

Combinat 2009 — International Sage Workshop on Free and Practical
Software for Algebraic Combinatorics

Difference Fields and Symbolic Summation (Part 1)

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Difference Fields and Symbolic Summation in Sage (Part 2)

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Simplify

$$\sum_{k=1}^n H_k$$

Telescoping

GIVEN $f(k) = H_k$.

FIND $g(k)$:

$$f(k) = g(k+1) - g(k)$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Telescoping

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FIND $g(k)$:

$$f(k) = g(k+1) - g(k)$$

for all $1 \leq k \leq n$ and $n \geq 0$.

We compute

$$g(k) = (H_k - 1)k.$$

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FIND $g(k)$:

$$f(k) = g(k + 1) - g(k)$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Summing this equation over k from 1 to n gives

$$\sum_{k=1}^n H_k = g(n + 1) - g(1)$$

$$= (H_{n+1} - 1)(n + 1).$$

Telescoping

FIND a closed form for

$$\sum_{k=1}^n H_k.$$

A difference field for the **summand**

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}$$

with the automorphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

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FIND a closed form for

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$$\sigma(k) = k + 1,$$

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Telescoping

FIND a closed form for

$$\sum_{k=1}^n H_k.$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

with the automorphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1,$$

$$\sigma(h) = h + \frac{1}{k+1},$$

$$S k = k + 1,$$

$$S H_k = H_k + \frac{1}{k+1}.$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

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This gives

$$g(k + 1) - g(k) = H_k$$

with

$$g(k) = (H_k - 1)k.$$

Hence,

$$(H_{n+1} - 1)(n + 1) = \sum_{k=1}^n H_k.$$

CONSTRUCT a $\Pi\Sigma$ -field (\mathbb{F}, σ) :

- ▶ a rational function field

$$\mathbb{F} := \mathbb{K}$$

- ▶ with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

such that $\{c \in \mathbb{K} \mid \sigma(c) = c\} = \mathbb{K}$.

CONSTRUCT a $\Pi\Sigma$ -field (\mathbb{F}, σ) :

- ▶ a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)$$

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$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

$$\sigma(t_1) = a_1 t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \quad f_1 \in \mathbb{K}$$

such that $\{c \in \mathbb{K}(t_1) \mid \sigma(c) = c\} = \mathbb{K}$.

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$$\sigma(t_2) = a_2 t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, \quad f_2 \in \mathbb{K}(t_1)$$

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CONSTRUCT a $\Pi\Sigma$ -field (\mathbb{F}, σ) :

- ▶ a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2) \dots (t_e)$$

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GIVEN $f \in \mathbb{F}$;

FIND $g \in \mathbb{F}$ such that

$$\sigma(g) - g = f.$$

Construction of Σ^* -extensions

- ▶ Let (\mathbb{F}, σ) be a difference field with constant field

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Such a difference field extension $(\mathbb{F}(t), \sigma)$ of (\mathbb{F}, σ) is called Σ^* -extension

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2. $\boxed{\exists g \in \mathbb{F} : \sigma(g) = g + f}$ No need for a Σ^* -extension!

Construction of Π -extensions

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3. $\boxed{\exists g \in \mathbb{F}^* : \sigma(g) = a^n g \text{ for } n > 1, \text{ but not for } n = 1}$ 

Telescoping

FIND a closed form for

$$\sum_{k=1}^n H_k.$$

A $\Pi\Sigma^*$ -field for the **summand**

Consider the rational function field

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with the automorphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1,$$

$$\sigma(h) = h + \frac{1}{k+1},$$

$$S k = k + 1,$$

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FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

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Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]^*$:

$$d = 1$$

$$\forall g \in \mathbb{Q}(k)(h) : \sigma(g) - g = h \Rightarrow g d \in \mathbb{Q}(k)[h].$$

FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma\left(\frac{g'}{d}\right) - \frac{g'}{d} = h.$$

FIND $g \in \mathbb{Q}(k)(h)$:

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FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma\left(\frac{g'}{d}\right) - \frac{g'}{d} = h.$$

Degree bound: COMPUTE $b \geq 0$:

$$b = 2$$

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \Rightarrow \deg(g) \leq b.$$

