above proof also tell us the necessary and sufficient conditions of the linear system for having rational general solutions. Namely,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = 0.$$

We can easily find all possibilities of the coefficients a, b, c, d. In fact,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = 0 \Longleftrightarrow \begin{cases} a^2 + bc = 0 \\ b(a+d) = 0 \\ c(a+d) = 0 \\ d^2 + bc = 0. \end{cases}$$

Solving this algebraic system we obtain the following cases

- if $b = 0$, then $a = d = 0$;
- if $b \neq 0$, then $a = -d$ and $c = -\frac{d^2}{b}$.

Thus the explicit polynomial solutions of the linear system are given by the following table, where C_1, C_2 are arbitrary constants.

System	Rational general solutions
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{cases} s(x) = ex + C_1 \\ t(x) = hx + C_2 \end{cases}$
$\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$	$\begin{cases} s(x) = ex + C_1 \\ t(x) = ce\dfrac{x^2}{2} + (cC_1 + h)x + C_2 \end{cases}$
$\begin{pmatrix} -d & b \\ -\dfrac{d^2}{b} & d \end{pmatrix}$	$\begin{cases} s(x) = \dfrac{hb - ed}{2}x^2 + (bC_1 + e)x + C_2 \\ t(x) = \dfrac{(hb - ed)d}{2b}x^2 + (dC_1 + h)x + \dfrac{d}{b}C_2 + C_1 \end{cases}$

Note that the last line of the table also covers the other symmetric cases, for instance

$$d = 0 \longmapsto \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}; \quad d = -a,\ b = -\frac{a^2}{c} \longmapsto \begin{pmatrix} a & -\dfrac{a^2}{c} \\ c & -a \end{pmatrix}.$$

We can prove that the solutions in the table are rational general solutions of the corresponding system. For instance, consider a simple system

$$\begin{cases} s' = e \\ t' = h \end{cases}$$

where e and h are constants but not all zero. It turns out that the system has a solution given by

$$s(x) = ex + C_1, \qquad t(x) = hx + C_2,$$

where C_1, C_2 are arbitrary constants. The implicit defining polynomial of $(s(x), t(x))$ is

$$H(s, t) = hs - et - hC_1 + eC_2.$$

Since the coefficients of $H(s, t)$ contain an arbitrary constant, namely $-hC_1 + eC_2$, using Lemma 3.13 we obtain that $(s(x), t(x))$ is a rational general solution.

Using a similar argument for the other systems in the table we prove that these solutions are rational general solutions of the corresponding systems.

7. Algorithm and example

We summarize the procedure in Section 3 by the following semi-algorithm. It depends on a method for solving a system (7).

Algorithm 1. *Input:* $F(x, y, y') = 0$ with $F \in \overline{\mathbb{Q}}[x, y, z]$ and a proper rational parametrization of $F(x, y, z) = 0$,

$$\mathcal{P}(s, t) = (\chi_1(s, t), \chi_2(s, t), \chi_3(s, t)),$$

where $\chi_1, \chi_2, \chi_3 \in \overline{\mathbb{Q}}(s, t)$.
Output: a rational general solution of $F(x, y, y') = 0$.

(1) Compute $f_1(s, t), f_2(s, t), g(s, t)$ as in (8).
(2) Solve the associated system of $F(x, y, y') = 0$ for a rational general solution $(s(x), t(x))$

$$\begin{cases} s' = \dfrac{f_1(s, t)}{g(s, t)} \\ t' = \dfrac{f_2(s, t)}{g(s, t)}. \end{cases}$$

(3) Compute the constant $c := \chi_1(s(x), t(x)) - x$.
(4) Return $y = \chi_2(s(x - c), t(x - c))$.

Note that we still have to solve the associated system for its rational general solutions in general cases. Currently we are developing a method for finding a rational general solution of such systems.

