STRUCTURAL THEOREMS FOR SYMBOLIC SUMMATION

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ABSTRACT. Starting with Karr's structural theorem for summation —the discrete version of Liouville's structural theorem for integration— we work out crucial properties of the underlying difference fields. This leads to new and constructive structural theorems for symbolic summation. E.g., these results can be applied for harmonic sums which arise frequently in particle physics.

1. INTRODUCTION

In [21, 22] M. Karr developed a summation algorithm in which indefinite nested sums and products can be simplified. More precisely, such expressions are rephrased in a $\Pi\Sigma$-field $\mathbb{F}$, a very general class of difference fields[1], and first order linear difference equations defined over $\mathbb{F}$ are solved by Karr's algorithm. In this way, one can decide constructively, if a given indefinite sum or product with a summand or multiplicand f from $\mathbb{F}$ can be expressed in terms of $\mathbb{F}$. For instance, given $\mathbb{F} = \mathbb{Q}(k)(S_1(k), S_2(k), S_3(k))$ where $S_r(k) = \sum_{i=1}^{k} \frac{1}{i^r}$ denotes the generalized harmonic numbers of order $r \geq 1$ and given

$$f(k) = \frac{\left(S_2(k)(k+1)^2+1\right) S_3(k) + S_1(k)((k+1)S_3(k) - S_2(k))}{S_3(k)\left(S_3(k)(k+1)^3+1\right)} \in \mathbb{F},$$

Karr's algorithm decides constructively if there is an antidifference $g \in \mathbb{F}$ for f, i.e.,

$$g(k+1) - g(k) = f(k); \tag{1}$$

In our concrete example, the algorithm produces the solution $g(k) = \frac{S_1(k)S_2(k)}{S_3(k)}$. Then summing the telescoping equation (1) over k leads to the simplification

$$\sum_{i=1}^{k} f(i) = \frac{S_2(k)(k+1)^2 + S_1(k)\left(S_2(k)(k+1)^3 + k + 1\right) + 1}{S_3(k)(k+1)^3+1} - 1 \in \mathbb{F}.$$

This framework and extensions [42, 43, 44, 23, 48, 24, 45, 25] generalize, e.g., the (q–)hypergeometric algorithms presented in [1, 18, 54, 34, 32, 35, 33, 5, 20, 3], they cover as special case the summation of (q–)harmonic sums [10, 51, 29, 11] arising, e.g., in particle physics, and they can treat classes of multi-sums that are out of scope of, e.g., the holonomic approach [53, 52, 15, 14].

Karr's algorithm can be considered as the discrete analogue of Risch's algorithm [36, 37] for indefinite integration. Here the essential building blocks of exponentials and logarithms can be expressed in terms of an elementary differential field $\mathbb{F}$, and Risch's algorithm can decide constructively, if for a given $f \in \mathbb{F}$ there exists an antiderivative $g \in \mathbb{F}$, i.e.,

$$D(g) = f; \tag{2}$$

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[1] Throughout this article all fields contain the rational numbers $\mathbb{Q}$ as subfield.

here D denotes the differential operator acting on the elements of $\mathbb{F}$. In this regard, Liouville's theorem of integration, see e.g. [28, 31, 38], plays an important role. In a nutshell, it states that for integration with elementary functions it suffices to restrict to logarithmic extensions, i.e., one can neglect exponential and algebraic function extensions; for an explicit formulation we refer to Section 2.1. In particular, Risch's algorithm provides a constructive version of Liouville's theorem: his algorithm finds such an extension in terms of logarithms for a given input integral, or it outputs that there does not exist such an extension in which the integral is expressible.

Inspired by Rosenlicht's algebraic proof [38] of Liouville's Theorem, Karr could derive a structural theorem for symbolic summation [21, 22]. To be more precise, he refined his $\Pi\Sigma$-difference field theory to the so-called reduced and normalized $\Pi\Sigma$-fields in which a discrete version of Liouville's theorem is applicable. For instance, given $\mathbb{F}$ from above and given $f(k) \in \mathbb{Q}(k)$, any solution $g(k) \in \mathbb{F}$ of (1) has the form

$$g(k) = w(k) + c_1 S_1(k) + c_2 S_2(k) + c_3 S_3(k) \tag{3}$$

for some $w(k) \in \mathbb{Q}(k)$ and $c_1, c_2, c_3 \in \mathbb{Q}$.

In the following we consider $\Pi\Sigma^*$-extensions and $\Pi\Sigma^*$-fields being slightly less general than Karr's $\Pi\Sigma$-fields [21], but covering all sums and products treated explicitly by Karr's work. For such $\Pi\Sigma^*$-extensions we shall be able to make Karr's structural theorem constructive: based on the algorithm given in [40] we show that any $\Pi\Sigma^*$-field can be transformed to a reduced $\Pi\Sigma^*$-field in which Karr's structural theorem can be applied. In addition, we complement Karr's structural results by taking into account the nested depth of the recursively defined $\Pi\Sigma^*$-extensions: we present in detail how Karr's reduced $\Pi\Sigma^*$-extensions can be used to simplify the nested depth of a given sum expression. Finally, we relate these results with the difference field theory of depth-optimal $\Pi\Sigma^*$-fields that have been introduced recently [41, 47, 49]. Comparing Karr's approach and depth-optimal $\Pi\Sigma^*$-extensions we obtain additional insight in $\Pi\Sigma$-difference field theory and we derive new structural theorems that contribute in the field of symbolic summation.

We stress that the suggested results and the underlying algorithms implemented in the summation package Sigma [46] play an important role in the simplification of d'Alembertian solutions [30, 2, 39], a subclass of Liouvillian solutions [19] of a given recurrence relation. In this regard, special emphasize is put on the simplification of harmonic sum expressions that arise frequently in particle physics; we refer to [6, 7, 8] for typical examples in the frame of difference fields.

