QCD: The Modern View of the Strong Interactions

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Modern summation technologies in computer algebra

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Indefinite summation

Simplify

$$\sum_{k=0}^{a}\left(1-(n-2k)\,S_1(k)\right)\binom{n}{k}^{-1} = \text{?},$$

where $S_1(k) := \sum_{i=1}^{k}\frac{1}{i}\ \ (= H_k)$.

GIVEN $f(k) = \big(1 - (n-2k)\, S_1(k)\big)\binom{n}{k}^{-1}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

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FIND $g(k)$:

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Sigma computes

$$g(k) = (n-k+1)\,S_1(k)\binom{n}{k}^{-1}$$

GIVEN $f(k) = \left(1 - (n-2k)\,S_1(k)\right)\binom{n}{k}^{-1}$

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Summing the telescoping equation over k from 0 to a gives

$$\boxed{\sum_{k=0}^{a}\left(1 - (n-2k)\,S_1(k)\right)\binom{n}{k}^{-1}} = \boxed{g(a+1) - g(0)}$$

$$= 1 + (1+a)S_1(a)\binom{n}{a}^{-1}.$$

GIVEN $f(k)=\left(1-(\mathbf{n}-2k)\,S_1(k)\right)\binom{n}{k}^{-1}$

FIND $g(k)$:

$$\boxed{f(k)=g(k+1)-g(k)}$$

A difference field for the **summand:**

Construct a rational function field

$$\mathbb{F}:=\mathbb{Q}(\mathbf{n})$$

and a field automorphism $\sigma:\mathbb{F}\to\mathbb{F}$ defined by

$$\sigma(\mathbf{c})=\mathbf{c}\quad\forall\mathbf{c}\in\mathbb{Q}(\mathbf{n}),$$

GIVEN $f(k) = \left(1 - (n - 2\mathbf{k})\, S_1(k)\right) \binom{n}{k}^{-1}$

FIND $g(k)$:

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A difference field for the **summand:**

Construct a rational function field

$$\mathbb{F} := \mathbb{Q}(n)(\mathbf{k})$$

and a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),$$

$$\sigma(\mathbf{k}) = \mathbf{k} + \mathbf{1}, \qquad\qquad \mathcal{S}\, k = k + 1,$$

GIVEN $f(k)=\left(1-(n-2k)\,\mathbf{S_1(k)}\right)\binom{n}{k}^{-1}$

FIND $g(k)$:

$$\boxed{f(k)=g(k+1)-g(k)}$$

A difference field for the **summand:**

Construct a rational function field

$$\mathbb{F}:=\mathbb{Q}(n)(k)(\mathbf{h})$$

and a field automorphism $\sigma:\mathbb{F}\to\mathbb{F}$ defined by

$$\sigma(c)=c\quad\forall c\in\mathbb{Q}(n),$$

$$\sigma(k)=k+1,\qquad\qquad \mathcal{S}\,k=k+1,$$

$$\sigma(\mathbf{h})=\mathbf{h}+\frac{\mathbf{1}}{\mathbf{k+1}},\qquad\qquad \mathcal{S}\,S_1(k)=S_1(k)+\frac{1}{k+1},$$

GIVEN $f(k) = \big(1 - (n-2k)\,S_1(k)\big)\binom{\mathbf{n}}{\mathbf{k}}^{-1}$

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A difference field for the **summand:**

Construct a rational function field

$(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-field

$$\mathbb{F} := \mathbb{Q}(n)(k)(h)(\mathbf{b})$$

Karr 1981

and a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),$$

$$\sigma(k) = k+1, \qquad \mathcal{S}\,k = k+1,$$

$$\sigma(h) = h + \frac{1}{k+1}, \qquad \mathcal{S}\,S_1(k) = S_1(k) + \frac{1}{k+1},$$

$$\sigma(\mathbf{b}) = \frac{\mathbf{n}-\mathbf{k}}{\mathbf{k}+\mathbf{1}}\,\mathbf{b}, \qquad \mathcal{S}\binom{n}{k} = \frac{n-k}{k+1}\binom{n}{k}.$$