Example 7.1. Consider the differential equation

$$F(x, y, y') \equiv y'^3 - 4xyy' + 8y^2 = 0. \tag{14}$$

This differential equation appears in the paper by Hubert (1996). Here we demonstrate our approach for this differential equation. The algebraic surface $F(x, y, z) = 0$ has a proper parametrization

$$\mathcal{P}(s, t) = (t, -4s^2 \cdot (2s - t), -4s \cdot (2s - t)).$$

The inverse map is

$$\mathcal{P}^{-1}(x, y, z) = \left(\frac{y}{z}, x\right).$$

We compute

$$g(s, t) = 8s \cdot (3s - t),$$
$$f_1(s, t) = 4s \cdot (3s - t), \qquad f_2(s, t) = 8s \cdot (3s - t).$$

In this case the associated system is simple

$$\begin{cases} s' = \dfrac{1}{2} \\ t' = 1. \end{cases}$$

Solving this system we obtain a rational general solution $s(x) = \frac{x}{2} + C_2,\ t(x) = x + C_1$ for arbitrary constants C_1, C_2. The above algorithm follows that the rational general solution of the differential equation $F(x, y, y') = 0$ is

$$y = -4s(x - C_1)^2 \cdot (2s(x - C_1) - t(x - C_1)) = -C(x + C)^2, \tag{15}$$

where $C = 2C_2 - C_1$.

Note that in this example $\gcd(g(s,t), f_1(s,t)) = 4s \cdot (3s - t)$. We may still find some solutions of the differential equation (14), whose solution curves are parametrizable by $\mathcal{P}$, by solving the system

$$\begin{cases} t(x) = x \\ -4s(x)^2 \cdot (2s(x) - t(x)) = f(x) \\ -4s(x) \cdot (2s(x) - t(x)) = f'(x) \\ 4s(x) \cdot (3s(x) - t(x)) = 0. \end{cases} \tag{16}$$

This system has two different solutions, namely

$$(s(x), t(x)) = (0, x) \quad \text{and} \quad (s(x), t(x)) = \left(\frac{x}{3}, x\right).$$

These solutions give us two other solutions of Eq. (14), namely $y = 0$ and $y = \frac{4}{27}x^3$. The solution $y = 0$ can be obtained by specifying the constant $C = 0$ in the general solution (15). However, we can not get the solution $y = \frac{4}{27}x^3$ from the general solution (15). Note that the separant of F is $S = 3y'^2 - 4xy$. We can prove that the common solutions of F and S, which are called *singular solutions* of $F(x, y, y') = 0$, are only $y = 0$ and $y = \frac{4}{27}x^3$.

8. Conclusion

We have shown the correspondence between a rational general solution of the non-autonomous differential equation $F(x, y, y') = 0$ and a rational general solution of its associated system of autonomous ODEs with respect to a proper rational parametrization $\mathcal{P}(s, t)$ of the solution surface $F(x, y, z) = 0$. The degrees of the components of a rational solution of the associated system are related to each other in the sense that one of them can be bounded by the degree of the other one and the appropriate degree of the first component of $\mathcal{P}(s, t)$. We have derived a criterion for the existence of a rational general solution of the associated system provided a degree bound of the solution. Currently we are working on turning the semi-algorithm Algorithm 1 into a full algorithm.

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References

Feng, R., Gao, X.-S., 2004. Rational general solutions of algebraic ordinary differential equations. In: Proc. ISSAC2004. ACM Press, New York, pp. 155–162.
Feng, R., Gao, X-S., 2006. A polynomial time algorithm for finding rational general solutions of first order autonomous ODEs. J. Symbolic Comput. 41, 739–762.
Hubert, E., 1996. The general solution of an ordinary differential equation. In: Proc. ISSAC1996. ACM Press, New York, pp. 189–195.
Kolchin, E.R., 1973. Differential Algebra and Algebraic Groups. Academic Press.
Ritt, J.F., 1950. Differential Algebra. In: Amer. Math. Society, vol. 33. Colloquium Publications.
Sendra, J.R., Winkler, F., Pérez-Díaz, S., 2008. Rational Algebraic Curves — A Computer Algebra Approach. Springer.
Walker, R.J., 1978. Algebraic Curves. Springer-Verlag.