The general structure of this article is as follows. In Section 2 we state Liouville's structural theorem, and we relate it to Karr's results in terms of reduced $\Pi\Sigma^*$-fields. In Section 3 we work out the crucial properties of reduced $\Pi\Sigma^*$-extensions, and in Section 4 we show that any $\Pi\Sigma^*$-field can be transformed algorithmically to a reduced $\Pi\Sigma^*$-field. In Section 5 reduced extensions are refined to complete-reduced extensions. In Section 6 we focus on structural theorems that bound the nested depth of a telescoping solution; it turns out that this is only possible if the reduced extensions are built up in a particular ordered way. Finally, in Section 7 we relate depth-optimal $\Pi\Sigma^*$-extensions to reduced and complete-reduced $\Pi\Sigma^*$-extensions. We present structural theorems that are independent of the order of the explicitly given tower of extensions.

2. Liouville's and Karr's structural theorems

We start with a short outline of Liouville's theorem for differential fields and relate it to Karr's achievements for the discrete analogue of difference fields.

2.1. **An outline of Liouville's Theorem.** Let $(\mathbb{F}, D)$ be a *differential field*, i.e., $\mathbb{F}$ is a field with a function $D\colon \mathbb{F} \to \mathbb{F}$ such that $D(a+b) = D(a) + D(b)$ and $D(a\,b) = D(a)b + aD(b)$ for all $a, b \in \mathbb{F}$; D is also called *differential operator*. The set of constants is defined by $\mathrm{const}_D\mathbb{F} = \{c \in \mathbb{F} | D(c) = 0\}$; note that $\mathrm{const}_D\mathbb{F}$ (also called constant field) forms a subfield of $\mathbb{F}$ which contains $\mathbb{Q}$. A differential field $(\mathbb{E}, \tilde{D})$ is called a *differential field extension* of a differential field $(\mathbb{F}, D)$ if $\mathbb{F}$ is a subfield of $\mathbb{E}$ and $\tilde{D}(a) = D(a)$ for all $a \in \mathbb{F}$; subsequently, we do not distinguish anymore between D and $\tilde{D}$. Finally, a differential field extension $(\mathbb{F}(t), D)$ of $(\mathbb{F}, D)$ is called *elementary*, see, e.g., [12, Def. 5.1.3] if t is algebraic over $\mathbb{F}$ or if t is transcendental over $\mathbb{F}$ and

(1) $D(t) = D(b)/b$ for some $b \in \mathbb{F}^*$ (a *logarithm*)
(2) $D(t)/t = D(b)$ for some $b \in \mathbb{F}$ (an *exponential*).

In addition, an extension $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ of $(\mathbb{F}, \sigma)$ is called *elementary*, if it is a tower of elementary extensions. Then Liouvillian's Theorem reads as follows.

Theorem 1. *[Liouville's Theorem] Let $(\mathbb{E}, D)$ be an elementary extension of $(\mathbb{F}, D)$ with $\mathrm{const}_D\mathbb{E} = \mathrm{const}_D\mathbb{F}$, and let $f \in \mathbb{F}$. If there is a $g \in \mathbb{E}$ with* (2)*, then there are $w \in \mathbb{F}$, $c_1, \dots, c_n \in \mathrm{const}_D\mathbb{F}$ and $f_1 \dots, f_n \in \mathbb{F}^*$ such that*

$$f = D(w) + \sum_{i=1}^{n} c_i \frac{D(f_i)}{f_i}.$$

In other words, it suffices to search for a solution g with (2) in logarithmic extensions, and one can neglect algebraic or exponential extensions.

Remark **2.** Liouville's Theorem has been observed already by Laplace [27, p.7] — but the first precise formulation together with a proof based on analytic arguments has been given by Liouville [28]. In particular, the first algebraic proof in terms of differential fields has been provided by [31]; a complete proof dealing also with algebraic extensions has been accomplished by Rosenlicht [38]. For an extensive list of literature and generalizations/refinements, like e.g. [50], we refer to [12].

To this end, Risch's algorithm [36, 37] can be considered as a constructive breakthrough of Liouville's structure theorem. For instance, let $(\mathbb{F}, D)$ be a differential field with $\mathbb{K} = \mathrm{const}_D\mathbb{F}$ given by a tower of elementary transcendental extensions over the differential field $(\mathbb{K}(x), D)$ with $D(x) = 1$. Then Risch's algorithm can decide in a finite number of steps, if for a given $f \in \mathbb{F}$ there exists a tower of elementary transcendental extensions $(\mathbb{F}(t_1)\dots(t_e), D)$ of $(\mathbb{F}, D)$ in which we have g with (2); in particular, if such an extension exists, it computes such w, f_i and c_i as given in Theorem 1. For a detailed description of this algorithm see [12].

2.2. **Karr's Summation theorems.** M. Karr [21, 22] developed a theory of $\Pi\Sigma$-difference fields which can be considered as the discrete version of elementary transcendental extensions (whose constant fields remain unchanged). In this context we need the following definitions. Let $(\mathbb{F}, \sigma)$ be a *difference field*, i.e., $\mathbb{F}$ is a field and $\sigma\colon \mathbb{F} \to \mathbb{F}$ is a field automorphism, and define the set of constants by $\mathrm{const}_\sigma\mathbb{F} := \{c \in \mathbb{F} | \sigma(c) = c\}$; as in the differential case, $\mathrm{const}_\sigma\mathbb{F}$ forms a subfield of $\mathbb{F}$ which contains $\mathbb{Q}$; $\text{const}_\sigma\mathbb{F}$ is also called the *constant field* of $(\mathbb{F},\sigma)$. In such a difference field we define the *forward difference operator* as follows: for $a \in \mathbb{F}$,

$$\Delta(a) := \sigma(a) - a.$$

A difference field $(\mathbb{E},\tilde{\sigma})$ is a *difference field extension* of a difference field $(\mathbb{F},\sigma)$ if $\mathbb{F}$ is a subfield of $\mathbb{E}$ and $\tilde{\sigma}(a) = \sigma(a)$ for all $a \in \mathbb{F}$; subsequently, we do not distinguish between σ and $\tilde{\sigma}$ anymore.

