Getting Ready for Physics at the LHC, RECAPP

Tutorial: Advanced Summation Algorithms and its Application in Particle Physics

Carsten Schneider

RISC-Linz, Austria

Some (selective) literature as starting point to read:

M. Karr. Summation in finite terms. *J. ACM*, 28:305–350, 1981.

M. Petkovšek, H. S. Wilf, and D. Zeilberger. $A = B$. A. K. Peters, Wellesley, MA, 1996.
Available at http://www.math.upenn.edu/ wilf/AeqB.html

C. Schneider. Symbolic summation assists combinatorics. *Sém. Lothar. Combin.*, 56:1–35, 2007.
Available at `http://www.mat.univie.ac.at/~slc/`

C. Schneider. A refined difference field theory for symbolic summation. *J. Symbolic Comput.*, 43(9):611–644, 2008.
Available at arXiv:0808.2543v1.

Telescoping

GIVEN $f(k) = k^2 H_{n+k}$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Telescoping

GIVEN $f(k) = k^2 H_{n+k}$.

FIND $g(k)$:

$$f(k) = g(k+1) - g(k)$$

for all $1 \le k \le n$ and $n \ge 0$.

Sigma computes

$$g(k) = \frac{1}{36}(k(-4k^2 + (6n+3)k - 12n(n+1) + 1) + 6(2k^3 - 3k^2 + k + n(n+1)(2n+1))H_{k+n}).$$

Telescoping

GIVEN $f(k) = k^2 H_{n+k}$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Summing this equation over k from 1 to n gives

$$\boxed{\sum_{k=1}^{n} k^2 \mathrm{H}_{n+k}} = \boxed{g(n+1) - g(1)}$$

$$= -\frac{1}{36} n(n+1)\left(10n + 6(2n+1)H_n - 12(2n+1)H_{2n} - 1\right).$$

Telescoping

FIND a closed form for

$$\sum_{k=1}^{n} H_k.$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{Q}, & & \\ \sigma(k) &= k+1, & \mathcal{S}\, k &= k+1, \\ \sigma(h) &= h + \frac{1}{k+1}, & \mathcal{S}\, H_k &= H_k + \frac{1}{k+1}. \end{aligned}$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

Sigma computes

$$g = (h-1)k \in \mathbb{F}.$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

Sigma computes

$$g = (h-1)k \in \mathbb{F}.$$

This gives

$$\boxed{g(k+1) - g(k) = H_k}$$

with

$$g(k) = (H_k - 1)k.$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

Sigma computes

$$g = (h-1)k \in \mathbb{F}.$$

This gives

$$\boxed{g(k+1) - g(k) = H_k}$$

with

$$g(k) = (H_k - 1)k.$$

Hence,

$$(H_{n+1} - 1)(n+1) = \sum_{k=1}^{n} H_k.$$

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]$:

$d = 1$

$$\forall g \in \mathbb{Q}(k)(h): \quad \sigma(g) - g = h \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[h].$$

FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]$:

$$\forall g \in \mathbb{Q}(k)(h): \quad \sigma(g) - g = h \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[h].$$

$d = 1$

FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

Degree bound: COMPUTE $b \geq 0$:

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq b.$$

$b = 2$

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]$:

$$\forall g \in \mathbb{Q}(k)(h): \quad \sigma(g) - g = h \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[h].$$

$d = 1$

FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

Degree bound: COMPUTE $b \geq 0$:

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq b.$$

$b = 2$

Polynomial Solution: FIND

$$g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h].$$

g=hk-k

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\sigma(g) - g = h$$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h+\tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\,h^2 + g_1 h + g_0\big] = h$$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\left[\sigma(g_2)\left(h+\tfrac{1}{k+1}\right)^2+\sigma(g_1h+g_0)\right] - \left[g_2\,h^2+g_1h+g_0\right]=h$$

coeff. comp.

