Combinat 2009 — International Sage Workshop on Free and Practical Software for Algebraic Combinatorics

Difference Fields and Symbolic Summation (Part 1)

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Difference Fields and Symbolic Summation in Sage (Part 2)

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Simplify

$$\sum_{k=1}^{n} H_k$$

Telescoping

GIVEN $f(k) = H_k$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

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We compute

$$g(k) = (H_k - 1)k.$$

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for all $1 \leq k \leq n$ and $n \geq 0$.

Summing this equation over k from 1 to n gives

$$\boxed{\sum_{k=1}^{n} H_k} = \boxed{g(n+1) - g(1)}$$

$$= (H_{n+1} - 1)(n+1).$$

Telescoping

FIND a closed form for

$$\sum_{k=1}^{n} H_k.$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

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$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

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$$\sum_{k=1}^{n} H_k.$$

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Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

$$\sigma(h) = h + \frac{1}{k+1}, \qquad \mathcal{S}\, H_k = H_k + \frac{1}{k+1}.$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

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Hence,

$$(H_{n+1} - 1)(n+1) = \sum_{k=1}^{n} H_k.$$

CONSTRUCT a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}$$

- with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

such that $\{c \in \mathbb{K} \qquad\qquad |\sigma(c) = c\} = \mathbb{K}$.

CONSTRUCT a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)$$

- with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$
$$\sigma(t_1) = a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \qquad f_1 \in \mathbb{K}$$

such that $\{c \in \mathbb{K}(t_1) \quad | \sigma(c) = c\} = \mathbb{K}$.

CONSTRUCT a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:

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$$\mathbb{F} := \mathbb{K}(t_1)(t_2)$$

- with an automorphism

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$$\sigma(t_1) = a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \qquad f_1 \in \mathbb{K}$$

$$\sigma(t_2) = a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, \qquad f_2 \in \mathbb{K}(t_1)$$

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CONSTRUCT a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:

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$$\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$$

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&\vdots \\
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GIVEN $f \in \mathbb{F}$;
FIND $g \in \mathbb{F}$ such that

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Construction of Σ^*-extensions

- Let $(\mathbb{F}, \sigma)$ be a difference field with constant field

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3. $\boxed{\exists g \in \mathbb{F}^* :\ \sigma(g) = a^n\, g \text{ for } n > 1, \text{ but not for } n = 1}$ 

Telescoping

FIND a closed form for

$$\sum_{k=1}^{n} H_k.$$

A $\Pi\Sigma^*$-field for the summand

Consider the rational function field

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with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{Q}, \\ \sigma(k) &= k + 1, & \mathcal{S}\, k &= k + 1, \\ \sigma(h) &= h + \frac{1}{k+1}, & \mathcal{S}\, H_k &= H_k + \frac{1}{k+1}. \end{aligned}$$

FIND $g \in \mathbb{Q}(k)(h)$:

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Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]^*$:

$$\boxed{d = 1}$$

$$\forall g \in \mathbb{Q}(k)(h) : \quad \sigma(g) - g = h \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[h].$$

FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

FIND $g \in \mathbb{Q}(k)(h)$:

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FIND $g' \in \mathbb{Q}(k)[h]$ with

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Degree bound: COMPUTE $b \geq 0$:

$$b = 2$$

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Polynomial Solution: FIND

$$g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h].$$

g=hk-k

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\sigma(g) - g = h$$

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$$\Big[\sigma(g_2)\big(h+\tfrac{1}{k+1}\big)^2+\sigma(g_1 h+g_0)\Big] \\ -\big[g_2\,h^2+g_1 h+g_0\big]=h$$

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coeff. comp.

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$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = h - c\Big[\tfrac{2h(k+1)+1}{(k+1)^2}\Big]$$

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$$\boxed{\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}}$$

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$$\boxed{\sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}}$$

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$$\boxed{\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}}$$

$c = 0,\quad g_1 = k + d,\quad d \in \mathbb{Q}$

$$g_0 = -k,\quad d = 0 \quad \leftarrow \quad \boxed{\sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}}$$

Creative telescoping (for order 2)

GIVEN

$$S(n) = \sum_{k=0}^{n} f(n,k).$$

FIND $c_0(n), c_1(n), c_2(n)$ and $g(n,k)$:

$$\boxed{g(n,k+1) - g(n,k)}$$

$$= \boxed{c_0(n)f(n,k) + c_1(n)f(n+1,k) + c_2(n)f(n+2,k).}$$

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$$=\boxed{c_0(n)f(n,k)+c_1(n)f(n+1,k)+c_2(n)f(n+2,k).}$$

THEN summation over k from 0 to n gives

$$\boxed{g(n,n+1)-g(n,0)}$$

$$=\boxed{\begin{array}{l} c_0(n)\,\mathsf{S}(n)+\\ c_1(n)\left[\mathsf{S}(n+1)-f(n+1,n+1)\right]\\ c_2(n)\left[\mathsf{S}(n+2)-f(n+2,n+1)-f(n+2,n+2)\right].\end{array}}$$

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$$= \boxed{c_0(n) f(n,k) + c_1(n) f(n+1,k) + c_2(n) f(n+2,k).}$$

HENCE

$$h(n) = c_0(n) S(n) + c_1(n) S(n+1) + c_2(n) S(n+2)$$

for some $h(n)$.

FIND a recurrence for

$$S(n) := \sum_{k=1}^{n} \binom{n}{k} H_k.$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(n)(k)(h)(b)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{Q}(n), & & \\
\sigma(k) &= k+1, & \mathcal{S}\,k &= k+1, \\
\sigma(h) &= h + \frac{1}{k+1}, & \mathcal{S}\,H_k &= H_k + \frac{1}{k+1}, \\
\sigma(b) &= \frac{n-k}{k+1}\,b, & \mathcal{S}\binom{n}{k} &= \frac{n-k}{k+1}\binom{n}{k}.
\end{aligned}$$

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FIND $c_0, c_1, c_2 \in \mathbb{Q}(n)$ and $g \in \mathbb{F}$:

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We compute

$$c_0 := 4\,(1+n), \quad c_1 := -2\,(3+2\,n), \quad c_2 := 2+n,$$

$$g := \frac{(1+n)\,(-2+k-n+(2\,k-2\,k^2+k\,n)h)\,b)}{(1-k+n)\,(2-k+n)}.$$

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This gives

$$\boxed{g(n,k+1) - g(n,k) = c_0(n) f(n,k) + c_1(n) f(n+1,k) + c_2(n) f(n+2,k)}$$

with

$$c_0(n) := 4\,(1+n), \quad c_1(n) := -2\,(3+2\,n), \quad c_2(n) := 2+n,$$

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Summing over k from 0 to n gives

$$\boxed{1 = 4(1+n)S(n) - 2(3+2n)S(n+1) + (2+n)S(n+2)}$$

for

$$S(n) = \sum_{k=0}^{n} \binom{n}{k} H_k$$

Example

1. Creative telescoping

GIVEN a definite sum

$$S(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum;
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FIND a recurrence for $S(n)$

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2. Recurrence solving

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$a_0(n),\dots,a_d(n),h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)S(n)+\dots+a_d(n)S(n+d)=h(n);$$

FIND all solutions expressible by indefinite nested products and sums
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NOTE: By construction, the solutions are highly nested.

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3. Indefinite summation

Simplify the solutions:

- No algebraic relations occur among the sums
- The sums have minimal nested depth.

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4. Find a "closed form"

S(n)=combined solutions.

Example