Summation/Integration/Differential/Difference Equations Workshop 2009

RISC, J. Kepler University

A Refined Difference Field Theory and Optimal Nested Sum Representations

Carsten Schneider
RISC/AMS, Austria

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Problem description

A sequence domain:

$$(X, \mathrm{ev}, \mathfrak{d})$$

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set of terms

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set of terms evaluation $X \times \mathbb{N} \to \mathbb{K}$

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- For $f = \frac{p}{q} \in \mathbb{K}(x)$ with $\gcd(p, q) = 1$,

$$\mathrm{ev}(f, k) = \begin{cases} 0 & \text{if } q(k) = 0 \\ \frac{p(k)}{q(k)} & \text{if } q(k) \neq 0; \end{cases}$$

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-

$$\mathfrak{d}(f) = \begin{cases} 0 & \text{if } f \in \mathbb{K} \\ 1 & \text{if } f \in \mathbb{K}(x) \setminus \mathbb{K} \end{cases}$$

Sum sequence domain

Given $(X, \mathrm{ev}, \mathfrak{d})$, we get the sum sequence domain $(\mathsf{Sum}(X), \mathrm{ev}, \mathfrak{d})$.

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$\mathsf{Sum}(X)$ is the term algebra over X with the signature

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\oplus: & \mathsf{Sum}(X) \times \mathsf{Sum}(X) & \rightarrow & \mathsf{Sum}(X) \\
\otimes: & \mathsf{Sum}(X) \times \mathsf{Sum}(X) & \rightarrow & \mathsf{Sum}(X) \\
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Example. Elements from $\mathsf{Sum}(\mathbb{Q}(x))$ are:

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$$\frac{1}{x} \qquad \mathsf{Sum}(7, \frac{1}{x})$$

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$$\mathsf{Sum}\Big(1, \tfrac{1}{x} \otimes \big(\mathsf{Sum}(1, \tfrac{1}{x} \otimes \big(\mathsf{Sum}(1, \tfrac{1}{x})^2 \oplus \mathsf{Sum}(1, \tfrac{1}{x^2})\big)\big) \oplus \mathsf{Sum}(1, \mathsf{Sum}(1, \tfrac{1}{x}))\big)\Big)$$

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Example.

$$\mathrm{ev}(\overbrace{\mathsf{Sum}(7, \frac{1}{x})}^{\mathsf{H}}, k) = \sum_{i=7}^{k} \mathrm{ev}(\frac{1}{x}, i) = \sum_{i=7}^{k} \frac{1}{i} \qquad \mathfrak{d}(H) = 1 + \mathfrak{d}(\frac{1}{x}) = 2$$

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Example.

$$\mathrm{ev}(\overbrace{\mathsf{Sum}\Big(1, \tfrac{1}{x} \otimes \big(\mathsf{Sum}\ (1, \tfrac{1}{x} \otimes \big(\mathsf{Sum}\ (1, \tfrac{1}{x})^2 \oplus \mathsf{Sum}(1, \tfrac{1}{x^2}))\big) \oplus \mathsf{Sum}(1, \mathsf{Sum}(1, \tfrac{1}{x})))\Big)}^{A}, k)$$

$$= \sum_{r=1}^{k} \frac{\displaystyle\sum_{l=1}^{r} \frac{\left(\displaystyle\sum_{i=1}^{l} \frac{1}{i}\right)^2 + \displaystyle\sum_{i=1}^{l} \frac{1}{i^2}}{l} + \sum_{l=1}^{r} \frac{\displaystyle\sum_{i=1}^{l} \frac{1}{i}}{l}}{r}$$

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Example.

$$\mathrm{ev}(\overbrace{\mathsf{Sum}_{1}\Big(1,\tfrac{1}{x}\otimes\big(\mathsf{Sum}_{2}(1,\tfrac{1}{x}\otimes\big(\mathsf{Sum}_{3}(1,\tfrac{1}{x})^2\oplus\mathsf{Sum}(1,\tfrac{1}{x^2})\big)\big)\oplus\mathsf{Sum}(1,\mathsf{Sum}(1,\tfrac{1}{x}))\big)\Big)}^{\mathsf{A}},k)$$

$$=\sum_{r=1}^{k}\frac{\displaystyle\sum_{l=1}^{r}\frac{\Big(\displaystyle\sum_{i=1}^{l}\frac{1}{i}\Big)^2+\displaystyle\sum_{i=1}^{l}\frac{1}{i^2}}{l}+\displaystyle\sum_{l=1}^{r}\frac{\displaystyle\sum_{i=1}^{l}\frac{1}{i}}{l}}{r}\qquad \mathfrak{d}(A)=4$$

The **optimal depth** of $A \in \mathsf{Sum}(X)$ is

$$\min\Big\{\mathfrak{d}(B) | B \in \mathsf{Sum}(X) \text{ such that } \\ \mathrm{ev}(A,k) = \mathrm{ev}(B,k) \text{ for some } k \text{ on}\Big\}.$$

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Problem: Depth Optimal Simplification.