GIVEN $f(k) = \left(1 - (n-2k)\, S_1(k)\right)\binom{n}{k}^{-1}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

$$\downarrow$$

GIVEN $f := (1-(n-2k)h)b^{-1} \in \mathbb{F}$.
FIND $g \in \mathbb{F}$:

$$\boxed{f = \sigma(g) - g}$$

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$\downarrow$ Sigma

$$g = (n-k+1)h\, b^{-1}$$

$h \equiv S_1(k)$
$b \equiv \binom{n}{k}$

A family of identities

$$\boxed{\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha} =?}$$

$\alpha = -1$:

$$\sum_{k=0}^{a}\left(1-(n-2k)S_1(k)\right)\binom{n}{k}^{-1} = \frac{(a+1)S_1(a)+1}{\binom{n}{a}}$$

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$$\sum_{k=0}^{n}\left(1-(n-2k)S_1(k)\right)\binom{n}{k}^{-1}=(n+1)S_1(n)+1$$

$\alpha=-2$:

$$\sum_{k=0}^{a}\left(1-2(n-2k)S_1(k)\right)\binom{n}{k}^{-2}$$

$$=\frac{(n+1)^2}{(n+2)^2}+\frac{(a+1)(-a+2n+2(a+1)(n+2)S_1(a)+3)}{(n+2)^2\binom{n}{a}^{-2}}$$

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$$=\frac{(n+1)^2}{(n+2)^2}+\frac{\left(n+2\left(n^2+3n+2\right)S_1(n)+3\right)(n+1)}{(n+2)^2}$$

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$\alpha=-3$:

$$\sum_{k=0}^{n}\left(1-3(n-2k)S_1(k)\right)\binom{n}{k}^{-3} = ?$$

Telescoping

GIVEN

$$\mathsf{SUM}(n) := \sum_{k=0}^{n} \underbrace{(1 - 3(n-2k)S_1(k))\binom{n}{k}^{-3}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{\quad f(n,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{SUM}(n) := \sum_{k=0}^{n} \underbrace{(1-3(n-2k)S_1(k))\binom{n}{k}^{-3}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)\,f(n+1,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

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$$\mathsf{SUM}(n) := \sum_{k=0}^{n} \underbrace{\left(1-3(n-2k)S_1(k)\right)\binom{n}{k}^{-3}}_{=:\,f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)\,f(n+1,k)+c_2(n)\,f(n+2,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

Sigma *computes*: $c_0(n)=(n+2)^4(n+3)^2$, $c_1(n)=(n+1)^3(n+3)^2(2n+5)$, $c_2(n)=(n+1)^3(n+2)^3$, and

$$g(n,k) := \binom{n}{k}^{-3} p_1(k,n,S_1(k)),$$

$$g(n,k+1) := \binom{n}{k}^{-3} p_2(k,n,S_1(k)).$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{SUM}(n) := \sum_{k=0}^{n} \underbrace{\left(1 - 3(n-2k)S_1(k)\right)\binom{n}{k}^{-3}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 0 to n gives:

$$\boxed{g(n,n+1) - g(n,0)} =$$

$$\boxed{\begin{array}{l} c_0(n)\,\mathsf{SUM}(n)+ \\ c_1(n)\left[\mathsf{SUM}(n+1) - f(n+1,n+1)\right]+ \\ c_2(n)\left[\mathsf{SUM}(n+2) - f(n+2,n+1) - f(n+2,n+2)\right]. \end{array}}$$

Zeilberger's creative telescoping paradigm

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$$\text{SUM}(n) := \sum_{k=0}^{n} \underbrace{\left(1 - 3(n-2k)S_1(k)\right)\binom{n}{k}^{-3}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

Summing this equation over k from 0 to n gives:

Solve recurrence

$$\boxed{g(n,n+1) - g(n,0)} =$$

$$\boxed{\begin{array}{l} c_0(n)\,\text{SUM}(n) + \\ c_1(n)\left[\text{SUM}(n+1) - f(n+1,n+1)\right] + \\ c_2(n)\left[\text{SUM}(n+2) - f(n+2,n+1) - f(n+2,n+2)\right]. \end{array}}$$

A family of identities

$$\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha}=?$$

$\alpha=-1$:

$$\sum_{k=0}^{n}\left(1-(n-2k)S_1(k)\right)\binom{n}{k}^{-1}=(n+1)S_1(n)+1$$

$\alpha=-2$:

$$\sum_{k=0}^{n}\left(1-2(n-2k)S_1(k)\right)\binom{n}{k}^{-2}$$

$$=\frac{(n+1)^2}{(n+2)^2}+\frac{\left(n+2\left(n^2+3n+2\right)S_1(n)+3\right)(n+1)}{(n+2)^2}$$

