FPSAC 2009

RISC, J. Kepler University

Symbolic Summation and its Application in Particle Physics

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Indefinite summation

Simplify

$$\sum_{k=0}^{a} \big(1 + (n - 2k)\, S_1(k)\big) \binom{n}{k} = \text{?},$$

where $S_1(k) := \displaystyle\sum_{i=1}^{k} \frac{1}{i} \;\; (= H_k)$.

GIVEN $f(k) = \big(1 + (n-2k)\,S_1(k)\big)\binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

GIVEN $f(k) = \big(1 + (n-2k)\,S_1(k)\big)\binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

Sigma computes

$$g(k) = (k\,S_1(k) - 1)\binom{n}{k}$$

GIVEN $f(k)=\big(1+(n-2k)\,S_1(k)\big)\binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k)=g(k+1)-g(k)}$$

Summing the telescoping equation over k from 0 to a gives

$$\boxed{\sum_{k=0}^{a}\big(1+(n-2k)\,S_1(k)\big)\binom{n}{k}}=\boxed{g(a+1)-g(0)}$$

$$=1+(n-a)S_1(a)\binom{n}{a}.$$

GIVEN $f(k) = \big(1 + (\mathbf{n} - 2k)\, S_1(k)\big) \binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

A difference field for the **summand:**

Construct a rational function field

$$\mathbb{F} := \mathbb{Q}(\mathbf{n})$$

and a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(\mathbf{c}) = \mathbf{c} \quad \forall \mathbf{c} \in \mathbb{Q}(\mathbf{n}),$$

GIVEN $f(k) = \big(1 + (n - 2\mathbf{k})\, S_1(k)\big)\binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

A difference field for the **summand:**

Construct a rational function field

$$\mathbb{F} := \mathbb{Q}(n)(\mathbf{k})$$

and a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),$$

$$\sigma(\mathbf{k}) = \mathbf{k} + \mathbf{1}, \qquad\qquad \mathcal{S}\, k = k + 1,$$

GIVEN $f(k) = (1 + (n - 2k)\,\mathbf{S_1(k)})\binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

A difference field for the **summand:**

Construct a rational function field

$$\mathbb{F} := \mathbb{Q}(n)(k)(\mathbf{h})$$

and a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),$$

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$$\sigma(\mathbf{h}) = \mathbf{h} + \frac{\mathbf{1}}{\mathbf{k+1}}, \qquad \mathcal{S}\, S_1(k) = S_1(k) + \frac{1}{k+1},$$

GIVEN $f(k) = \big(1 + (n-2k)\, S_1(k)\big) \binom{\mathbf{n}}{\mathbf{k}}$

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$$\boxed{f(k) = g(k+1) - g(k)}$$

A difference field for the **summand:**

Construct a rational function field

$(\mathbb{F}, \sigma)$ is a $\Pi\Sigma$-field

$$\mathbb{F} := \mathbb{Q}(n)(k)(h)(\mathbf{b})$$ Karr 1981

and a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),$$

$$\sigma(k) = k + 1, \qquad\qquad \mathcal{S}\, k = k + 1,$$

$$\sigma(h) = h + \frac{1}{k+1}, \qquad\qquad \mathcal{S}\, S_1(k) = S_1(k) + \frac{1}{k+1},$$

$$\sigma(\mathbf{b}) = \frac{\mathbf{n}-\mathbf{k}}{\;}\,\mathbf{b} \qquad\qquad \mathcal{S}\binom{n}{\;} = \frac{n-k}{\;}\binom{n}{\;}$$