$$\sigma(g_2)-g_2=0$$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\big[\sigma(g_2)\big(h+\tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\big] - \big[g_2\,h^2 + g_1 h + g_0\big] = h$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\, h^2 + g_1 h + g_0\big] = h$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = h - c\Big[\tfrac{2h(k+1)+1}{(k+1)^2}\Big]$$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h+\tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\,h^2 + g_1 h + g_0\big] = h$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\,h + g_0) - (g_1\,h + g_0) = h - c\Big[\tfrac{2h(k+1)+1}{(k+1)^2}\Big]$$

coeff. comp.

$$\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}$$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h+\tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\,h^2 + g_1 h + g_0\big] = h$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\,h + g_0) - (g_1\,h + g_0) = h - c\Big[\tfrac{2h(k+1)+1}{(k+1)^2}\Big]$$

coeff. comp.

$$\sigma(g_1) - g_1 = 1 - c\,\frac{2}{k+1}$$

$c = 0$, $g_1 = k + d$, $d \in \mathbb{Q}$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\left[\sigma(g_2)\left(h+\tfrac{1}{k+1}\right)^2 + \sigma(g_1 h + g_0)\right] - \left[g_2\,h^2 + g_1 h + g_0\right] = h$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\,h + g_0) - (g_1\,h + g_0) = h - c\left[\tfrac{2h(k+1)+1}{(k+1)^2}\right]$$

coeff. comp.

$$\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}$$

$c = 0,\quad g_1 = k + d,\ d \in \mathbb{Q}$

$$\sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}$$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\left[\sigma(g_2)\left(h+\tfrac{1}{k+1}\right)^2 + \sigma(g_1 h + g_0)\right] - \left[g_2\,h^2 + g_1 h + g_0\right] = h$$

coeff. comp.

g=hk-k

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\,h + g_0) - (g_1\,h + g_0) = h - c\left[\tfrac{2h(k+1)+1}{(k+1)^2}\right]$$

coeff. comp.

$$\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}$$

$c = 0,\quad g_1 = k + d,\ d \in \mathbb{Q}$

$$g_0 = -k,\ d = 0 \quad \leftarrow \quad \sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}$$

The general case (Karr's algorithm)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:
a rational function field $\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$ with an automorphism

The general case (Karr's algorithm)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:
a rational function field $\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$ with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

The general case (Karr's algorithm)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:
a rational function field $\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$ with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$
$$\sigma(t_1) = \alpha_1\, t_1 + \beta_1, \qquad \alpha_1 \in \mathbb{K}^*, \qquad \beta_1 \in \mathbb{K}$$

The general case (Karr's algorithm)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F},\sigma)$:
a rational function field $\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$ with an automorphism

$$\begin{array}{lll} \sigma(c) = c \quad \forall c \in \mathbb{K} & & \\ \sigma(t_1) = \alpha_1\, t_1 + \beta_1, & \alpha_1 \in \mathbb{K}^*, & \beta_1 \in \mathbb{K} \\ \sigma(t_2) = \alpha_2\, t_2 + \beta_2, & \alpha_2 \in \mathbb{K}(t_1)^*, & \beta_2 \in \mathbb{K}(t_1) \end{array}$$

The general case (Karr's algorithm)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:
a rational function field $\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$ with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= \alpha_1\, t_1 + \beta_1, & \alpha_1 &\in \mathbb{K}^*, & \beta_1 &\in \mathbb{K} \\
\sigma(t_2) &= \alpha_2\, t_2 + \beta_2, & \alpha_2 &\in \mathbb{K}(t_1)^*, & \beta_2 &\in \mathbb{K}(t_1) \\
\sigma(t_3) &= \alpha_3\, t_3 + \beta_3, & \alpha_3 &\in \mathbb{K}(t_1, t_2)^*, & \beta_3 &\in \mathbb{K}(t_1, t_2) \\
&\vdots
\end{aligned}$$

The general case (Karr's algorithm)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:
a rational function field $\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$ with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

$$\begin{aligned}
\sigma(t_1) &= \alpha_1\, t_1 + \beta_1, & \alpha_1 &\in \mathbb{K}^*, & \beta_1 &\in \mathbb{K}\\
\sigma(t_2) &= \alpha_2\, t_2 + \beta_2, & \alpha_2 &\in \mathbb{K}(t_1)^*, & \beta_2 &\in \mathbb{K}(t_1)\\
\sigma(t_3) &= \alpha_3\, t_3 + \beta_3, & \alpha_3 &\in \mathbb{K}(t_1, t_2)^*, & \beta_3 &\in \mathbb{K}(t_1, t_2)\\
&\vdots
\end{aligned}$$

such that $\{c \in \mathbb{K}(t_1)\dots(t_e) | \sigma(c) = c\} = \mathbb{K}$.