$\alpha=-3$:

$$\sum_{k=0}^{n}\left(1-3(n-2k)S_1(k)\right)\binom{n}{k}^{-3}=$$

$$=5(-1)^nS_{-3}(n)(n+1)^3$$

$$-6(-1)^nS_{-2,1}(n)(n+1)^3+6S_1(n)(n+1)+1$$

A family of identities

$$\sum_{k=0}^{n} \left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha} = ?$$

$\alpha = -4$:

$$\sum_{k=0}^{n}\left(1-4(n-2k)S_1(k)\right)\binom{n}{k}^{-4} = \frac{(10(n+1)S_1(n)+3)(n+1)}{2n+3}$$
$$+\frac{(-1)^n\binom{2n}{n}^{-1}(n+1)^5}{(4n(n+2)+3)}\left(\frac{7}{2}\sum_{i=1}^{n}\frac{(-1)^i\binom{2i}{i}}{i^3}-5\sum_{i=1}^{n}\frac{(-1)^i\binom{2i}{i}S_1(i)}{i^2}\right)$$

A family of identities discovered by S. Ahlgren (see Paule/Schneider 03):

$$\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha} =?$$

extended → Krattenthaler/Rivoal 07

$\alpha = 1$:
$$\sum_{k=0}^{n}\left(1+(n-2k)S_1(k)\right)\binom{n}{k} = 1$$

$\alpha = 2$:
$$\sum_{k=0}^{n}\left(1+2(n-2k)S_1(k)\right)\binom{n}{k}^{2} = 0$$

$\alpha = 3$:
$$\sum_{k=0}^{n}\left(1+3(n-2k)S_1(k)\right)\binom{n}{k}^{3} = (-1)^n$$

$\alpha = 4$:
$$\sum_{k=0}^{n}\left(1+4(n-2k)S_1(k)\right)\binom{n}{k}^{4} = (-1)^n\binom{2\,n}{n}$$

$\alpha = 5$:
$$\sum_{k=0}^{n}\left(1+5(n-2k)S_1(k)\right)\binom{n}{k}^{5} = (-1)^n\sum_{j=0}^{n}\binom{n}{j}^{2}\binom{n+j}{j}$$

1. Creative telescoping

GIVEN a definite sum

$$S(n)=\sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum;
n: extra parameter

FIND a recurrence for $S(n)$

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FIND a recurrence for $S(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n),\dots,a_d(n),h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)S(n)+\dots+a_d(n)S(n+d)=h(n);$$

FIND all solutions expressible by indefinite nested products and sums
(Nörlund 24, Abramov/Petkovšek 94, Hendriks/Singer 99/Sigma 01)

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NOTE: By construction, the solutions are highly nested.

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3. Indefinite summation

Simplify the solutions:

- No algebraic relations occur among the sums
- The sums have minimal nested depth.

1. Creative telescoping

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$a_0(n), \dots, a_d(n), h(n)$:
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$$a_0(n)S(n) + \cdots + a_d(n)S(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products and sums
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4. Find a "closed form"

S(n)=combined solutions.

Example 1: In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon)=\int_0^1 dx_1\dots\int_0^1 dx_7\sum_{i=1}^{K}\frac{p_i(x_1,x_2,\dots,x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1,x_2,\dots,x_7)^{\dots+s_i\varepsilon+\dots}}$$

where $K\in\mathbb{N}$, $r_i,s_i\in\mathbb{Q}$, and p_i,q_i are polynomials in $x_1,\dots,x_7$.

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$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + \boxed{F_{-1}(n)}\varepsilon^{-1} + \dots$$

The 3-loop anomalous dimensions can be derived from the single pole part of $F(n,\varepsilon)$. The other poles are needed for the renormalization.