GIVEN $f(k) = \big(1 + (n-2k)\, S_1(k)\big)\binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

$$\downarrow$$

GIVEN $f := (1 + (n-2k)h)b \in \mathbb{F}$.
FIND $g \in \mathbb{F}$:

$$\boxed{f = \sigma(g) - g}$$

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$$g = (k\,h - 1)b$$

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GIVEN $f := (1 + (n-2k)h)b \in \mathbb{F}$.
FIND $g \in \mathbb{F}$:

$$\boxed{f = \sigma(g) - g}$$

$\downarrow$ Sigma

$h \equiv S_1(k)$
$b \equiv \binom{n}{k}$

$$g = (k\,h - 1)b$$

A family of identities discovered by S. Ahlgren (see Paule/Schneider 03):

$$\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha}=?$$

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$\alpha=1$:

$$\sum_{k=0}^{a}\left(1+(n-2k)S_1(k)\right)\binom{n}{k}=1+(n-a)S_1(a)\binom{n}{a}$$

A family of identities discovered by S. Ahlgren (see Paule/Schneider 03):

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$\alpha = 1$:

$$\sum_{k=0}^{n}\left(1+(n-2k)S_1(k)\right)\binom{n}{k} = 1$$

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$\alpha = 1$:

$$\sum_{k=0}^{n}\left(1+(n-2k)S_1(k)\right)\binom{n}{k} = 1$$

$\alpha = 2$:

$$\sum_{k=0}^{a}\left(1+2(n-2k)S_1(k)\right)\binom{n}{k}^{2} = \frac{(a-n)^2(1+2nS_1(a))}{n^2}\binom{n}{a}$$

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$\alpha=3$:

$$\sum_{k=0}^{n}\left(1+3(n-2k)S_1(k)\right)\binom{n}{k}^{3}=?$$

$\alpha=4$:

$$\sum_{k=0}^{n}\left(1+4(n-2k)S_1(k)\right)\binom{n}{k}^{4}=?$$

$\alpha=5$:

$$\sum_{k=0}^{n}\left(1+5(n-2k)S_1(k)\right)\binom{n}{k}^{5}=?$$

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$\alpha = 4$:
$$\sum_{k=0}^{n}\left(1+4(n-2k)S_1(k)\right)\binom{n}{k}^{4} = (-1)^n\binom{2n}{n}$$

$\alpha = 5$:
$$\sum_{k=0}^{n}\left(1+5(n-2k)S_1(k)\right)\binom{n}{k}^{5} =?$$

Telescoping

GIVEN

$$\mathsf{SUM}(n) := \sum_{k=0}^{n} \underbrace{(1 + 5(n-2k)S_1(k))\binom{n}{k}^5}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{\qquad f(n,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

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$$\text{SUM}(n) := \sum_{k=0}^{n} \underbrace{\left(1 + 5(n-2k)S_1(k)\right)\binom{n}{k}^5}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n)f(n,k) + c_1(n)\,f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

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$$\mathsf{SUM}(n) := \sum_{k=0}^{n} \underbrace{(1+5(n-2k)S_1(k))\binom{n}{k}^5}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)\,f(n+1,k)+c_2(n)\,f(n+2,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

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for all $0 \leq k \leq n$ and all $n \geq 0$.

No solutions implies that the sequences

$$\langle S_1(a)\rangle_{a\geq 0}, \quad \langle \binom{n}{a} \rangle_{a\geq 0}, \quad \langle \sum_{k=0}^{a} f(n,k)\rangle_{a\geq 0}, \dots, \langle \sum_{k=0}^{a} f(n+3,k)\rangle_{a\geq 0} \in \mathbb{Q}(n)^{\mathbb{N}}$$

are algebraically independent over the field of rational sequences.