The general case (Karr's algorithm)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:
a rational function field $\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$ with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

$$\begin{aligned}
\sigma(t_1) &= \alpha_1\, t_1 + \beta_1, & \alpha_1 &\in \mathbb{K}^*, & \beta_1 &\in \mathbb{K}\\
\sigma(t_2) &= \alpha_2\, t_2 + \beta_2, & \alpha_2 &\in \mathbb{K}(t_1)^*, & \beta_2 &\in \mathbb{K}(t_1)\\
\sigma(t_3) &= \alpha_3\, t_3 + \beta_3, & \alpha_3 &\in \mathbb{K}(t_1,t_2)^*, & \beta_3 &\in \mathbb{K}(t_1,t_2)\\
&\vdots
\end{aligned}$$

such that $\{c \in \mathbb{K}(t_1)\dots(t_e) | \sigma(c) = c\} = \mathbb{K}$.

GIVEN $f \in \mathbb{F}$;

FIND $g \in \mathbb{F}$ with

$$\sigma(g) - g = f.$$

Construction of Σ^*-extensions

- Let $(\mathbb{F}, \sigma)$ be a difference field with constant field

$$\text{const}_\sigma \mathbb{F} := \{k \in \mathbb{F} | \sigma(k) = k\}.$$

Construction of Σ^*-extensions

- Let $(\mathbb{F}, \sigma)$ be a difference field with constant field

$$\text{const}_\sigma \mathbb{F} := \{k \in \mathbb{F} | \sigma(k) = k\}.$$

- Adjoin a new variable t to $\mathbb{F}$, i.e., let $\mathbb{F}(t)$ be a rational function field.

Construction of Σ^*-extensions

- Let $(\mathbb{F}, \sigma)$ be a difference field with constant field

$$\text{const}_\sigma \mathbb{F} := \{k \in \mathbb{F} | \sigma(k) = k\}.$$

- Adjoin a new variable t to $\mathbb{F}$, i.e., let $\mathbb{F}(t)$ be a rational function field.
- Extend the shift operator s.t. $\sigma(t) = t + f$ for some $f \in \mathbb{F}$.

Construction of Σ^*-extensions

- Let $(\mathbb{F}, \sigma)$ be a difference field with constant field

$$\mathrm{const}_\sigma\mathbb{F} := \{k \in \mathbb{F} | \sigma(k) = k\}.$$

- Adjoin a new variable t to $\mathbb{F}$, i.e., let $\mathbb{F}(t)$ be a rational function field.
- Extend the shift operator s.t. $\sigma(t) = t + f$ for some $f \in \mathbb{F}$.

(Karr 1981) Then $\mathrm{const}_\sigma\mathbb{F}(t) = \mathrm{const}_\sigma\mathbb{F}$ iff

$$\nexists g \in \mathbb{F} : \quad \boxed{\sigma(g) - g = f}$$

Construction of Σ^*-extensions

- Let $(\mathbb{F}, \sigma)$ be a difference field with constant field

$$\text{const}_\sigma\mathbb{F} := \{k \in \mathbb{F} | \sigma(k) = k\}.$$

- Adjoin a new variable t to $\mathbb{F}$, i.e., let $\mathbb{F}(t)$ be a rational function field.
- Extend the shift operator s.t. $\sigma(t) = t + f$ for some $f \in \mathbb{F}$.

(Karr 1981) Then $\text{const}_\sigma\mathbb{F}(t) = \text{const}_\sigma\mathbb{F}$ iff

$$\nexists g \in \mathbb{F} : \quad \boxed{\sigma(g) - g = f}$$

Such an extension is called Σ^*-extension (the product case can be handled analogously with Π-extensions).