Given $A \in \mathsf{Sum}(X)$;
find $B \in \mathsf{Sum}(X)$ and $\lambda \in \mathbb{N}$ such that

$$\operatorname{ev}(A,k) = \operatorname{ev}(B,k) \text{ for all } k \geq \lambda$$

and such that $\mathfrak{d}(B)$ is the **optimal depth** of A.

The **optimal depth** of $A \in \mathsf{Sum}(X)$ is

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Running example. *Given*

$$A = \mathsf{Sum}\Big(1, \tfrac{1}{x}\otimes\big(\mathsf{Sum}(1, \tfrac{1}{x}\otimes\big(\mathsf{Sum}(1, \tfrac{1}{x})^2\oplus\mathsf{Sum}(1, \tfrac{1}{x^2})\big)\big)\oplus\mathsf{Sum}(1, \mathsf{Sum}(1, \tfrac{1}{x}))\big)\Big)$$

with

$$\mathrm{ev}(A,k)=\sum_{r=1}^{k}\frac{\displaystyle\sum_{l=1}^{r}\frac{\Big(\displaystyle\sum_{i=1}^{l}\frac{1}{i}\Big)^2+\displaystyle\sum_{i=1}^{l}\frac{1}{i^2}}{l}+\sum_{l=1}^{r}\frac{\displaystyle\sum_{i=1}^{l}\frac{1}{i}}{l}}{r}$$

Find $B \in \mathsf{Sum}(\mathbb{Q}(x))$ and $\lambda \in \mathbb{N}$ such that

$$\mathrm{ev}(A,k)=\mathrm{ev}(B,k) \quad \forall k \geq \lambda$$

and such that the depth of B is optimal.

Karr's $\Pi\Sigma^*$-fields

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- A difference field $(\mathbb{F}, \sigma)$ is a field $\mathbb{F}$ plus a field automorphism σ.

Example. $(\mathbb{Q}(x), \sigma)$ with the shift operator $\sigma(f) = f(x+1)$.

- The constant field is

$$\text{const}_\sigma \mathbb{F} = \{k \in \mathbb{F} | \sigma(k) = k\}.$$

Example. $\text{const}_\sigma \mathbb{Q}(x) = \mathbb{Q}$

Karr's $\Pi\Sigma^*$-fields

- A difference field $(\mathbb{E}, \sigma')$ is a difference field extension of $(\mathbb{F}, \sigma)$ if

$$\mathbb{F} \leq \mathbb{E} \quad \text{and} \quad \sigma'|_{\mathbb{F}} = \sigma.$$

Note: From now on, we do not distinguish σ and σ' any longer

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- **Lemma.** Let $(\mathbb{F}, \sigma)$ be a DF, and $a \in \mathbb{F}^*$ and $b \in \mathbb{F}$.
Let t be transcendental over $\mathbb{F}$ ($\mathbb{F}(t)$ is a rational function field).
Then there is a unique DF-extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ such that
$$\sigma(t) = a\,t + b.$$

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Example. There is exactly one DF-extension $(\mathbb{Q}(x), \sigma)$ of $(\mathbb{Q}, \sigma)$ with $\sigma(x) = x + 1$.

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- **Definition** Such an extension in which the constants remain unchanged $(\mathrm{const}_\sigma \mathbb{F}(t) = \mathrm{const}_\sigma \mathbb{F})$ is called
  - Π-extension if $\sigma(t) = a\,t$
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$$\left.\begin{array}{l} \text{– } \Pi\text{-extension if } \sigma(t) = a\,t \\ \text{– } \Sigma^*\text{-extension if } \sigma(t) = t + b \end{array}\right\} \Pi\Sigma^*\text{-extension}$$

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- $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ is a (nested) $\Pi\Sigma^*$-extension (resp. Π-extension, Σ^*-extension) of $(\mathbb{F}, \sigma)$ if it is a tower of such extensions.