For more details see: Parameterized telescoping proves algebraic independence of sums, FPSAC'07

Zeilberger's creative telescoping paradigm

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$$\mathsf{SUM}(n) := \sum_{k=0}^{n} \underbrace{(1+5(n-2k)S_1(k))\binom{n}{k}^5}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n), c_3(n), c_4(n)$

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+\dots+c_4(n)\,f(n+4,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Sigma *computes*: $c_0(n),\dots,c_4(n) \in \mathbb{Q}[n]$

$$g(n,k) := \binom{n}{k}^5 \frac{p_1(k,n,S_1(k))}{(k-n-4)^5(k-n-3)^5(k-n-2)^5(k-n-1)^5},$$

$$g(n,k+1) := \binom{n}{k}^5 \frac{p_2(k,n,S_1(k))}{(k-n-3)^5(k-n-2)^5(k-n-1)^5}.$$

Zeilberger's creative telescoping paradigm

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FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n), c_3(n), c_4(n)$

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+\dots+c_4(n)\,f(n+4,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 0 to n gives:

$$\boxed{g(n,n+1)-g(n,0)} =$$

$$\boxed{\begin{array}{l} c_0(n)\,\mathsf{SUM}(n)+ \\ c_1(n)\,\big[\mathsf{SUM}(n+1)-f(n+1,n+1)\big] \\ \vdots \\ c_4(n)\,\big[\mathsf{SUM}(n+4)-f(n+4,n+1)-f(n+4,n+2)-\dots-f(n+4,n+4)\big]. \end{array}}$$

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A family of identities discovered by S. Ahlgren (see Paule/Schneider 03):

$$\sum_{k=0}^{n}(1+\alpha(n-2k)S_1(k))\binom{n}{k}^{\alpha}=?$$

extended → Krattenthaler/Rivoal 07

$\alpha=1$:
$$\sum_{k=0}^{n}(1+(n-2k)S_1(k))\binom{n}{k}=1$$

$\alpha=2$:
$$\sum_{k=0}^{n}(1+2(n-2k)S_1(k))\binom{n}{k}^2=0$$

$\alpha=3$:
$$\sum_{k=0}^{n}(1+3(n-2k)S_1(k))\binom{n}{k}^3=(-1)^n$$

$\alpha=4$:
$$\sum_{k=0}^{n}(1+4(n-2k)S_1(k))\binom{n}{k}^4=(-1)^n\binom{2\,n}{n}$$

$\alpha=5$:
$$\sum_{k=0}^{n}(1+5(n-2k)S_1(k))\binom{n}{k}^5=(-1)^n\sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{j}$$

Apéry's proof (1979) of the irrationality of $\zeta(3)$ relies on the following fact:

$$a(n) = \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2$$

and

$$b(n) = \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2 \left(S_3(n) + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n+m}{m} \binom{n}{m}} \right)$$

satisfy both the recurrence relation

$$(n+1)^3 A(n) - (2n+3)\left(17n^2 + 51n + 39\right) A(n+1) + (n+2)^3 A(n+2) = 0.$$

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Van der Poorten (1979) points out that Henri Cohen and Don Zagier showed this fact by

"some rather complicated but ingenious explanations"

based on the creative telescoping method.

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$a(n)$*-case:* trivial exercise by Zeilberger's algorithm (1991)

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$$(n+1)^3 A(n) - (2n+3)\left(17n^2 + 51n + 39\right) A(n+1) + (n+2)^3 A(n+2) = 0.$$

$b(n)$-case: skilful application of computer algebra

1. Generalization of the Cohen/Zagier method in the WZ-setting (Zeilberger, 1993)
2. Multi-summation + holonomic closure properties (Chyzak/Salvy, 1998)

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satisfy both the recurrence relation

$$(n+1)^3 A(n) - (2n+3)\left(17n^2+51n+39\right) A(n+1) + (n+2)^3 A(n+2) = 0.$$

$b(n)$-case: plain sailing (and not plane sailing) by Sigma

The other direction:

$$\boxed{\sum_{k=0}^{n}\big(1+\alpha(n-2k)S_1(k)\big)\binom{n}{k}^{\alpha} =?}$$

$\alpha=-1$:
$$\sum_{k=0}^{a}\big(1-(n-2k)S_1(k)\big)\binom{n}{k}^{-1}=\frac{(a+1)S_1(a)+1}{\binom{n}{a}}$$