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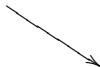
Polynomial Solution: FIND

$$g = hk - k$$

$$g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h].$$

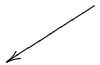
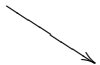
ANSATZ $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]$

$$\sigma(g) - g = h$$



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$$\left[\sigma(g_2) \left(h + \frac{1}{k+1} \right)^2 + \sigma(g_1 h + g_0) \right] - [g_2 h^2 + g_1 h + g_0] = h$$



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coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

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$$\sigma(g_2) - g_2 = 0$$

$$g_2 = c \in \mathbb{Q}$$

$$\sigma(g_1 h + g_0) - (g_1 h + g_0) = h - c \left[\frac{2h(k+1)+1}{(k+1)^2} \right]$$

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$$\sigma(g_1) - g_1 = 1 - c \frac{2}{k+1}$$

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$$\sigma(g_0) - g_0 = -1 - d \frac{1}{k+1}$$

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$$g = hk - k$$

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$$\sigma(g_1) - g_1 = 1 - c \frac{2}{k+1}$$

$$c = 0, \quad g_1 = k + d \\ d \in \mathbb{Q}$$

$$g_0 = -k \\ d = 0$$

$$\sigma(g_0) - g_0 = -1 - d \frac{1}{k+1}$$

Creative telescoping (for order 2)

GIVEN

$$S(n) = \sum_{k=0}^n f(n, k).$$

FIND $c_0(n), c_1(n), c_2(n)$ and $g(n, k)$:

$$\boxed{g(n, k+1) - g(n, k)}$$

$$= \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k) + c_2(n)f(n+2, k)}.$$

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THEN summation over k from 0 to n gives

$$\boxed{g(n, n+1) - g(n, 0)} \\ = \boxed{c_0(n)S(n) + c_1(n)[S(n+1) - f(n+1, n+1)] + c_2(n)[S(n+2) - f(n+2, n+1) - f(n+2, n+2)]}.$$

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HENCE

$$h(n) = c_0(n)S(n) + c_1(n)S(n+1) + c_2(n)S(n+2)$$

for some $h(n)$.

FIND a recurrence for

$$S(n) := \sum_{k=1}^n \binom{n}{k} H_k.$$

A difference field for the **summand**

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(n)(k)(h)(b)$$

with the automorphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),$$

$$\sigma(k) = k + 1,$$

$$\sigma(h) = h + \frac{1}{k+1},$$

$$\sigma(b) = \frac{n-k}{k+1} b,$$

$$S k = k + 1,$$

$$S H_k = H_k + \frac{1}{k+1},$$

$$S \binom{n}{k} = \frac{n-k}{k+1} \binom{n}{k}.$$

REPRESENT $f(n+i, k)$ in \mathbb{F} :

$$f(n, k) = H_k \binom{n}{k} \longleftrightarrow hb =: f_0$$

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FIND $c_0, c_1, c_2 \in \mathbb{Q}(n)$ and $g \in \mathbb{F}$:

$$\sigma(g) - g = c_0 f_0 + c_1 f_1 + c_2 f_2.$$

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We compute

$$c_0 := 4(1+n), \quad c_1 := -2(3+2n), \quad c_2 := 2+n,$$

$$g := \frac{(1+n)(-2+k-n+(2k-2k^2+kn)h)b}{(1-k+n)(2-k+n)}.$$

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This gives

$$g(n, k+1) - g(n, k) = c_0(n)f(n, k) + c_1(n)f(n+1, k) + c_2(n)f(n+2, k)$$

with

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Summing over k from 0 to n gives

$$1 = 4(1+n)S(n) - 2(3+2n)S(n+1) + (2+n)S(n+2)$$

for

$$S(n) = \sum_{k=0}^n \binom{n}{k} H_k$$

Example

1. Creative telescoping

GIVEN a **definite** sum

$$S(n) = \sum_{k=0}^n f(n, k);$$

$f(n, k)$: indefinite nested product-sum;
 n : extra parameter

FIND a **recurrence** for $S(n)$

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2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
 indefinite nested product-sum expressions.

$$a_0(n)S(n) + \dots + a_d(n)S(n+d) = h(n);$$

FIND **all solutions** expressible by indefinite nested products and sums
 (Nörlund 24, Abramov/Petkovšek 94, Hendriks/Singer 99/Sigma 01)

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NOTE: By construction, the solutions are highly nested.

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3. Indefinite summation

Simplify the solutions:

- ▶ **No algebraic relations** occur among the sums
- ▶ The sums have **minimal nested depth**.

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4. Find a “closed form”

 $S(n)$ =combined solutions.

Example