J. Vermaseren, S. Moch: 3-5 CPU years (2004)

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$$=F_{-3}(n)\varepsilon^{-3}+F_{-2}(n)\varepsilon^{-2}+\boxed{F_{-1}(n)}\varepsilon^{-1}+\dots$$

$\downarrow$ (J. Blümlein, DESY)

Initial values $F_{-1}(i)$, $i=1,\dots,450$ (difficult, unsolved task)

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$\downarrow$ Recurrence finder (M. Kauers)

$$a_0(n)F_{-1}(n) + a_1(n)F_{-1}(n+1) + \dots + a_7(n)F_{-1}(n+7) = 0$$

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$\downarrow$ Sigma

CLOSED FORM

Example 2: I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

$$\text{GIVEN } F(N)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\times$$
$$\times\underbrace{\left(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+N)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+N)\Gamma(k+j+2)}\right.
+\left.\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+N)\Gamma(k+j+2+\frac{\varepsilon}{2})}\right)}_{f(N,k,j)}.$$

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GIVEN $F(N) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\times$

$$\times\underbrace{\left(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+N)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+N)\Gamma(k+j+2)} + \frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+N)\Gamma(k+j+2+\frac{\varepsilon}{2})}\right)}_{f(N,k,j)}.$$

FIND the ϵ-expansion

$$F(N) = F_0(N) + \varepsilon F_1(N) + \dots$$

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$$\times \underbrace{\left(\frac{\Gamma(k+1)}{\Gamma(k+2+N)} \frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+N)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+N)\Gamma(k+j+2)} + \frac{\Gamma(k+1)}{\Gamma(k+2+N)} \frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+N)\Gamma(k+j+2+\frac{\varepsilon}{2})} \right)}_{f(N,k,j)}.$$

Step 1: FIND the ϵ-expansion

$$f(N,k,j) = \quad f_0(N,k,j) + \quad \varepsilon f_1(N,k,j) + \dots$$

Example 2: I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

$$\text{GIVEN } F(N) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)} \times$$
$$\times \underbrace{\left(\frac{\Gamma(k+1)}{\Gamma(k+2+N)} \frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+N)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+N)\Gamma(k+j+2)} + \frac{\Gamma(k+1)}{\Gamma(k+2+N)} \frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+N)\Gamma(k+j+2+\frac{\varepsilon}{2})} \right)}_{f(N,k,j)}.$$

Step 2: Simplify the sums in

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f(N,k,j) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j) + \varepsilon \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_1(N,k,j) + \dots$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{j=0}^{a} f_0(N,k,j) =$$ Sigma

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{j=0}^{a} f_0(N,k,j) = \frac{(a+1)!(k-1)!(a+k+N+1)!(S_1(a)-S_1(a+k)-S_1(a+N)+S_1(a+k+N))}{N(a+k+1)!(a+N+1)!(k+N+1)!}$$

$$\underbrace{+\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!} + \frac{(2a+k+N+2)a!k!(a+k+N)!}{(a+k+1)(a+N+1)(a+k+1)!(a+N+1)!(k+N+1)!}}_{a\ \longrightarrow\ \infty}$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{j=0}^{\infty} f_0(N,k,j) = \frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{k=1}^{a}\sum_{j=0}^{\infty} f_0(N,k,j) = \sum_{k=1}^{a}\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

$$=$$ Sigma

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j) = \sum_{k=1}^{\infty}\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

$$= \frac{S_1(N)^2+S_2(N)}{2N(N+1)!}$$

where

$$S_2(N)=\sum_{i=1}^{N}\frac{1}{i^2}$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+N)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+N)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+N)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)+$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)=\frac{S_1(N)^2+3S_1(N)}{2N(N+1)!}.$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\,\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+N)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+N)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\,\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+N)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)+$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)=\frac{S_1(N)^2+3S_1(N)}{2N(N+1)!}.$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)=\frac{-S_1(N)^3-3S_2(N)S_1(N)-8S_3(N)}{6N(N+1)!}.$$

Automatic machinery

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\,\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+N)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+N)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\,\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+N)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)+\varepsilon^2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)+$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_2(N,k,j)=\frac{1}{96N(N+1)}\Big(S_1(N)^4+(12\zeta_2+54S_2(N))S_1(N)^2$$