Construction of Σ^*-extensions

- Let $(\mathbb{F},\sigma)$ be a difference field with constant field

$$\mathrm{const}_\sigma\mathbb{F} := \{k \in \mathbb{F} | \sigma(k) = k\}.$$

- Adjoin a new variable t to $\mathbb{F}$, i.e., let $\mathbb{F}(t)$ be a rational function field.
- Extend the shift operator s.t. $\sigma(t) = t + f$ for some $f \in \mathbb{F}$.

(Karr 1981) Then $\mathrm{const}_\sigma\mathbb{F}(t) = \mathrm{const}_\sigma\mathbb{F}$ iff

$$\nexists g \in \mathbb{F}: \quad \boxed{\sigma(g) - g = f}$$

In a nutshell: Either the sum t with $\sigma(t) = t + f$ can be expressed by $g \in \mathbb{F}$ with telescoping, or we can adjoin a new sum t to our difference field in form of a Σ^*-extension.

telescoping

- GIVEN

$$\sum_{k=0}^{n} f(k).$$

- FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

telescoping

- GIVEN

$$\sum_{k=0}^{n} f(k).$$

- FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

- THEN (with some mild extra conditions)

$$\sum_{k=0}^{n} f(k) = g(n+1) - g(0)$$

Refined telescoping

- GIVEN

$$\sum_{k=0}^{n} f(k).$$

- FIND $g(k)$ and $f^*(k)$:

$$\boxed{f(k) = g(k+1) - g(k) + f^*(k)}$$

where $f^*(k)$ is simpler than $f(k)$.

- THEN (with some mild extra conditions)

$$\sum_{k=0}^{n} f(k) = g(n+1) - g(0) + \sum_{k=0}^{n} f^*(k).$$

Simpler w.r.t. the depth

$$\sum_{k=1}^{n} S_1(k)^2 S_2(k) =$$

$$\sum_{k=0}^{a} \left(\sum_{i=0}^{k} \binom{n}{i} \right)^2 =$$

Sigma

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{S_1(j)}{j^2}}{k^3} =$$

Simpler w.r.t. the depth

$$\sum_{k=1}^{n} S_1(k)^2 S_2(k) = \frac{1}{3} S_3(n) - \frac{1}{3} S_1(n)^3 + ((n+1)S_2(n)+1)\, S_1(n)^2 + (2n+1)\,(1-S_1(n))\, S_2(n) - 2S_1(n)$$

$$\sum_{k=0}^{a}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^2 = (n-a)\binom{n}{a}\sum_{i=0}^{a}\binom{n}{i} - \frac{n-2a-2}{2}\left(\sum_{i=0}^{a}\binom{n}{i}\right)^2 - \frac{n}{2}\sum_{i=1}^{a}\binom{n}{i}^2$$

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{S_1(j)}{j^2}}{k^3} = S_3(n)\sum_{j=1}^{n}\frac{S_1(j)}{j^2} - \sum_{j=1}^{n}\frac{S_1(j)S_3(j)}{j^2} + \sum_{j=1}^{n}\frac{S_1(j)}{j^5}$$

Simpler w.r.t. the depth

$$\sum_{k=1}^{n} S_1(k)^2 S_2(k) = \frac{1}{3}S_3(n) - \frac{1}{3}S_1(n)^3 + \left((n+1)S_2(n)+1\right)S_1(n)^2 + (2n+1)\left(1-S_1(n)\right)S_2(n) - 2S_1(n)$$

$$\sum_{k=0}^{a}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^2 = (n-a)\binom{n}{a}\sum_{i=0}^{a}\binom{n}{i} - \frac{n-2a-2}{2}\left(\sum_{i=0}^{a}\binom{n}{i}\right)^2 - \frac{n}{2}\sum_{i=1}^{a}\binom{n}{i}^2$$