In the following we introduce $\Pi\Sigma^*$-extensions being slightly less general than Karr's $\Pi\Sigma$-fields [21], but covering all sums and products treated explicitly by Karr's work. A difference field extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ is a $\Pi\Sigma^*$*-extension* if t is transcendental over $\mathbb{F}$, $\text{const}_\sigma\mathbb{F}(t) = \text{const}_\sigma\mathbb{F}$ and one of the following holds:

(1) $\Delta(t) = b$ for some $b \in \mathbb{F}^*$ (a Σ^**-extension*)
(2) $\sigma(t)/t = b$ for some $b \in \mathbb{F}^*$ (a Π*-extension*).

$(\mathbb{F}(t_1)\dots(t_e),\sigma)$ is a $\Pi\Sigma^*$*-extension* (resp. Σ^*-extension, Π-extension) of $(\mathbb{F},\sigma)$ if it is a tower of such extensions (this implies that $\text{const}_\sigma\mathbb{F}(t_1)\dots(t_e) = \text{const}_\sigma\mathbb{F}$). A difference field $(\mathbb{K}(t_1)\dots(t_e),\sigma)$ is a $\Pi\Sigma^*$*-field over* $\mathbb{K}$ if $(\mathbb{K}(t_1)\dots(t_e),\sigma)$ is a $\Pi\Sigma^*$-extension of $(\mathbb{K},\sigma)$ and $\text{const}_\sigma\mathbb{K} = \mathbb{K}$.

Example **3.** We rephrase $\mathbb{Q}(k)(S_1(k), S_2(k), S_3(k))$ from Section 1 in terms of a $\Pi\Sigma^*$-field $(\mathbb{F},\sigma)$ as follows. Consider the difference field $(\mathbb{Q},\sigma)$ with $\sigma(q) = q$ for all $q \in \mathbb{Q}$, i.e., $\text{const}_\sigma\mathbb{Q} = \mathbb{Q}$. Now take the rational function field $\mathbb{Q}(k)$ and extend the field automorphism σ to $\sigma\colon \mathbb{Q}(k) \to \mathbb{Q}(k)$ by $\sigma(k) = k + 1$; note that σ is uniquely determined in this way. Since $\text{const}_\sigma\mathbb{Q}(k) = \text{const}_\sigma\mathbb{Q} = \mathbb{Q}$, $(\mathbb{Q}(k),\sigma)$ forms a Σ^*-extension of $(\mathbb{Q},\sigma)$. Similarly, we can define (uniquely) the difference field extension $(\mathbb{Q}(k)(s_1),\sigma)$ of $(\mathbb{Q}(k),\sigma)$ such that s_1 is transcendental and $\sigma(s_1) = s_1 + \frac{1}{k+1}$. Again, since $(\mathbb{Q}(k)(s_1),\sigma) = \mathbb{Q}$, this forms a Σ^*-extension. We can continue in this way and obtain the Σ^*-extension $(\mathbb{F},\sigma)$ of $(\mathbb{Q},\sigma)$ with the rational function field $\mathbb{F} = \mathbb{Q}(k)(s_1)(s_2)(s_3)$ and with the field automorphism $\sigma\colon \mathbb{F} \to \mathbb{F}$ uniquely defined by

$$\sigma(k) = k+1, \quad \sigma(s_1) = s_1 + \tfrac{1}{k+1}, \quad \sigma(s_2) = s_2 + \tfrac{1}{(k+1)^2}, \quad \sigma(s_3) = s_3 + \tfrac{1}{(k+1)^3}; \tag{4}$$

since $\text{const}_\sigma\mathbb{F} = \mathbb{Q}$, $(\mathbb{F},\sigma)$ is a $\Pi\Sigma^*$-field over $\mathbb{Q}$.

Remark **4.** Note that, e.g., $\log(x)$ with $D\log(x) = \frac{1}{x}$ and the harmonic numbers $S_1(k) = \sum_{i=1}^k \frac{1}{i}$ with $\Delta(S_1(k)) = \frac{1}{k+1}$ are closely related; in particular $\lim_{k\to\infty}(H_k - log(k)) = \gamma$ where $\gamma = 0.5772...$ denotes Euler's constant. Similarities between elementary unimonomial extensions and $\Pi\Sigma^*$-extensions in the algebraic setting of difference/differential fields are worked out, e.g., in [13].

As it turns out, the discrete version of Liouville's structural theorem in the context of $\Pi\Sigma^*$-extensions can be stated in the following surprisingly simple form: a sum of $f \in \mathbb{F}$ is either expressible in $\mathbb{F}$ or it can be represented by one Σ^*-extension; in particular, one can neglect Π-extensions. This follows by the following result.

Theorem 5 ([21])**.** *Let $(\mathbb{F}(t),\sigma)$ be an extension of $(\mathbb{F},\sigma)$ with $\sigma(t) = t+f$ for some $f \in \mathbb{F}$. Then this is a Σ^*-extension iff there is no $g \in \mathbb{F}$ such that $\sigma(g) = g + f$.*

Namely, let $(\mathbb{F},\sigma)$ be a difference field with $f \in \mathbb{F}$. Then either there exists a solution[2] $g \in \mathbb{F}$ of the telescoping equation

$$\Delta(g) = f; \tag{5}$$

[2]Note that the telescoping problem (1) is rephrased in the algebraic setting of difference fields.

or if not, there is the Σ^*-extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ with $\sigma(t) = t+f$ by Theorem 5, i.e., t forms a solution of (5).