$$+104S_3(N)S_1(N)-48S_{2,1}(N)S_1(N)+51S_2(N)^2+36\zeta_2S_2(N)$$

$$+126S_4(N)-48S_{3,1}(N)-96S_{1,1,2}(N)\Big)$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\,\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+N)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+N)\Gamma(k+j+2)}$$
$$+\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\,\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+N)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$
$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)+\varepsilon^2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)+\varepsilon^3\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)+\dots$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_3(N,k,j)=\frac{1}{960N(N+1)}\Big(S_1(N)^5+(20\zeta_2+130S_2(N))S_1(N)^3+$$
$$(40\zeta_3+380S_3(N))S_1(N)^2+\left(135S_2(N)^2+60\zeta_2S_2(N)+510S_4(N)\right)S_1(N)$$
$$-240S_{3,1}(N)S_1(N)-240S_{1,1,2}(N)S_1(N)+160\zeta_2S_3(N)+S_2(N)(120\zeta_3$$
$$+380S_3(N))+624S_5(N)+\left(-120S_1(N)^2-120S_2(N)\right)S_{2,1}(N)$$
$$-240S_{4,1}(N)-240S_{1,1,3}(N)+240S_{2,2,1}(N)\Big)$$

Four fold multi-sums derive from:

- massive 2-loop operator matrix elements at general values of the Mellin variable (e.g., I. Bierenbaum, J. Blümlein, S. Klein, C.S., Nucl.Phys., 2008)
- general Mellin-Barnes representations in $2 \to k, k \geq 2$ scattering amplitudes in one- and higher loops

  (e.g., M. Czakon, J. Gluza, T. Riemann, Nucl.Phys., 2006)
- and similar problems in particle physics

Four fold multi-sums derive from:

- massive 2-loop operator matrix elements at general values of the Mellin variable (e.g., I. Bierenbaum, J. Blümlein, S. Klein, C.S., Nucl.Phys., 2008)

$$\int_0^1 dx_1 \dots dx_4 \frac{((1-x_4)x_3+x_4x_1)^{N-1}(1-x_4)x_4^{-\epsilon/2}(1-x_2)^{\epsilon/2}x_2^{N-1-\epsilon/2}(1-(1-x_4)x_3-x_4)^{N-}}{(x_2+x_4-x_4x_2)^{N-\epsilon}}$$

$$\downarrow$$

$$\sum_{s=0}^{\infty}\sum_{j=0}^{N-2}\sum_{k=0}^{j}\sum_{i=0}^{N-j-2}(-1)^{i+k}\frac{\binom{j}{k}\binom{N-j-2}{i}\left(1-\frac{\epsilon}{2}\right)_i\left(\frac{\epsilon}{2}+1\right)_s(-\epsilon)_s(i+k+2)_s}{(1)_s\left(3-\frac{\epsilon}{2}\right)_{i+k}(i+2)_s\left(-\frac{\epsilon}{2}+i+k+3\right)_s}\frac{\Gamma(j+2)\Gamma(i+k+1)}{\Gamma(j+1)\Gamma(i+2)}$$
$$=\epsilon^2F_2(N)+\epsilon^3F_3(N)+\dots$$

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$$\int_0^1 dx_1 \dots dx_4 \frac{((1-x_4)x_3+x_4x_1)^{N-1}(1-x_4)x_4^{-\epsilon/2}(1-x_2)^{\epsilon/2}x_2^{N-1-\epsilon/2}(1-(1-x_4)x_3-x_4)^{N-}}{(x_2+x_4-x_4x_2)^{N-\epsilon}}$$

$\downarrow$

$$\sum_{s=0}^{\infty}\sum_{j=0}^{N-2}\sum_{k=0}^{j}\sum_{i=0}^{N-j-2}(-1)^{i+k}\frac{\binom{j}{k}\binom{N-j-2}{i}\left(1-\frac{\epsilon}{2}\right)_i\left(\frac{\epsilon}{2}+1\right)_s(-\epsilon)_s(i+k+2)_s}{(1)_s\left(3-\frac{\epsilon}{2}\right)_{i+k}(i+2)_s\left(-\frac{\epsilon}{2}+i+k+3\right)_s}\frac{\Gamma(j+2)\Gamma(i+k+1)}{\Gamma(j+1)\Gamma(i+2)}$$
$$= \epsilon^2 F_2(N) + \epsilon^3 F_3(N) + \dots$$

$F_2(N) =$?