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{S_1(j)}{j^2}}{k^3} = S(3,2,1,N) \qquad \text{(Harmonic sum)}$$

$$= S_3(n)\sum_{j=1}^{n}\frac{S_1(j)}{j^2} - \sum_{j=1}^{n}\frac{S_1(j)S_3(j)}{j^2} + \sum_{j=1}^{n}\frac{S_1(j)}{j^5} \qquad \text{(Euler sums)}$$

Further examples

$$\sum_{k=1}^{n} S_1(k)^3 = \tfrac{1}{2}\Big(2(n+1)S_1(n)^3 - 3(2n+1)S_1(n)^2 + 6(2n+1)S_1(n) - 12n - 1 + S$$

$$\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j}\sum_{i=1}^{j}\frac{1}{i} = \frac{1}{6}\left[S_1(n)^3 + 3S_1(n)\,S_2(n) + 2\,S_3(n)\right]$$

$$\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\frac{S_2(j)}{j^3}\right)^2 = -\left(S_2(n)^2 + S_4(n)\right)\sum_{j=1}^{n}\frac{S_2(j)}{j^3} + (n+1)\left(\sum_{j=1}^{n}\frac{S_2(j)}{j^3}\right)^2 + \sum_{j=1}^{n}\frac{S_2(j)^3}{j^3} - \sum_{j=1}^{n}\frac{S_2(j)^2}{j^5} + \sum_{j=1}^{n}\frac{S_2(j)S_4(j)}{j^3}.$$

Creative telescoping (for order 2)

GIVEN

$$S(n) = \sum_{k=0}^{n} f(n,k).$$

FIND $c_0(n), c_1(n), c_2(n)$ and $g(n,k)$:

$$\boxed{g(n,k+1) - g(n,k)}$$

$$= \boxed{c_0(n)f(n,k) + c_1(n)f(n+1,k) + c_2(n)f(n+2,k).}$$

Creative telescoping (for order 2)

GIVEN

$$S(n)=\sum_{k=0}^{n} f(n,k).$$

GIVEN $c_0(n), c_1(n), c_2(n)$ and $g(n,k)$:

$$\boxed{g(n,k+1)-g(n,k)}$$
$$=\boxed{c_0(n)f(n,k)+c_1(n)f(n+1,k)+c_2(n)f(n+2,k).}$$

THEN summation over k from 0 to n gives

$$\boxed{g(n,n+1)-g(n,0)}$$
$$=\boxed{\begin{array}{l} c_0(n)\,\mathsf{S}(n)+\\ c_1(n)\,\big[\mathsf{S}(n+1)-f(n+1,n+1)\big]\\ c_2(n)\,\big[\mathsf{S}(n+2)-f(n+2,n+1)-f(n+2,n+2)\big].\end{array}}$$

Creative telescoping (for order 2)

GIVEN

$$S(n) = \sum_{k=0}^{n} f(n,k).$$

GIVEN $c_0(n), c_1(n), c_2(n)$ and $g(n,k)$:

$$\boxed{g(n,k+1) - g(n,k)}$$
$$= \boxed{c_0(n)f(n,k) + c_1(n)f(n+1,k) + c_2(n)f(n+2,k).}$$

HENCE

$$h(n) = c_0(n)S(n) + c_1(n)S(n+1) + c_2(n)S(n+2)$$

for some $h(n)$.

Creative telescoping (for order 2)

GIVEN

$$S(n) = \sum_{k=0}^{n} f(n,k).$$

GIVEN $c_0(n), c_1(n), c_2(n)$ and $g(n,k)$:

$$\boxed{g(n,k+1) - g(n,k)}$$
$$= \boxed{c_0(n) f(n,k) + c_1(n) f(n+1,k) + c_2(n) f(n+2,k).}$$

HENCE

$$h(n) = c_0(n) S(n) + c_1(n) S(n+1) + c_2(n) S(n+2)$$

for some $h(n)$.