Similarly to Risch, Karr developed an algorithm in [21] which makes these observations constructive. Given a $\Pi\Sigma^*$-field $(\mathbb{F}, \sigma)$ over $\mathbb{K}$ and given $f \in \mathbb{F}$, decide in finitely many steps if there exists a $g \in \mathbb{F}$ such that (5); if yes, output such a g.

Example **6.** We start with the $\Pi\Sigma^*$-field $(\mathbb{F}, \sigma)$ from Example 3. Now take $f = \sigma(\frac{s_3}{k}) \in \mathbb{F}$. Using, e.g., Karr's algorithm, or the simplified version [44] implemented in the summation package Sigma, one can check that there is no $g \in \mathbb{F}$ such that (5). Hence we can construct the Σ^*-extension $(\mathbb{F}(s_{1,3}), \sigma)$ of $(\mathbb{F}, \sigma)$ with $\sigma(s_{1,3}) = s_{1,3} + f$. Completely analogously, we can construct the difference field extension $(\mathbb{F}(s_{1,3})(e)(s_{6,1,3}), \sigma)$ of $(\mathbb{F}, \sigma)$ with the rational function field $\mathbb{F}(s_{1,3})(e)(s_{6,1,3})$ and with

$$\sigma(s_{1,3}) = s_{1,3} + \frac{\sigma(s_3)}{k+1},\ \sigma(e) = e + \frac{\sigma(s_2)\,\sigma(s_3)}{k+1},\ \sigma(s_{6,1,3}) = s_{6,1,3} + \frac{\sigma(s_{1,3})}{(k+1)^6}; \quad (6)$$

in particular, we can check algorithmically that this extension forms a tower of Σ^*-extensions by verifying iteratively the non-existence of solutions of the corresponding telescoping problems. Note also that one can verify by the same mechanism that the base field $(\mathbb{F}, \sigma)$ constructed in Ex. 3 forms a $\Pi\Sigma^*$-field over $\mathbb{Q}$.

Remark **7.** The extensions s_3, $s_{1,3}$ and $s_{6,1,3}$ in Example 6 represent the harmonic sums $S_3(k)$, $S_{1,3}(k)$ and $S_{6,1,3}(k)$, respectively, which are defined as follows [10, 51]: for positive integers $m_1, \dots, m_r \in \mathbb{N} \setminus \{0\}$,

$$S_{m_1,\dots,m_r}(k) = \sum_{i_1=1}^{k} \frac{1}{i_1^{m_1}} \sum_{i_2=1}^{i_1} \frac{1}{i_2^{m_2}} \cdots \sum_{i_r=1}^{i_{r-1}} \frac{1}{i_r^{m_r}};$$

e.g., the shift $S_{1,3}(k+1) = S_{1,3}(k) + \frac{S_3(k+1)}{k+1}$ is reflected by $\sigma(s_{1,3}) = s_{1,3} + \frac{\sigma(s_3)}{k}$. In this way, also the truncated Euler sum [17] $\sum_{i=1}^{k} \frac{S_2(i)S_3(i)}{i}$ is rephrased by e. Similarly, q–analogues of harmonic sums [4, 16, 11] can be formulated in $\Pi\Sigma^*$-fields.

2.3. **Karr's Structural theorem.** In [21, 22] Karr arrives at the following conclusion: one can predict the structure of a solution g for (5) in a refined version of $\Pi\Sigma$-fields; see [22, page 314]. For $\Pi\Sigma^*$-extensions this refinement reads as follows.

Definition **8.** A $\Pi\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ of $(\mathbb{F}, \sigma)$ is called *reduced over* $\mathbb{F}$ or in short *reduced* if for any Σ^*-extension t_i with $f := \Delta(t_i) \in \mathbb{F}(t_1)\dots(t_{i-1}) \setminus \mathbb{F}$ the following property holds: there do not exist a $g \in \mathbb{F}(t_1)\dots(t_{i-1})$ and an $f' \in \mathbb{F}$ such that

$$\Delta(g) + f' = f. \quad (7)$$

The following special case is immediate.

Lemma 9. *Let* $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ *be a* $\Pi\Sigma^*$*-extension of* $(\mathbb{F}, \sigma)$ *with* $\sigma(t_i) - t_i \in \mathbb{F}$ *or* $\sigma(t_i)/t_i \in \mathbb{F}$ *for* $1 \le i \le e$*. Then this extensions is reduced.*

In Section 4 we provide an algorithmic approach which enables one to check whether a $\Pi\Sigma^*$-extension is reduced. In particular, if this is not the case, this machinery automatically transforms the given extension to an isomorphic difference field which is built by a tower of reduced $\Pi\Sigma^*$-extensions; see Theorem 24. In other words, one can always apply the following structural theorem (in a given reduced $\Pi\Sigma^*$-extension or in an isomorphic extension which is reduced).