The other direction:

$$\boxed{\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha} =?}$$

$\alpha=-1$:
$$\sum_{k=0}^{n}\left(1-(n-2k)S_1(k)\right)\binom{n}{k}^{-1}=(n+1)S_1(n)+1$$

The other direction:

$$\sum_{k=0}^{n}(1+\alpha(n-2k)S_1(k))\binom{n}{k}^{\alpha}=?$$

$\alpha=-1$:

$$\sum_{k=0}^{n}(1-(n-2k)S_1(k))\binom{n}{k}^{-1}=(n+1)S_1(n)+1$$

$\alpha=-2$:

$$\sum_{k=0}^{a}(1-2(n-2k)S_1(k))\binom{n}{k}^{-2}$$

$$=\frac{(n+1)^2}{(n+2)^2}+\frac{(a+1)(-a+2n+2(a+1)(n+2)S_1(a)+3)}{(n+2)^2\binom{n}{a}^{-2}}$$

The other direction:

$$\boxed{\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha}=?}$$

$\alpha=-1$:

$$\sum_{k=0}^{n}\left(1-(n-2k)S_1(k)\right)\binom{n}{k}^{-1}=(n+1)S_1(n)+1$$

$\alpha=-2$:

$$\sum_{k=0}^{n}\left(1-2(n-2k)S_1(k)\right)\binom{n}{k}^{-2}$$

$$=\frac{(n+1)^2}{(n+2)^2}+\frac{\left(n+2\left(n^2+3n+2\right)S_1(n)+3\right)(n+1)}{(n+2)^2}$$

The other direction:

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$$=\frac{(n+1)^2}{(n+2)^2}+\frac{\left(n+2\left(n^2+3n+2\right)S_1(n)+3\right)(n+1)}{(n+2)^2}$$

$$\sum_{k=0}^{n}\left(1-3(n-2k)S_1(k)\right)\binom{n}{k}^{-3}= ?$$

The other direction:

$$\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha} =?$$

$\alpha=-1$:

$$\sum_{k=0}^{n}\left(1-(n-2k)S_1(k)\right)\binom{n}{k}^{-1} = (n+1)S_1(n)+1$$

$\alpha=-2$:

$$\sum_{k=0}^{n}\left(1-2(n-2k)S_1(k)\right)\binom{n}{k}^{-2}$$

$$=\frac{(n+1)^2}{(n+2)^2}+\frac{\left(n+2\left(n^2+3n+2\right)S_1(n)+3\right)(n+1)}{(n+2)^2}$$

$$\sum_{k=0}^{n}\left(1-3(n-2k)S_1(k)\right)\binom{n}{k}^{-3} =$$

$$=5(-1)^nS_{-3}(n)(n+1)^3$$

$$-6(-1)^nS_{-2,1}(n)(n+1)^3+6S_1(n)(n+1)+1$$

The other direction:

$$\boxed{\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha} =?}$$

$\alpha=-4$:

$$\sum_{k=0}^{n}\left(1-4(n-2k)S_1(k)\right)\binom{n}{k}^{-4} = \frac{(10(n+1)S_1(n)+3)(n+1)}{2n+3}$$

$$+\frac{(-1)^n\binom{2n}{n}^{-1}(n+1)^5}{(4n(n+2)+3)}\left(\frac{7}{2}\sum_{i=1}^{n}\frac{(-1)^i\binom{2i}{i}}{i^3}-5\sum_{i=1}^{n}\frac{(-1)^i\binom{2i}{i}S_1(i)}{i^2}\right)$$

1. Creative telescoping

GIVEN a definite sum

$$S(n)=\sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum;
n: extra parameter

FIND a recurrence for $S(n)$

1. Creative telescoping

GIVEN a definite sum

$$S(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum;
n: extra parameter

FIND a recurrence for $S(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)S(n) + \dots + a_d(n)S(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products and sums
(Nörlund 24, Abramov/Petkovšek 94, Hendriks/Singer 99/Sigma 01)

1. Creative telescoping

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NOTE: By construction, the solutions are highly nested.