Example with $S(n) = \sum_{k=0}^{n} \binom{n}{k} S_1(k)$

FIND a recurrence for

$$S(n) := \sum_{k=1}^{n} \binom{n}{k} S_1(k).$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(n)(k)(h)(b)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

$$\sigma(h) = h + \frac{1}{k+1}, \qquad \mathcal{S}\, S_1(k) = S_1(k) + \frac{1}{k+1},$$

$$\sigma(b) = \frac{n-k}{k+1} b, \qquad \mathcal{S} \binom{n}{k} = \frac{n-k}{k+1} \binom{n}{k}.$$

REPRESENT $f(n+i,k)$ in $\mathbb{F}$:

$$f(n,k) = S_1(k)\binom{n}{k} \longleftrightarrow h\,b =: f_0$$

REPRESENT $f(n+i,k)$ in $\mathbb{F}$:

$$f(n,k) = S_1(k)\binom{n}{k} \longleftrightarrow h\,b =: f_0$$

$$f(n+1,k) = \frac{(n+1)\,S_1(k)\binom{n}{k}}{n+1-k} \longleftrightarrow \frac{(n+1)\,h\,b}{n+1-k} =: f_1$$

$$f(n+2,k) = \frac{(n+1)\,(n+2)\,S_1(k)\binom{n}{k}}{(n+1-k)\,(n+2-k)} \longleftrightarrow \frac{(n+1)\,(n+2)\,h\,b}{(n+1-k)\,(n+2-k)} =: f_2.$$

REPRESENT $f(n+i,k)$ in $\mathbb{F}$:

$$f(n,k)=S_1(k)\binom{n}{k}\quad\longleftrightarrow\quad h\,b=:f_0$$

$$f(n+1,k)=\frac{(n+1)\,S_1(k)\binom{n}{k}}{n+1-k}\quad\longleftrightarrow\quad\frac{(n+1)\,h\,b}{n+1-k}=:f_1$$

$$f(n+2,k)=\frac{(n+1)\,(n+2)\,S_1(k)\binom{n}{k}}{(n+1-k)\,(n+2-k)}\quad\longleftrightarrow\quad\frac{(n+1)\,(n+2)\,h\,b}{(n+1-k)\,(n+2-k)}=:f_2.$$

FIND $c_0,c_1,c_2\in\mathbb{Q}(n)$ and $g\in\mathbb{F}$:

$$\boxed{\sigma(g)-g=c_0f_0+c_1f_1+c_2f_2.}$$

REPRESENT $f(n+i,k)$ in $\mathbb{F}$:

$$f(n,k)=S_1(k)\binom{n}{k} \longleftrightarrow h\,b =: f_0$$

$$f(n+1,k)=\frac{(n+1)\,S_1(k)\,\binom{n}{k}}{n+1-k} \longleftrightarrow \frac{(n+1)\,h\,b}{n+1-k} =: f_1$$

$$f(n+2,k)=\frac{(n+1)\,(n+2)\,S_1(k)\,\binom{n}{k}}{(n+1-k)\,(n+2-k)} \longleftrightarrow \frac{(n+1)\,(n+2)\,h\,b}{(n+1-k)\,(n+2-k)} =: f_2.$$

FIND $c_0, c_1, c_2 \in \mathbb{Q}(n)$ and $g \in \mathbb{F}$:

$$\boxed{\sigma(g)-g=c_0 f_0+c_1 f_1+c_2 f_2.}$$

Sigma computes

$$c_0 := 4\,(1+n),\quad c_1 := -2\,(3+2\,n),\quad c_2 := 2+n,$$

$$g := \frac{(1+n)\,(-2+k-n+(2\,k-2\,k^2+k\,n)h)\,b)}{(1-k+n)\,(2-k+n)}.$$

REPRESENT $f(n+i,k)$ in $\mathbb{F}$:

$$f(n,k) = S_1(k)\binom{n}{k} \longleftrightarrow h\,b =: f_0$$

$$f(n+1,k) = \frac{(n+1)\,S_1(k)\,\binom{n}{k}}{n+1-k} \longleftrightarrow \frac{(n+1)\,h\,b}{n+1-k} =: f_1$$

$$f(n+2,k) = \frac{(n+1)\,(n+2)\,S_1(k)\,\binom{n}{k}}{(n+1-k)\,(n+2-k)} \longleftrightarrow \frac{(n+1)\,(n+2)\,h\,b}{(n+1-k)\,(n+2-k)} =: f_2.$$