Theorem 10. *[Karr's structural theorem] Let $(\mathbb{E},\sigma)$ be a reduced $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with $\mathbb{E}=\mathbb{F}(t_1)\dots(t_e)$ and $\sigma(t_i)=a_i\,t_i+f_i$, and define*[3]

$$S:=\{1\leq i\leq e|\Delta(t_i)\in\mathbb{F}\};\tag{8}$$

let $f\in\mathbb{F}$. If there is a $g\in\mathbb{E}$ with (5), there are $w\in\mathbb{F}$ and $c_i\in\text{const}_\sigma\mathbb{F}$ s.t.

$$f=\Delta(w)+\sum_{i\in S}c_if_i;\tag{9}$$

in particular, for any such g there is some $c\in\text{const}_\sigma\mathbb{F}$ such that

$$g=c+w+\sum_{i\in S}c_it_i.\tag{10}$$

For a proof in the context of $\Pi\Sigma$-fields we refer the reader to [22, Result, page 315], and for the corresponding proof for reduced $\Pi\Sigma^*$-extensions as given in Theorem 10 we refer to [39, Thm 4.2.1]; the proofs follow Rosenlicht's proof strategy [38] of Liouville's Theorem.

We emphasize that Karr's result exceeds Liouville's theorem in the following sense: given a reduced $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ and given $f\in\mathbb{F}$ one can forecast to a certain extend how the solution $g\in\mathbb{E}$ is composed; for a typical application see e.g. page (4).

Example **11.** Consider the $\Pi\Sigma^*$-field $(\mathbb{F},\sigma)$ over $\mathbb{Q}$ with $\mathbb{F}=\mathbb{Q}(k)(s_1)(s_2)(s_3)$ and (4). Note that $(\mathbb{F},\sigma)$ is a reduced $\Pi\Sigma^*$-extension of $(\mathbb{Q}(k),\sigma)$; see Lemma 9. Hence by Theorem 10 any solution $g\in\mathbb{F}$ of (5) for a given $f\in\mathbb{Q}(k)$ is of the form

$$g=w+c_1\,s_1+c_2\,s_2+c_3\,s_3\qquad\text{for some }w\in\mathbb{Q}(k)\text{ and }c_1,c_2,c_3\in\mathbb{Q}.\tag{11}$$

for a precise formulation, how (3) and (11) are related we refer to [48, 49]

Example **12.** Start with the $\Pi\Sigma^*$-field $(\mathbb{F},\sigma)$ over $\mathbb{Q}$ from Example 11, and consider the Σ^*-extension $(\mathbb{F}(s_{1,3})(e)(s_{6,1,3}),\sigma)$ of $(\mathbb{F},\sigma)$ with (6) from Example 6; later we can check that this extension is reduced over $\mathbb{F}$; see Example 30. Hence for any $g\in\mathbb{F}(s_{1,3})(e)(s_{6,1,3})$ with (5) for some $f\in\mathbb{F}$ it follows that

$$g=w+c_1\,s_{1,3}+c_2\,e\qquad\text{for some }c_1,c_2\in\mathbb{Q}\text{ and }w\in\mathbb{F}.\tag{12}$$

Example **13.** Again, start with the $\Pi\Sigma^*$-field $(\mathbb{F},\sigma)$ over $\mathbb{Q}$ from Example 11, and consider the Σ^*-extension $(\mathbb{F}(s_{1,3})(e)(s_{2,1,3}),\sigma)$ of $(\mathbb{F},\sigma)$ with

$$\sigma(s_{1,3})=s_{1,3}+\frac{\sigma(s_3)}{k+1},\quad\sigma(e)=e+\frac{\sigma(s_2)\,\sigma(s_3)}{k+1},\quad s_{2,1,3}=s_{2,1,3}+\frac{\sigma(s_{1,3})}{(k+1)^2}.$$

In this instance, the extension is not reduced. E.g., for $f=\frac{(k+1)^5+1}{(k+1)^6}$ there is

$$g=-s_3^2+2e+s_1-2s_{1,3}s_2+2\,s_{2,1,3}\tag{13}$$

s.t. (5): if this extension were reduced, g should be free of $s_{2,1,3}$ and g should contain $s_{1,3}$ only in the form $c\,s_{1,3}$ for some $c\in\mathbb{Q}$ by Theorem 10.

Remark **14.** Reinterpreting the variables in f and g of the previous example as harmonic sums and summing (1) over k lead to the following identity: for $k\geq 0$,

$$\sum_{i=1}^{k}\frac{i^5+1}{i^6}=-S_3(k)^2+2\sum_{i=1}^{k}\frac{S_2(i)S_3(i)}{i}+S_1(k)-2S_{1,3}(k)S_2(k)+2\,S_{2,1,3}(k).$$

[3]Note that S consists of exactly those i such that t_i with $f_i=\Delta(t_i)\in\mathbb{F}$ is a Σ^*-extension.

Obviously, the obtained right hand side is more complicated (i.e., consists of sums with higher nesting depth) than the given left hand side. In Sections 6 and 7 we work out in details why this is the case in general; for our particular case see Ex. 36.

2.4. **A simple structure theorem for $\Pi\Sigma^*$-extensions.** We conclude this section with the following simple "structural theorem" which is valid for any $\Pi\Sigma^*$-extension. Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with the rational function field $\mathbb{E}:=\mathbb{F}(t_1)\dots(t_e)$ and $\sigma(t_i)=a_i\,t_i$ or $\sigma(t_i)=t_i+a_i$ for $1\leq i\leq e$; let $f\in\mathbb{E}$. Then we define the set of leaf extensions which are free of f by

$$\mathrm{Leaf}_{\mathbb{F}\leq\mathbb{E}}(f):=\{t_i|t_i\text{ does not occur in }f\text{ and }a_{i+1},\dots,a_e\},$$

and we define the set of inner node extensions or extensions that occur in f by

$$\mathrm{InnerNode}_{\mathbb{F}\leq\mathbb{E}}(f):=\{t_1,\dots,t_e\}\setminus\mathrm{Leaf}_{\mathbb{F}\leq\mathbb{E}}(f);$$

those extensions which are Σ^*-extensions are denoted by

$$\mathrm{SumLeaf}_{\mathbb{F}\leq\mathbb{E}}(f):=\{t\in\mathrm{Leaf}_{\mathbb{F}\leq\mathbb{E}}(f)|t\text{ is a }\Sigma^*\text{-extension}\}.$$