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3. Indefinite summation

Simplify the solutions:

- No algebraic relations occur among the sums
- The sums have minimal nested depth.

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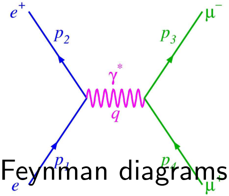
4. Find a “closed form”

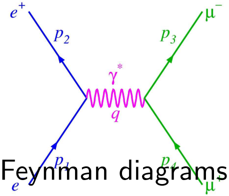
S(n)=combined solutions.

EXAMPLES: Evaluation of Feynman Integrals



Feynman diagrams

EXAMPLES: Evaluation of Feynman Integrals



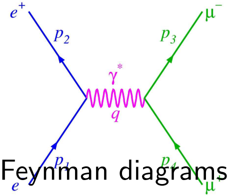
Feynman diagrams

$\longrightarrow$

$$\int \Phi(x)dx$$

Feynman integrals

EXAMPLES: Evaluation of Feynman Integrals



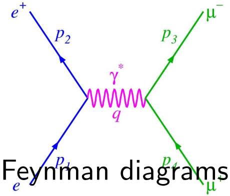
Feynman diagrams ⟶ $\int \Phi(x)dx$

Feynman integrals

↓ Reduction

multi-sums/
linear recurrences

EXAMPLES: Evaluation of Feynman Integrals



Feynman diagrams → $\int \Phi(x)dx$

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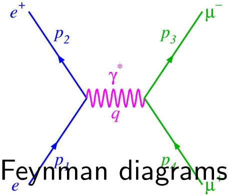
Sigma ←

sum expressions
processable by physicists

EXAMPLES: Evaluation of Feynman Integrals



Feynman diagrams

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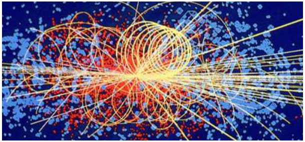

Reduction

multi-sums/
linear recurrences

Sigma

sum expressions
processable by physicists

Example 1: In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon) = \int_0^1 dx_1 \dots \int_0^1 dx_7 \sum_{i=1}^{K} \frac{p_i(x_1,x_2,\dots,x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1,x_2,\dots,x_7)^{\dots+s_i\varepsilon+\dots}}$$

where $K \in \mathbb{N}$, $r_i, s_i \in \mathbb{Q}$, and p_i, q_i are polynomials in $x_1,\dots,x_7$.

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$$=F_{-3}(n)\varepsilon^{-3}+F_{-2}(n)\varepsilon^{-2}+\boxed{F_{-1}(n)}\varepsilon^{-1}+\dots$$

The 3-loop anomalous dimensions can be derived from the single pole part of $F(n,\varepsilon)$. The other poles are needed for the renormalization.

Vermaseren, Moch: 3-5 CPU years (2004)

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$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + \boxed{F_{-1}(n)}\varepsilon^{-1} + \dots$$

$\downarrow$ (J. Blümlein, DESY)

Initial values $F_{-1}(i)$, $i = 1,\dots,450$ (difficult, unsolved task)

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$$a_0(n)F_{-1}(n)+a_1(n)F_{-1}(n+1)+\dots+a_7(n)F_{-1}(n+7)=0$$

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$\downarrow$ Sigma

CLOSED FORM

Example 2: I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

$$\text{GIVEN } F(N)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\times$$
$$\times\underbrace{\left(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+N)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+N)\Gamma(k+j+2)}\right.
+\left.\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+N)\Gamma(k+j+2+\frac{\varepsilon}{2})}\right)}_{f(N,k,j)}.$$

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GIVEN $F(N) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)} \times$

$$\times \underbrace{\left(\frac{\Gamma(k+1)}{\Gamma(k+2+N)} \frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+N)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+N)\Gamma(k+j+2)} + \frac{\Gamma(k+1)}{\Gamma(k+2+N)} \frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+N)\Gamma(k+j+2+\frac{\varepsilon}{2})}\right)}_{f(N,k,j)}.$$