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- It is a $\Pi\Sigma^*$-field if $\mathrm{const}_\sigma\mathbb{F} = \mathbb{F}$.

SUMMARY Let $(\mathbb{F}(t),\sigma)$ be a DF-extension of $(\mathbb{F},\sigma)$ with

$$\sigma(t)=t+f \qquad \text{with } f\in\mathbb{F}.$$

Then this is a Σ^*-extension

$$\Leftrightarrow \quad t \text{ is transcendental over } \mathbb{F} \text{ and } \mathrm{const}_\sigma\mathbb{F}(t)=\mathrm{const}_\sigma\mathbb{F}$$

Theorem (Karr 81) Let $(\mathbb{F}(t), \sigma)$ be a DF-extension of $(\mathbb{F}, \sigma)$ with

$$\sigma(t) = t + f \qquad \text{with } f \in \mathbb{F}.$$

Then this is a Σ^*-extension

$\Leftrightarrow$ t is transcendental over $\mathbb{F}$ and $\mathrm{const}_\sigma \mathbb{F}(t) = \mathrm{const}_\sigma \mathbb{F}$

$\Leftrightarrow$ $\nexists g \in \mathbb{F}$:

$$\sigma(g) = g + f$$

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$$\begin{aligned} &\Leftrightarrow \quad t \text{ is transcendental over } \mathbb{F} \text{ and } \mathrm{const}_\sigma \mathbb{F}(t) = \mathrm{const}_\sigma \mathbb{F} \\ &\Leftrightarrow \quad \nexists g \in \mathbb{F} : \end{aligned}$$

$$\sigma(g) = g + f$$

Karr's summation algorithm solves (among others) the telescoping problem:

Given a $\Pi\Sigma^*$-field $(\mathbb{F}, \sigma)$ over $\mathbb{K}$ and $f \in \mathbb{F}$.
Compute, if possible, a $g \in \mathbb{F}$ such that

$$\sigma(g) = g + f.$$

Start with the $\Pi\Sigma^*$-field $(\mathbb{Q}(x),\sigma)$ over $\mathbb{Q}$ with $\sigma(x)=x+1$.

$$A(k)=\sum_{r=1}^{k}\frac{\displaystyle\sum_{l=1}^{r}\frac{\Big(\displaystyle\sum_{i=1}^{l}\frac{1}{i}\Big)^2+\displaystyle\sum_{i=1}^{l}\frac{1}{i^2}}{l}+\sum_{l=1}^{r}\frac{\boxed{\displaystyle\sum_{i=1}^{l}\frac{1}{i}}}{l}}{r}$$

1 By Gosper's, Karr's, or Sigma's algorithm:

$$\nexists g\in\mathbb{Q}(x):\sigma(g)=g+\frac{1}{x+1}.$$

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Note: $\boxed{\sum_{i=1}^{l+1} \frac{1}{i} = \sum_{i=1}^{l} \frac{1}{i} + \frac{1}{l+1}}$

$$A(k)=\sum_{r=1}^{k}\frac{\displaystyle\sum_{l=1}^{r}\frac{\overbrace{\Big(\sum_{i=1}^{l}\frac{1}{i}\Big)^2}^{h^2}+\displaystyle\sum_{i=1}^{l}\frac{1}{i^2}}{l}+\boxed{\displaystyle\sum_{l=1}^{r}\frac{\overbrace{\sum_{i=1}^{l}\frac{1}{i}}^{\frac{h}{x}}}{l}}}{r}$$

2 By Karr or Sigma:

$$\nexists g\in\mathbb{Q}(x)(h):\sigma(g)=g+\frac{\sigma(h)}{x+1}.$$

$$A(k)=\sum_{r=1}^{k}\frac{\displaystyle\sum_{l=1}^{r}\frac{\overbrace{\Big(\sum_{i=1}^{l}\frac{1}{i}\Big)^2}^{h^2}+\displaystyle\sum_{i=1}^{l}\frac{1}{i^2}}{l}+\boxed{\displaystyle\sum_{l=1}^{r}\frac{\overbrace{\sum_{i=1}^{l}\frac{1}{i}}^{\frac{h}{x}}}{l}}}{r}$$

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is a Σ^*-extension.