This gives

$$\boxed{g(n,k+1) - g(n,k) = c_0(n)f(n,k) + c_1(n)f(n+1,k) + c_2(n)f(n+2,k)}$$

with

$$c_0(n) := 4\,(1+n),\quad c_1(n) := -2\,(3+2\,n),\quad c_2(n) := 2+n,$$

$$g(n,k) := \frac{(1+n)\,(-2+k-n+(2\,k-2\,k^2+k\,n)S_1(k))\,\binom{n}{k})}{(1-k+n)\,(2-k+n)}.$$

REPRESENT $f(n+i,k)$ in $\mathbb{F}$:

$$f(n,k) = S_1(k)\binom{n}{k} \longleftrightarrow h\,b =: f_0$$

$$f(n+1,k) = \frac{(n+1)\,S_1(k)\,\binom{n}{k}}{n+1-k} \longleftrightarrow \frac{(n+1)\,h\,b}{n+1-k} =: f_1$$

$$f(n+2,k) = \frac{(n+1)\,(n+2)\,S_1(k)\,\binom{n}{k}}{(n+1-k)\,(n+2-k)} \longleftrightarrow \frac{(n+1)\,(n+2)\,h\,b}{(n+1-k)\,(n+2-k)} =: f_2.$$

Summing over k from 0 to n gives

$$\boxed{1 = 4(1+n)S(n) - 2(3+2n)S(n+1) + (2+n)S(n+2)}$$

for

$$S(n) = \sum_{k=0}^{n}\binom{n}{k}S_1(k).$$

Algorithmic summation paradigms

GIVEN a definite sum

$$S(n) := \sum_{k=0}^{n} f(n, k).$$

FIND a recurrence for $S(n)$ in n (creative extension of telescoping).

Algorithmic summation paradigms

GIVEN a definite sum

$$S(n) := \sum_{k=0}^{n} f(n, k).$$

FIND a recurrence for $S(n)$ in n (creative extension of telescoping).

$$\downarrow$$

GIVEN a recurrence

$$a_d(n)S(n+d) + \cdots + a_1(n)S(n+1) + a_0(n)S(n) = h(n).$$

FIND all solutions in $\Pi\Sigma$-extensions (d'Alembertian solutions).

Algorithmic summation paradigms

GIVEN a definite sum

$$S(n) := \sum_{k=0}^{n} f(n,k).$$

FIND a recurrence for $S(n)$ in n (creative extension of telescoping).

$$\downarrow$$

GIVEN a recurrence

$$a_d(n)S(n+d) + \cdots + a_1(n)S(n+1) + a_0(n)S(n) = h(n).$$

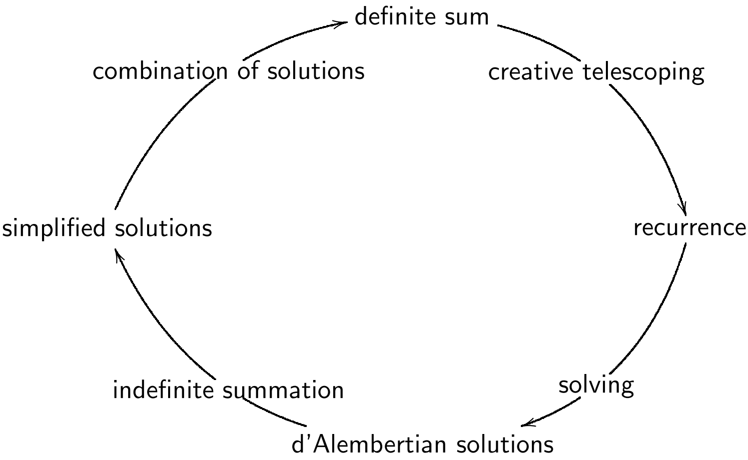
FIND all solutions in $\Pi\Sigma$-extensions (d'Alembertian solutions).

$$\downarrow$$

FIND an alternative representation:

$$S(n) = \text{combined solutions.}$$

The Sigma-summation spiral:



Examples