Four fold multi-sums derive from:

- massive 2-loop operator matrix elements at general values of the Mellin variable (e.g., I. Bierenbaum, J. Blümlein, S. Klein, C.S., Nucl.Phys., 2008)

$$\int_0^1 dx_1 \dots dx_4 \frac{((1-x_4)x_3+x_4x_1)^{N-1}(1-x_4)x_4^{-\epsilon/2}(1-x_2)^{\epsilon/2}x_2^{N-1-\epsilon/2}(1-(1-x_4)x_3-x_4)^{N-}}{(x_2+x_4-x_4x_2)^{N-\epsilon}}$$

$$\downarrow$$

$$\sum_{s=0}^{\infty}\sum_{j=0}^{N-2}\sum_{k=0}^{j}\sum_{i=0}^{N-j-2}(-1)^{i+k}\frac{\binom{j}{k}\binom{N-j-2}{i}\left(1-\frac{\epsilon}{2}\right)_i\left(\frac{\epsilon}{2}+1\right)_s(-\epsilon)_s(i+k+2)_s}{(1)_s\left(3-\frac{\epsilon}{2}\right)_{i+k}(i+2)_s\left(-\frac{\epsilon}{2}+i+k+3\right)_s}\frac{\Gamma(j+2)\Gamma(i+k+1)}{\Gamma(j+1)\Gamma(i+2)}$$

$$= \epsilon^2 F_2(N) + \epsilon^3 F_3(N) + \dots$$

$$F_2(N) = -\frac{2S_1(N)N}{N+1} + \frac{3}{2}S_2(N)N + S_3(N)N - 2S_{2,1}(N)N + \frac{N}{2}$$

Four fold multi-sums derive from:

- massive 2-loop operator matrix elements at general values of the Mellin variable (e.g., I. Bierenbaum, J. Blümlein, S. Klein, C.S., Nucl.Phys., 2008)

$$\int_0^1 dx_1 \dots dx_4 \frac{((1-x_4)x_3+x_4x_1)^{N-1}(1-x_4)x_4^{-\epsilon/2}(1-x_2)^{\epsilon/2}x_2^{N-1-\epsilon/2}(1-(1-x_4)x_3-x_4)^{N-}}{(x_2+x_4-x_4x_2)^{N-\epsilon}}$$

$\downarrow$

$$\sum_{s=0}^{\infty}\sum_{j=0}^{N-2}\sum_{k=0}^{j}\sum_{i=0}^{N-j-2}(-1)^{i+k}\frac{\binom{j}{k}\binom{N-j-2}{i}\left(1-\frac{\epsilon}{2}\right)_i\left(\frac{\epsilon}{2}+1\right)_s(-\epsilon)_s(i+k+2)_s}{(1)_s\left(3-\frac{\epsilon}{2}\right)_{i+k}(i+2)_s\left(-\frac{\epsilon}{2}+i+k+3\right)_s}\frac{\Gamma(j+2)\Gamma(i+k+1)}{\Gamma(j+1)\Gamma(i+2)}$$
$$=\epsilon^2F_2(N)+\epsilon^3F_3(N)+\dots$$

$$F_3(N)=-\frac{NS_1(N)^2}{2(N+1)}+\frac{\left(-3N^2-5N+2\right)S_1(N)}{2(N+1)^2}+\frac{1}{2}NS_2(N)^2-\frac{N}{4}$$
$$+\frac{1}{4}(4N\zeta_2-N)+\frac{\left(-2\zeta_2N^2+3N^2-2\zeta_2N+2N-2\right)S_2(N)}{2(N+1)}$$
$$+\frac{3}{4}NS_3(N)+NS_4(N)-\frac{1}{2}NS_{2,1}(N)-NS_{2,1,1}(N)$$

computation time: 14 minutes

A massive 3–loop integral for general N

$$\Gamma(2-3\epsilon/2)\int_0^1 dx_1\ldots\int_0^1 dx_7\Theta(1-x_1-x_2)\frac{x_1^{-\epsilon/2}x_2^{-\epsilon/2}(1-x_1-x_2)}{(1+x_1\frac{1-x_3}{x_3}+x_2\frac{1-x_4}{x_4})^{2-3\epsilon/2}}$$
$$x_3^{-1+\epsilon/2}(1-x_3)^{\epsilon/2}x_4^{-1+\epsilon/2}(1-x_4)^{\epsilon/2}(1-x_5x_1+\ldots x_7)^N$$