$$\boxed{\sum_{l=1}^{r+1}\frac{\sum_{i=1}^{l}\frac{1}{i}}{l}=\sum_{l=1}^{r}\frac{\sum_{i=1}^{l}\frac{1}{i}}{l}+\frac{\sum_{i=1}^{r+1}\frac{1}{i}}{r+1}}$$

$$A(k) = \sum_{r=1}^{k} \frac{\displaystyle\sum_{l=1}^{r} \frac{\overbrace{\Big(\sum_{i=1}^{l} \frac{1}{i}\Big)^2}^{h^2} + \boxed{\displaystyle\sum_{i=1}^{l} \frac{1}{i^2}}}{l} + \sum_{l=1}^{r} \frac{\overbrace{\displaystyle\sum_{i=1}^{l} \frac{1}{i}}^{s}}{l}}{r}$$

3 Given the telescoping problem

$$\sigma(g) = g + \frac{1}{(x+1)^2},$$

Sigma computes the solution

$$g = 2s - h^2 \in \mathbb{Q}(x)(h)(s).$$

$$A(k)=\sum_{r=1}^{k}\frac{\displaystyle\sum_{l=1}^{r}\frac{\overbrace{\Big(\sum_{i=1}^{l}\frac{1}{i}\Big)^2}^{h^2}+\overset{2s-h^2}{\boxed{\sum_{i=1}^{l}\frac{1}{i^2}}}}{l}+\overbrace{\sum_{l=1}^{r}\frac{\sum_{i=1}^{l}\frac{1}{i}}{l}}^{s}}{r}$$

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Note:

$$\boxed{\sum_{i=1}^{l}\frac{1}{i^2}=2\sum_{i=1}^{l}\frac{\sum_{j=1}^{i}\frac{1}{j}}{i}-\left(\sum_{i=1}^{l}\frac{1}{i}\right)^2}$$

$$A(k)=\sum_{r=1}^{k}\frac{\boxed{\sum_{l=1}^{r}\frac{\overbrace{\Big(\sum_{i=1}^{l}\frac{1}{i}\Big)^2+\sum_{i=1}^{l}\frac{1}{i^2}}^{\frac{2s}{x}}}{l}}+\sum_{l=1}^{r}\frac{\overbrace{\sum_{i=1}^{l}\frac{1}{i}}^{s}}{l}}{r}$$

4 By Sigma:

$$\nexists g\in\mathbb{Q}(x)(h)(s):\sigma(g)=g+2\frac{\sigma(s)}{x+1};$$

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5 By Sigma:

$$\nexists g \in \mathbb{Q}(x)(h)(s)(t) : \sigma(g) = g + \frac{\sigma(s+t)}{x+1}$$

a

$$A(k)=\boxed{\sum_{r=1}^{k}\frac{\overbrace{\sum_{l=1}^{r}\frac{\Big(\sum_{i=1}^{l}\frac{1}{i}\Big)^2+\sum_{i=1}^{l}\frac{1}{i^2}}{l}+\sum_{l=1}^{r}\frac{\sum_{i=1}^{l}\frac{1}{i}}{l}}^{\frac{t+s}{x}}}{r}}$$

5 By Sigma:

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SUMMARY $A(k)$ is represented by a in the $\Pi\Sigma^*$-field $(\mathbb{Q}(x)(h)(s)(t)(a),\sigma)$

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\forall k\in\mathbb{Q}\quad \sigma(k)&=k\\
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SUMMARY $A(k)$ is represented by a in the $\Pi\Sigma^*$-field $(\mathbb{Q}(x)(h)(s)(t)(a),\sigma)$

$$\begin{aligned}
\forall k\in\mathbb{Q}\quad \sigma(k)&=k & \forall k\in\mathbb{Q}\quad \delta(k)&=0\\
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\sigma(h)&=h+\tfrac{1}{x+1} & \delta(h)&=2\\
\sigma(s)&=s+\tfrac{\sigma(h)}{x+1} & \delta(s)&=3\\
\sigma(t)&=t+2\tfrac{\sigma(s)}{x+1} & \delta(t)&=4\\
\sigma(a)&=a+\tfrac{\sigma(s+t)}{x+1} & \delta(a)&=5
\end{aligned}$$

$$A(k) = \boxed{\sum_{r=1}^{k} \frac{\overbrace{\sum_{l=1}^{r} \frac{\left(\sum_{i=1}^{l} \frac{1}{i}\right)^2 + \sum_{i=1}^{l} \frac{1}{i^2}}{l} + \sum_{l=1}^{r} \frac{\sum_{i=1}^{l} \frac{1}{i}}{l}}^{\color{blue}{\mathsf{a}}\ \ \frac{\underline{t+s}}{x}}}{r}}$$

$$\|$$

$$\sum_{r=1}^{n} \frac{\sum_{l=1}^{r} \frac{2 \sum_{i=1}^{l} \frac{\sum_{j=1}^{i} \frac{1}{j}}{i}}{l} + \sum_{l=1}^{r} \frac{\sum_{j=1}^{l} \frac{1}{j}}{l}}{r} \qquad \text{depth} = 5$$