We denote all Σ^*-extensions being leafs by $\mathrm{SumLeaf}_{\mathbb{F}\leq\mathbb{E}}:=\mathrm{SumLeaf}_{\mathbb{F}\leq\mathbb{E}}(1)$.
At this point the following remark is in place. If there is a permutation $\tau\in S_e$ such that $a_{\tau(i)}\in\mathbb{F}(t_{\tau(1)})\dots(t_{\tau(i-1)})$ for all i with $1\leq i\leq e$, then $(\mathbb{F}(t_{\tau(1)})\dots(t_{\tau(e)}),\sigma)$ forms again a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$. In particular, one can reorder the $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ with $f\in\mathbb{E}$ to $(\mathbb{F}(x_1)\dots(x_r)(y_1)\dots(y_s),\sigma)$ such that

$$\mathrm{InnerNode}_{\mathbb{F}\leq\mathbb{E}}(f)=\{x_1,\dots,x_r\}\tag{14}$$

and $\mathrm{Leaf}_{\mathbb{F}\leq\mathbb{E}}(f)=\{y_1,\dots,y_s\}$; note that $\sigma(y_i)/y_i\in\mathbb{F}(x_1)\dots(x_r)$ or $\sigma(y_i)-y_i\in\mathbb{F}(x_1)\dots(x_r)$ for $1\leq i\leq s$. Hence by Lemma 9 $(\mathbb{F}(x_1)\dots(x_r)(y_1)\dots(y_s),\sigma)$ is a reduced $\Pi\Sigma^*$-extension of $(\mathbb{F}(x_1)\dots(x_r),\sigma)$. Thus we can apply Theorem 10, and we arrive at the following result.

Theorem 15. *Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with $f\in\mathbb{E}$ and define $\{x_1,\dots,x_r\}$ by (14). If there is a $g\in\mathbb{E}$ such that (5), then*

$$g=\sum_{a\in SumLeaf_{\mathbb{F}\leq\mathbb{E}}(f)}c_a a+w\quad\text{for some }c_a\in\mathrm{const}_\sigma\mathbb{F}\text{ and }w\in\mathbb{F}(x_1,\dots,x_r).$$

Example **16.** Consider the $\Pi\Sigma^*$-field $(\mathbb{E},\sigma)$ over $\mathbb{Q}$ from Example 13 with $\mathbb{E}=\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(e)(s_{2,1,3})$, and have a look at the solution (13) of (5) for $f=\frac{(k+1)^5+1}{(k+1)^6}$. Then, as predicted in Theorem 15, the solution (13) is given by a linear combination over $\mathbb{Q}$ in terms of the variables $\mathrm{SumLeaf}_{\mathbb{Q}\leq\mathbb{E}}(f)=\{s_1,e,s_{2,1,3}\}$ plus one expression from $\mathbb{Q}(k,s_2,s_3,s_{1,3})$.

Combining Theorem 20 with Theorem 15 we arrive at

Theorem 17 (A refinement of Karr's structural theorem)**.** *Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$, let $(\mathbb{E}(t_1)\dots(t_e),\sigma)$ be a reduced $\Pi\Sigma^*$-extension of $(\mathbb{E},\sigma)$ and let $f\in\mathbb{E}$. Define $S=\{1\leq i\leq e|\Delta(t_i)\in\mathbb{E}\}=\{i_1,\dots,i_u\}$ and consider the Σ^*-extension $(\mathbb{H},\sigma)$ of $(\mathbb{E},\sigma)$ with $\mathbb{H}=\mathbb{E}(t_{i_1})\dots(t_{i_u})$; define $\{x_1,\dots,x_r\}:=$ $InnerNode_{\mathbb{F}\leq\mathbb{H}}(f)$. If there is a $g\in\mathbb{E}(t_1,\dots,t_e)$ such that (5), then*[4]

$$g=\sum_{a\in SumLeaf_{\mathbb{F}\leq\mathbb{H}}(f)}c_a a+w\qquad\text{for some }c_a\in\mathrm{const}_\sigma\mathbb{F}\text{ and }w\in\mathbb{F}(x_1,\dots,x_r).$$

[4]Note that $S\subseteq\mathrm{SumLeaf}_{\mathbb{E}\leq\mathbb{H}}(f)$, i.e., Theorem 17 refines Theorem 20.

Example **18.** Consider the $\Pi\Sigma^*$-field $(\mathbb{F}(s_{1,3})(e)(s_{6,1,3}),\sigma)$ from Example 12, and let

$$f=\frac{k^3+3k^2+3k-s_2-(k+1)(k(k+2)(s_2-4)+s_2-5)s_3+5}{(k+1)^4}\in\mathbb{F}.$$

Following Theorem 17, we define $S=\{s_{1,3},e\}$, and we get $\text{SumLeaf}_{\mathbb{Q}\leq\mathbb{F}(s_{1,3})(e)}(f)=\{s_1,s_{1,3},e\}$ and $\text{InnerNode}_{\mathbb{Q}\leq\mathbb{F}(s_{1,3})(e)}(f)=\{k,s_2,s_3\}$. Hence, if there is a $g\in\mathbb{F}(s_{1,3})(e)(s_{6,1,3})$ such that (5), then $g=w+c_1\,s_1+c_2\,s_{1,3}+c_3\,e$ for some $c_1,c_2,c_3\in\mathbb{Q}$ and $w\in\mathbb{Q}(k,s_2,s_3)$. Note that our prediction refines the version given in (12). Indeed, we find $g=s_3^2+s_1+4s_{1,3}-e$.