FIND the ϵ-expansion

$$F(N) = F_0(N) + \varepsilon F_1(N) + \dots$$

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$$\times\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+N)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+N)\Gamma(k+j+2)}$$
$$\underbrace{+\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+N)\Gamma(k+j+2+\frac{\varepsilon}{2})}}_{f(N,k,j)}\Bigg).$$

Step 1: FIND the ϵ-expansion

$$f(N,k,j) = \quad f_0(N,k,j) + \quad \varepsilon f_1(N,k,j) + \dots$$

Example 2: I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

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$$\times\underbrace{\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+N)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+N)\Gamma(k+j+2)} + \frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+N)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg)}_{f(N,k,j)}.$$

Step 2: Simplify the sums in

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f(N,k,j)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)+\dots$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{j=0}^{a} f_0(N,k,j) =$$ ▸ Sigma

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{j=0}^{a} f_0(N,k,j) = \frac{(a+1)!(k-1)!(a+k+N+1)!(S_1(a)-S_1(a+k)-S_1(a+N)+S_1(a+k+N))}{N(a+k+1)!(a+N+1)!(k+N+1)!}$$
$$\underbrace{+\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}+\frac{(2a+k+N+2)a!k!(a+k+N)!}{(a+k+1)(a+N+1)(a+k+1)!(a+N+1)!(k+N+1)!}}_{a\,\to\,\infty}$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{j=0}^{\infty} f_0(N,k,j) = \frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{k=1}^{a}\sum_{j=0}^{\infty} f_0(N,k,j) = \sum_{k=1}^{a}\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

$$= \text{Sigma}$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j) = \sum_{k=1}^{\infty}\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$
$$= \frac{S_1(N)^2+S_2(N)}{2N(N+1)!}$$

where

$$S_2(N) = \sum_{i=1}^{N}\frac{1}{i^2}$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+N)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+N)\Gamma(k+j+2)}$$
$$+\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+N)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$
$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)+$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)=\frac{S_1(N)^2+3S_1(N)}{2N(N+1)!}.$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+N)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+N)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+N)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)+$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)=\frac{S_1(N)^2+3S_1(N)}{2N(N+1)!}.$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)=\frac{-S_1(N)^3-3S_2(N)S_1(N)-8S_3(N)}{6N(N+1)!}.$$

Automatic machinery

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+N)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+N)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+N)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)+\varepsilon^2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)+$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_2(N,k,j)=\frac{1}{96N(N+1)}\Big(S_1(N)^4+(12\zeta_2+54S_2(N))S_1(N)^2$$

$$+104S_3(N)S_1(N)-48S_{2,1}(N)S_1(N)+51S_2(N)^2+36\zeta_2S_2(N)$$

$$+126S_4(N)-48S_{3,1}(N)-96S_{1,1,2}(N)\Big)$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+N)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+N)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+N)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)+\varepsilon^2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)+\varepsilon^3\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)+\dots$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_3(N,k,j)=\frac{1}{960N(N+1)}\Big(S_1(N)^5+(20\zeta_2+130S_2(N))S_1(N)^3+$$

$$(40\zeta_3+380S_3(N))S_1(N)^2+\left(135S_2(N)^2+60\zeta_2S_2(N)+510S_4(N)\right)S_1(N)$$

$$-240S_{3,1}(N)S_1(N)-240S_{1,1,2}(N)S_1(N)+160\zeta_2S_3(N)+S_2(N)(120\zeta_3$$