- For a given $\Pi\Sigma^*$-field $(\mathbb{F}, \sigma)$ the depth function

$$\delta : \mathbb{F} \to \mathbb{N}$$

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Example. Take our $\Pi\Sigma^*$-field $(\mathbb{Q}(x)(h)(s)(t)(a),\sigma)$

$$\begin{aligned} \sigma(x) &= x+1 & \delta(x) &= 1\\ \sigma(h) &= h+\tfrac{1}{x+1} & \delta(h) &= 2\\ \sigma(s) &= s+\tfrac{\sigma(h)}{x+1} & \delta(s) &= 3\\ \sigma(t) &= t+2\tfrac{\sigma(s)}{x+1} & \delta(t) &= 4\\ \sigma(a) &= a+\tfrac{\sigma(s+t)}{x+1} & \delta(a) &= 5 \end{aligned}$$

For any $f \in \mathbb{Q}(k,h,s,t,a)$,

$$\delta(f) = \max(\{\delta(x) | x \in \{k,h,s,t,a\} \text{ occurs in } f\} \cup \{0\})$$

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\end{aligned}$$

The extension depth of the Σ^*-extension $(\mathbb{Q}(x)(h)(s)(t)(a),\sigma)$ of $(\mathbb{Q}(x)(h)(s),\sigma)$ is $\max(\delta(t),\delta(a)) = 5$.

Theorem (Karr 81) Let $(\mathbb{F}(t), \sigma)$ be a DF-extension of $(\mathbb{F}, \sigma)$ with

$$\sigma(t) = t + f \qquad \text{with } f \in \mathbb{F}.$$

This is a Σ^*-extension iff

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- A $\Pi\Sigma^\delta$-field is a tower of Π- or Σ^δ-extensions over the constant field.

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- Note: A Σ^δ-extension is a Σ^*-extension.
- A $\Pi\Sigma^\delta$-field is a tower of Π- or Σ^δ-extensions over the constant field.

Sigma solves the following problem:

Given a $\Pi\Sigma^\delta$-field $(\mathbb{F},\sigma)$ over $\mathbb{K}$; $f\in\mathbb{F}$.
Compute, if possible, a Σ^δ-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ with extension depth $\leq\delta(f)$ and $g\in\mathbb{E}$ such that

$$\sigma(g)=g+f.$$

Start with the $\Pi\Sigma^*$-field $(\mathbb{Q}(x),\sigma)$ over $\mathbb{Q}$ with $\sigma(x)=x+1$ and $\delta(x)=1$.

$$A(k)=\sum_{r=1}^{k}\frac{\displaystyle\sum_{l=1}^{r}\frac{\Big(\displaystyle\sum_{i=1}^{l}\frac{1}{i}\Big)^2+\displaystyle\sum_{i=1}^{l}\frac{1}{i^2}}{l}+\sum_{l=1}^{r}\frac{\boxed{\displaystyle\sum_{i=1}^{l}\frac{1}{i}}}{l}}{r}$$

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$$\nexists g\in\mathbb{Q}(x):\sigma(g)=g+\frac{1}{x+1}.$$

Note: there is no Σ^*-extension of $(\mathbb{Q}(x),\sigma)$ with extension depth 1.