$\downarrow$ straightforward

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{N}\sum_{s=0}^{N-r}\sum_{t=0}^{N-r-s}-(-1)^{r+s+t}$$
$$\times\frac{\binom{N}{r}\binom{N-r}{s}\binom{N-r-s}{t}\left(-\frac{\epsilon}{2}+s+1\right)_n\left(-\frac{\epsilon}{2}+t+1\right)_m\left(\frac{\epsilon}{2}+r+s+t+2\right)_{m+n}}{(-\epsilon+r+s+t+4)_{m+n}}$$
$$\times\Gamma\left[\frac{\frac{4-3\epsilon}{2},\frac{\epsilon}{2}+m+1,\frac{\epsilon}{2}+n+1,r+1,s+1,s+1,-\frac{\epsilon}{2}+s+1,t+1,t+1,-\frac{\epsilon}{2}+t+1}{m+1,n+1,s+2,\frac{\epsilon}{2}+n+s+2,t+2,\frac{\epsilon}{2}+m+t+2,-\epsilon+r+s+t+4}\right]$$

A massive 3–loop integral for general N

$$\Gamma(2-3\epsilon/2)\int_0^1 dx_1\dots\int_0^1 dx_7\Theta(1-x_1-x_2)\frac{x_1^{-\epsilon/2}x_2^{-\epsilon/2}(1-x_1-x_2)}{(1+x_1\frac{1-x_3}{x_3}+x_2\frac{1-x_4}{x_4})^{2-3\epsilon/2}}$$
$$x_3^{-1+\epsilon/2}(1-x_3)^{\epsilon/2}x_4^{-1+\epsilon/2}(1-x_4)^{\epsilon/2}(1-x_5x_1+\dots x_7)^N$$

$\downarrow$ S. Klein

$$\frac{e^{-\frac{3\epsilon\gamma}{2}}\Gamma(2-\frac{3\epsilon}{2})}{(N+1)(N+2)(N+3)}\times$$
$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{t=1}^{N+2}\frac{\binom{N+3}{t}\frac{\epsilon}{2}+N+2)_{m+n}(-\frac{\epsilon}{2}+N-t+3)_n(t-\frac{\epsilon}{2})_m}{(-\epsilon+N+4)_{m+n}}$$
$$\times\Gamma\left[\frac{\frac{\epsilon}{2}+m+1,\frac{\epsilon}{2}+n+1,N-t+3,-\frac{\epsilon}{2}+N-t+3,t,t-\frac{\epsilon}{2}}{m+1,n+1,-\epsilon+N+4,\frac{\epsilon}{2}+n+N-t+4,\frac{\epsilon}{2}+m+t+1}\right]$$
$$-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{s=1}^{N+3}\sum_{r=1}^{s-1}\binom{N+3}{s}\binom{s}{r}(-1)^s\frac{\left(r-\frac{\epsilon}{2}\right)_m\left(\frac{\epsilon}{2}+s-1\right)_{m+n}\left(-\frac{\epsilon}{2}-r+s\right)_n}{(-\epsilon+s+1)_{m+n}}$$
$$\times\Gamma\left[-\frac{\frac{\epsilon}{2}+m+1,\frac{\epsilon}{2}+n+1,r,r-\frac{\epsilon}{2},s-r,-\frac{\epsilon}{2}-r+s}{m+1,n+1,\frac{\epsilon}{2}+m+r+1,-\epsilon+s+1,\frac{\epsilon}{2}+n-r+s+1}\right]$$

A massive 3–loop integral for general N

$$\Gamma(2-3\epsilon/2)\int_0^1 dx_1\ldots\int_0^1 dx_7\Theta(1-x_1-x_2)\frac{x_1^{-\epsilon/2}x_2^{-\epsilon/2}(1-x_1-x_2)}{(1+x_1\frac{1-x_3}{x_3}+x_2\frac{1-x_4}{x_4})^{2-3\epsilon/2}}$$
$$x_3^{-1+\epsilon/2}(1-x_3)^{\epsilon/2}x_4^{-1+\epsilon/2}(1-x_4)^{\epsilon/2}(1-x_5x_1+\ldots x_7)^N$$