$$+380S_3(N))+624S_5(N)+\left(-120S_1(N)^2-120S_2(N)\right)S_{2,1}(N)$$

$$-240S_{4,1}(N)-240S_{1,1,3}(N)+240S_{2,2,1}(N)\Big)$$

A difficult sum from a 2–loop integral

$$\sum_{s=0}^{\infty}\sum_{j=0}^{N-2}\sum_{k=0}^{j}\sum_{i=0}^{N-j-2}\frac{(-1)^{i+k}\binom{j}{i}\binom{-j+N-2}{k}\Gamma(j+2)\Gamma(i+k+1)\left(\frac{\epsilon}{2}+2\right)_s(-\epsilon)_s\left(-\frac{\epsilon}{2}\right)_k(i+k+2)_s}{\Gamma(j+1)\Gamma(k+2)(1)_s\left(3-\frac{\epsilon}{2}\right)_{i+k}(k+2)_s\left(-\frac{\epsilon}{2}+i+k+3\right)_s}$$
$$= \epsilon^2F_2(N)+\epsilon^3F_3(N)+\dots$$

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$$=\epsilon^2F_2(N)+\epsilon^3F_3(N)+\dots$$

$$F_2(N)=\frac{1}{6}NS_1(N)^3-\frac{3}{4}S_1(N)^2+\frac{1}{16}\left(-29N^2+89N-8\right)$$
$$+\left(\frac{1}{8}\left(3N^2-3N-14\right)+\frac{1}{2}(N+2)S_2(N)\right)S_1(N)$$
$$-\frac{3}{4}S_2(N)+\frac{1}{3}(N+3)S_3(N)+(-N-2)S_{2,1}(N)$$

A difficult sum from a 2–loop integral

$$\sum_{s=0}^{\infty}\sum_{j=0}^{N-2}\sum_{k=0}^{j}\sum_{i=0}^{N-j-2}\frac{(-1)^{i+k}\binom{j}{i}\binom{-j+N-2}{k}\Gamma(j+2)\Gamma(i+k+1)\left(\frac{\epsilon}{2}+2\right)_s(-\epsilon)_s\left(-\frac{\epsilon}{2}\right)_k(i+k+2)_s}{\Gamma(j+1)\Gamma(k+2)(1)_s\left(3-\frac{\epsilon}{2}\right)_{i+k}(k+2)_s\left(-\frac{\epsilon}{2}+i+k+3\right)_s}$$
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$$\begin{aligned}F_2(N)=&\frac{1}{6}NS_1(N)^3-\frac{3}{4}S_1(N)^2+\frac{1}{16}\left(-29N^2+89N-8\right)\\&+\left(\frac{1}{8}\left(3N^2-3N-14\right)+\frac{1}{2}(N+2)S_2(N)\right)S_1(N)\\&-\frac{3}{4}S_2(N)+\frac{1}{3}(N+3)S_3(N)+(-N-2)S_{2,1}(N)\end{aligned}$$

$F_3(N) =$ coming soon (Sigma combined with MultiSum)
Joint work with Flavia Stan

A concluding email (2004)

```
From: Doron Zeilberger
To: Robin Pemantle, Herbert Wilf
CC:Carsten Schneider


Robin and Herb,

I am willing to bet that Carsten Schneider's SIGMA package
for handling sums with harmonic numbers (among others)
can do it in a jiffy. I am Cc-ing this to Carsten.

Carsten: please do it, and Cc- the  answer to me.
             -Doron
```

```
From: Robin Pemantle
To: herb wilf; doron zeilberger

Herb, Doron,

I have a sum that, when I evaluate numerically, looks
suspiciously like it comes out to exactly 1.
Is there a way I can automatically decide this?
The sum may be written in many ways, but one is:
```

$$\sum_{j,k=1}^{\infty} \frac{S_1(j)(S_1(k+1)-1)}{jk(k+1)(j+k)}$$

After one week of (hard) work Sigma found/proved:

$$\sum_{j,k=1}^{\infty} \frac{S_1(j)(S_1(k+1)-1)}{jk(k+1)(j+k)} = -4\zeta(2) - 2\zeta(3) + 4\zeta(2)\zeta(3) + 2\zeta(5)$$

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Now it is possible in a jiffy!