3. Equivalent characterizations of reduced $\Pi\Sigma^*$-extensions

We work out alternative characterizations whether a $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ is reduced. Here we need the following lemma.

Lemma 19. *Let $(\mathbb{F}(t),\sigma)$ be a Σ^*-extension of $(\mathbb{F},\sigma)$ with $\sigma(t)=t+f$ and $\mathbb{K}:=\text{const}_\sigma\mathbb{F}$, and let $f'\in\mathbb{F}$. Then there are $c\in\mathbb{K}$ and $g\in\mathbb{F}$ such that*

$$\Delta(g)+c\,f'=f \tag{15}$$

iff there is a Σ^*-extension $(\mathbb{F}(s),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(s)=s+f'$ in which we find $g\in\mathbb{F}(s)$ such that (5)*.*

Proof. Suppose that there are a $g\in\mathbb{F}$ and $c\in\mathbb{K}$ such that (15), and assume in addition that there is a $g'\in\mathbb{F}$ such that $\Delta(g')=f'$. Then $\Delta(q)=f$ with $q:=g+c\,g'\in\mathbb{F}$, a contradiction that $(\mathbb{F}(t),\sigma)$ is a Σ^*-extension of $(\mathbb{F},\sigma)$ by Theorem 5. Hence $(\mathbb{F}(s),\sigma)$ is a Σ^*-extension of $(\mathbb{F},\sigma)$ by Theorem 5. Besides this, for $h:=g+c\,s$ we have $\Delta(h)=\Delta(g)+c\,f'=f$.
Conversely, suppose that there is a Σ^*-extension $(\mathbb{F}(s),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(s)=s+f'$ together with a $g\in\mathbb{F}(s)$ as in (5). By Theorem 10, $g=c\,s+w$ for some $w\in\mathbb{F}$ and $c\in\mathbb{K}$. Thus, $f=\Delta(g)=w+c\,f'$. ◻

Theorem 20. *Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with $\mathbb{E}=\mathbb{F}(t_1)\dots(t_e)$ and define S as in* (8)*. Then the following statements are equivalent.*

(1) This extension is reduced.
(2) For any $g\in\mathbb{E}$ with $\Delta(g)\in\mathbb{F}$ we have (10) *for some $c_i\in\text{const}_\sigma\mathbb{F}$ and $w\in\mathbb{F}$.*
(3) For any Σ^*-extension t_i with $f:=\Delta(t_i)$ and $i\notin S$ the following property holds: There does not exist a Σ^*-extension $(\mathbb{F}(t_1)\dots(t_{i-1})(s),\sigma)$ of $(\mathbb{F}(t_1)\dots(t_{i-1}),\sigma)$ with $\Delta(s)\in\mathbb{F}$ in which we have g with (5)*.*

Proof. **(1)** $\Rightarrow$ **(2)** follows by Theorem 10. Now suppose that $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ is not a reduced $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$. Then there is an i with $1\leq i\leq e$ such that $f:=\Delta(t_i)\in\mathbb{F}(t_1,\dots,t_{i-1})\setminus\mathbb{F}$ and (7) for some $f'\in\mathbb{F}$ and $g\in\mathbb{F}(t_1)\dots(t_{i-1})$. Hence, we obtain $\Delta(g')=f'$ with $g':=t_i-g$, and thus **(2)** does not hold. This proves the equivalence of **(1)** and **(2)**. Equivalence **(1)** $\Leftrightarrow$ **(3)** is an immediate consequence of Lemma 19. ◻

Example **21.** Consider the $\Pi\Sigma^*$-extension $(\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(e)(s_{2,1,3}),\sigma)$ of $(\mathbb{Q}(k)(s_1)(s_2)(s_3),\sigma)$ from Ex. 13 which is not reduced. Thm. 20 explains why we can find, e.g., $f=\frac{(k+1)^5+1}{(k+1)^6}$ with (13) s.t. (5). Equivalently, we can take the Σ^*-extension $(\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(e)(s),\sigma)$ of $(\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(e),\sigma)$ with $\sigma(s)=s+f$ such that we get $\Delta(h)=f$ with $h=\frac{1}{2}(s+s_3^2-2e-s_1+2s_{1,3}s_2)$.

In summary, exactly reduced $\Pi\Sigma^*$-extension guarantee that Theorem 10 holds (equivalence (1)$\Leftrightarrow$(2)). In particular, it relates reduced $\Pi\Sigma^*$-extensions to certain refined Σ^*-extensions (equivalence (1)$\Leftrightarrow$(3)). This observation will be crucial to relate reduced $\Pi\Sigma^*$-extension to depth-optimal $\Pi\Sigma^*$-extension; see Section 7.

4. Constructive aspects of reduced $\Pi\Sigma^*$-extensions

In [21] it has been outlined that any $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ can be transformed in principle to a reduced version. Subsequently, we make this more precise in terms of difference field isomorphisms, and we show how such a transformation can be carried out algorithmically. As a consequence, one can always apply Karr's structural Theorem 10 constructively in the given extension or in the corresponding transformed one.

$\tau\colon \mathbb{F} \to \mathbb{F}'$ is called a *σ-isomorphism* (resp. *σ-monomorphism*) between two difference fields $(\mathbb{F},\sigma)$ and $(\mathbb{F}',\sigma')$ if τ is a field isomorphism (resp. field monomorphism) and $\tau(\sigma(f)) = \sigma'(\tau(f))$ for all $f \in \mathbb{F}$. In particular, let $(\mathbb{E},\sigma)$ and $(\mathbb{E}',\sigma')$ be difference field extensions of $(\mathbb{F},\sigma)$. Then a σ-isomorphism (resp. σ-monomorphism) $\tau\colon \mathbb{E} \to \mathbb{E}'$ is a an *$\mathbb{F}$-isomorphism* (resp. *$\mathbb{F}$-monomorphism*) if $\tau(a) = a$ for all $a \in \mathbb{F}$. We start with the following two lemmas.