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Note: there is no Σ^*-extension of $(\mathbb{Q}(x),\sigma)$ with extension depth 1.
Hence the DF-extension $(\mathbb{Q}(x)(h),\sigma)$ of $(\mathbb{Q}(x),\sigma)$ with

$$\sigma(h)=h+\frac{1}{x+1},\quad \delta(h)=2$$

is a Σ^δ-extension.

$$A(k)=\sum_{r=1}^{k}\frac{\displaystyle\sum_{l=1}^{r}\frac{\overbrace{\Big(\sum_{i=1}^{l}\frac{1}{i}\Big)^2}^{h^2}+\sum_{i=1}^{l}\frac{1}{i^2}}{l}+\boxed{\displaystyle\sum_{l=1}^{r}\frac{\overbrace{\sum_{i=1}^{l}\frac{1}{i}}^{\frac{h}{x}}}{l}}}{r}$$

2 Given the telescoping problem

$$\sigma(g)-g=\frac{\sigma(h)}{x+1},$$

$$A(k)=\sum_{r=1}^{k}\frac{\displaystyle\sum_{l=1}^{r}\frac{\overbrace{\Big(\sum_{i=1}^{l}\frac{1}{i}\Big)^2}^{h^2}+\sum_{i=1}^{l}\frac{1}{i^2}}{l}+\boxed{\displaystyle\sum_{l=1}^{r}\frac{\overbrace{\sum_{i=1}^{l}\frac{1}{i}}^{\frac{h}{x}}}{l}}}{r}$$

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$$\sigma(g)-g=\frac{\sigma(h)}{x+1},$$

Sigma finds the Σ^{δ}-extension h_2 over $\mathbb{Q}(x)(h)$ with

$$\sigma(h_2)=h_2+\frac{1}{(x+1)^2},\quad \delta(h_2)=2$$

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Sigma finds the Σ^{δ}-extension h_2 over $\mathbb{Q}(x)(h)$ with

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together with the solution

$$g=\frac{1}{2}(h^2+h_2)$$

$$A(k)=\sum_{r=1}^{k}\frac{\displaystyle\sum_{l=1}^{r}\frac{\overbrace{\Big(\sum_{i=1}^{l}\frac{1}{i}\Big)^2}^{h^2}+\boxed{\displaystyle\sum_{i=1}^{l}\frac{1}{i^2}}}{l}+\overbrace{\sum_{l=1}^{r}\frac{\displaystyle\sum_{i=1}^{l}\frac{1}{i}}{l}}^{\frac{1}{2}(h^2+h_2)}}{r}$$

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$$\sigma(g)=g+\frac{1}{(x+1)^2},$$

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Sigma finds

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$$\sigma(g)-g=\frac{\sigma(h^2+h_2)}{x+1}$$

$$A(k) = \sum_{r=1}^{k} \frac{\boxed{\sum_{l=1}^{r} \frac{\overbrace{\Big(\sum_{i=1}^{l} \frac{1}{i}\Big)^2 + \sum_{i=1}^{l} \frac{1}{i^2}}^{\frac{h^2+h_2}{x}}}{l}} + \sum_{l=1}^{r} \frac{\overbrace{\sum_{i=1}^{l} \frac{1}{i}}^{\frac{1}{2}(h^2+h_2)}}{l}}{r}$$

4 Given the telescoping problem

$$\sigma(g) - g = \frac{\sigma(h^2 + h_2)}{x+1}$$

Sigma computes the Σ^δ-extension h_3 over $(\mathbb{Q}(x)(h)(h_2), \sigma)$ with

$$\sigma(h_3) = h_3 + \frac{1}{(x+1)^3}, \quad \delta(h_3) = 2$$

$$\frac{1}{3}(h^3+3hh_2+2h_3)$$

$$A(k)=\sum_{r=1}^{k}\frac{\boxed{\displaystyle\sum_{l=1}^{r}\frac{\overbrace{\Big(\sum_{i=1}^{l}\frac{1}{i}\Big)^2+\sum_{i=1}^{l}\frac{1}{i^2}}^{\frac{h^2+h_2}{x}}}{l}}+\overbrace{\displaystyle\sum_{l=1}^{r}\frac{\sum_{i=1}^{l}\frac{1}{i}}{l}}^{\frac{1}{2}(h^2+h_2)}}{r}$$

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Sigma computes the Σ^δ-extension h_3 over $(\mathbb{Q}(x)(h)(h_2),\sigma)$ with

$$\sigma(h_3)=h_3+\frac{1}{(x+1)^3},\quad \delta(h_3)=2$$

together with solution $g=\frac{1}{3}(h^3+3hh_2+2h_3)$

$$A(k)=\boxed{\sum_{r=1}^{k}\frac{\overbrace{\sum_{l=1}^{r}\frac{\Big(\sum_{i=1}^{l}\frac{1}{i}\Big)^2+\sum_{i=1}^{l}\frac{1}{i^2}}{l}+\sum_{l=1}^{r}\frac{\sum_{i=1}^{l}\frac{1}{i}}{l}}^{\frac{1}{3}(h^3+3hh_2+2h_3)+\frac{1}{2}(h^2+h_2)}}{r}}$$