constant term $= -\frac{S_1(N)^4}{4(N+1)(N+2)(N+3)}+\frac{NS_1(N)^3}{(N+1)^2(N+2)(N+3)}+$
$\left(\frac{2(3N+5)}{(N+1)^2(N+2)^2(N+3)}-\frac{5S_2(N)}{2(N+1)(N+2)(N+3)}\right)S_1(N)^2$
$+S_1(N)\Big(\big(-\frac{4\left(\text{z3}N^4+7\zeta_3N^3+17\zeta_3N^2+17\zeta_3N-2N+6\zeta_3-3\right)}{(N+1)^3(N+2)^2(N+3)}$
$+\frac{5NS(2,N)}{(N+1)^2(N+2)(N+3)}+\frac{2(2N+3)S_3(N)}{(N+1)(N+2)(N+3)}+\frac{4S_{2,1}(N)}{(N+1)(N+2)(N+3)}\big)$
$+\frac{(4N+9)S_2(N)^2}{4(N+1)(N+2)(N+3)}-\frac{4\left(\zeta_3N^4+7\zeta_3N^3+17\zeta_3N^2+17\zeta_3N-4N+6\zeta_3-6\right)}{(N+1)^4(N+2)^2(N+3)}$
$+\frac{2(7N+11)S_2(N)}{(N+1)^2(N+2)^2(N+3)}+\frac{2(5N+6)S_3(N)}{(N+1)^2(N+2)(N+3)}-\frac{(2N+3)S_4(N)}{2(N+1)(N+2)(N+3)}$
$-\frac{4NS_{2,1}(N)}{(N+1)^2(N+2)(N+3)}-\frac{2(3N+5)S_{3,1}(N)}{(N+1)(N+2)(N+3)}+\frac{2S_{2,1,1}(N)}{(N+2)(N+3)}$

computation time: 3 minutes

A massive 3–loop integral for general N

linear term $= \frac{1}{(N+1)(N+2)(N+3)}\Big(-\frac{S_1(N)^5}{12}+\frac{(13N+3)S_1(N)^4}{24(N+1)}+\Big(\frac{22N^2+55N+30}{6(N+1)^2(N+2)}-S_2(N$

$+S_1(N)^2\Big(\frac{(17N+5)S_2(N)}{4(N+1)}-S_{2,1}(N)-\frac{\zeta_3(N^5+6N^4+13N^3+12N^2+4N)-12N^2-40N-34}{(N+1)^2(N+2)^2}$

$+\frac{(6N-1)}{6}S_3(N)\Big)+\Big(\frac{1}{4}(4N-3)S_2(N)^2+\frac{\left(38N^2+97N+58\right)S_2(N)}{2(N+1)^2(N+2)}-\frac{2S_{2,1}(N)}{N+1}+\frac{(34N+27)S_3(N)}{3(N+1)}$

$-\frac{2\left(9\zeta_2^2N^6+90\zeta_2^2N^5+20\zeta_3N^5+360\zeta_2^2N^4+135\zeta_3N^4+738\zeta_2^2N^3+350\zeta_3N^3+819\zeta_2^2N^2+435\zeta_3N^2-5\right.}{5(N+1)^3(N+2)^2}$

$+\frac{(41N+27)S_2(N)^2}{8(N+1)}-\frac{2\left(3N^2+8N+6\right)S_{2,1}(N)}{(N+1)^2(N+2)}+\frac{\left(76N^2+205N+138\right)S_3(N)}{3(N+1)^2(N+2)}+\frac{(83N+105)S_4(N)}{4(N+1)}$

$-\frac{2\left(9\zeta_2^2N^6+90\zeta_2^2N^5+10\zeta_3N^5+360\zeta_2^2N^4+60\zeta_3N^4+738\zeta_2^2N^3+130\zeta_3N^3+819\zeta_2^2N^2+120\zeta_3N^2-11\right.}{5(N+1)^4(N+2)^2}$

$+\frac{3\zeta_3N^5+24\zeta_3N^4+75\zeta_3N^3+114\zeta_3N^2+26N^2+84\zeta_3N+84N+24\zeta_3+70}{(N+1)^2(N+2)^2}+(N+4)S_5(N)$

$+S_2(N)\left(5S_{2,1}(N)+\frac{1}{2}(8N+19)S_3(N)\right)-\frac{2(4N+5)S_{3,1}(N)}{N+1}-2(3N+8)S_{3,2}(N)$

$+(-9N-13)S_{4,1}(N)-\frac{2(2N-1)S_{2,1,1}(N)}{N+1}-2(N+4)S_{2,2,1}(N)+2(N+4)S_{3,1,1}(N)$

$+(N-3)S_{2,1,1,1}(N)\Big)$

computation time: 70 minutes