a

5 Finally: Sigma finds the Σ^{δ}-extension h_4 over $(\mathbb{Q}(x)(h)(h_2)(h_3),\sigma)$ with

$$\sigma(h_4)=h_4+\tfrac{1}{(x+1)^4},\quad \delta(h_4)=2$$

and represents $A(k)$ by

$$a=\tfrac{1}{12}(h^4+2h^3+6(h+1)h_2h+3h_2^2+(8h+4)h_3+6h_4)$$

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SUMMARY $A(k)$ is represented in the $\Pi\Sigma^{\delta}$-field $(\mathbb{Q}(x)(h)(h_2)(h_3)(h_4),\sigma)$:

$$\begin{aligned}\forall k\in\mathbb{Q}\quad \sigma(k)&=k & \forall k\in\mathbb{Q}\quad \delta(k)&=0\\ \sigma(x)&=x+1 & \delta(x)&=1\\ \sigma(h)&=h+\tfrac{1}{x+1} & \delta(h)&=2\\ \sigma(h_2)&=h_2+\tfrac{1}{(x+1)^2} & \delta(h_2)&=2\\ \sigma(h_3)&=h_3+\tfrac{1}{(x+1)^3} & \delta(h_3)&=2\\ \sigma(h_4)&=h_4+\tfrac{1}{(x+1)^4} & \delta(h_4)&=2\end{aligned}$$

$$\frac{1}{12}(h^4+2h^3+6(h+1)h_2h+3h_2^2+(8h+4)h_3+6h_4)$$

$$A(k)=\boxed{\sum_{r=1}^{k}\frac{\displaystyle\sum_{l=1}^{r}\frac{\Big(\sum_{i=1}^{l}\frac{1}{i}\Big)^2+\sum_{i=1}^{l}\frac{1}{i^2}}{l}+\sum_{l=1}^{r}\frac{\sum_{i=1}^{l}\frac{1}{i}}{l}}{r}}$$

$$||$$

$$\mathrm{ev}(B,k)$$

with

$$B=\frac{1}{12}\Big(\mathsf{Sum}(1,\tfrac{1}{x})^4+2\mathsf{Sum}(1,\tfrac{1}{x})^3+6(\mathsf{Sum}(1,\tfrac{1}{x})+1)\mathsf{Sum}(1,\tfrac{1}{x^2})\mathsf{Sum}(1,\tfrac{1}{x})$$
$$+3\mathsf{Sum}(1,\tfrac{1}{x^2})^2+(8\mathsf{Sum}(1,\tfrac{1}{x})+4)\mathsf{Sum}(1,\tfrac{1}{x^3})+6\mathsf{Sum}(1,\tfrac{1}{x^4})\Big)$$

B has the **optimal depth** 2

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   - $\dots$

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Problem: Depth Optimal Simplification.

Given $A \in \mathsf{Sum}(X)$;
find $B \in \mathsf{Sum}(X)$ and $\lambda \in \mathbb{N}$ such that

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3. We exploit the following property of a depth-optimal $\Pi\Sigma^*$-field $(\mathbb{F}, \sigma)$:

$$\boxed{\forall f, g \in \mathbb{F}: \quad \sigma(g) - g = f \quad \Rightarrow \quad \delta(g) \leq \delta(f) + 1}$$

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Given $A \in \mathsf{Sum}(X)$;
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$$\mathrm{ev}(A, k) = \mathrm{ev}(B, k) \text{ for all } k \geq \lambda$$

and such that $\mathfrak{d}(B)$ is the optimal depth of A.

3. We exploit the following property of a depth-optimal $\Pi\Sigma^*$-field $(\mathbb{F}, \sigma)$:

$$\boxed{\forall f, g \in \mathbb{F}: \quad \sigma(g) - g = f \quad \Rightarrow \quad \delta(g) \leq \delta(f) + 1} = \text{“}\mathfrak{d}(\textstyle\sum f)\text{”}$$

MAIN RESULTS:

1. Any element $A \in \mathsf{Sum}(X)$ can be represented in a depth-optimal $\Pi\Sigma^*$-field.

2. **Any such representation** delivers a solution for our
Problem: Depth Optimal Simplification.

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The notion depth-optimal $\Pi\Sigma^*$-extension makes sense ☺