Lemma 22. *Let $(\mathbb{F}(t),\sigma)$ be a Σ^*-extension of $(\mathbb{F},\sigma)$ with $\sigma(t) = t + f$, and let $f' \in \mathbb{F}$ and $g \in \mathbb{F}$ such that (7). Then there is a Σ^*-extension $(\mathbb{F}(s),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(s) = s+f'$ together with an $\mathbb{F}$-isomorphism $\tau\colon \mathbb{F}(t) \to \mathbb{F}(s)$ with $\tau(t) = s+g$.*

Proof. By Lemma 19 there is the Σ^*-ext. $(\mathbb{F}(s),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(s) = s+f'$. Take the field isomorphism $\tau\colon \mathbb{F}(t) \to \mathbb{F}(s)$ with $\tau(h) = h$ for all $h \in \mathbb{F}$ and $\tau(t) = s + g$. By $\tau(\sigma(t)) = \tau(t + f) = \tau(t) + f = s + g + f = s + c\,f' + \sigma(g) = \sigma(s + g) = \sigma(\tau(t))$ it follows that τ is an $\mathbb{F}$-isomorphism. □

Lemma 23. *[[47, Prop. 18]] Let $(\mathbb{F},\sigma)$, $(\mathbb{F}',\sigma')$ be difference fields with a σ-isomorphism $\tau\colon \mathbb{F} \to \mathbb{F}'$; let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma^*$-ext. of $(\mathbb{F},\sigma)$ with $\sigma(t) = \alpha\,t + \beta$. Then there is a $\Pi\Sigma^*$-extension $(\mathbb{F}'(t'),\sigma)$ of $(\mathbb{F}',\sigma)$ with $\sigma(t') = \tau(\alpha)t' + \tau(\beta)$ together with an σ-isomorphism $\tau'\colon \mathbb{E} \to \mathbb{E}'$ s.t. $\tau'|_{\mathbb{F}} = \tau$ and $\tau'(t) = t'$.*

By iterative applications of Lemmas 22 and 23 each $\Pi\Sigma^*$-extension can be transformed to an isomorphic reduced $\Pi\Sigma^*$-extension; see Theorem 24. In particular, this construction can be given explicitly if one can solve the following problem.

Problem RS (*R*educed *S*ummation): **Given** a $\Pi\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$, and given $f \in \mathbb{F}$; **find** $g \in \mathbb{F}(t_1)\dots(t_e)$ and $f' \in \mathbb{F}(t_1)\dots(t_i)$ as in (7) such that i with $0 \le i \le e$ is minimal.

In the following we call a difference field $(\mathbb{F},\sigma)$ *RS-computable*, if one can solve problem RS for any $\Pi\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ and for any $f \in \mathbb{F}$.

Theorem 24. *For any $\Pi\Sigma^*$-extension $(\mathbb{H},\sigma)$ of $(\mathbb{F},\sigma)$ there is a reduced $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ and an $\mathbb{F}$-isomorphism $\tau\colon \mathbb{H} \to \mathbb{E}$. Such a $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ and τ can be given explicitly, if $(\mathbb{F},\sigma)$ is RS-computable.*

Proof. The induction base is trivial. Suppose that we are given a $\Pi\Sigma^*$-extension $(\mathbb{H},\sigma)$ of $(\mathbb{F},\sigma)$ with $\mathbb{H} := \mathbb{F}(x_1)\dots(x_e)$ and a reduced $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ with $\mathbb{E} := \mathbb{F}(t_1)\dots(t_e)$ together with an $\mathbb{F}$-isomorphism $\tau\colon \mathbb{H} \to \mathbb{E}$. Now consider the $\Pi\Sigma^*$-extension $(\mathbb{H}(x),\sigma)$ of $(\mathbb{H},\sigma)$ with $\sigma(x) = \alpha\,x + \beta$, and take

Algorithm 1 $\texttt{ToReducedField}(\mathbb{F}(t_1)\dots(t_e), k)$

In: A $\Pi\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(t_i)=a_i\,t_i+b_i$ for $1\leq i\leq e$; $(\mathbb{F},\sigma)$ is RS-computable.
Out: A reduced $\Pi\Sigma^*$-extension $(\mathbb{F}(x_1)\dots(x_e),\sigma)$ of $(\mathbb{F},\sigma)$, and an $\mathbb{F}$-isomorphism $\tau\colon\mathbb{F}(t_1)\dots(t_e)\to\mathbb{F}(x_1)\dots(x_e)$.

```
Let τ: F → F be the identity map.
FOR i = 1 to e DO
    Set α := τ(a_i); f := τ(b_i); h := x_i.
    IF t_i is a Σ*-extension (a_i = α = 1) THEN
        Let f' ∈ F(x_1)...(x_j) \ F(x_1)...(x_{j-1}) and g ∈ F(x_1)...(x_{i-1})
        be the result of problem RS for f and (F(x_1)...(x_{i-1}), σ).
        IF j = 0, THEN Set f := f'; h := x_i + g FI
    FI
    Construct the ΠΣ*-extension (F(x_1)...(x_i), σ) of (F(x_1)...(x_{i-1}), σ) with
    σ(x_i) = α x_i + f; extend τ: F(t_1)...(t_{i-1}) → F(x_1)...(x_{i-1}) to the
    F-isomorphism τ: F(t_1)...(t_i) → F(x_1)...(x_i) by τ(t_i) = h. OD
RETURN ((F(x_1)...(x_e), σ), τ).
```

the $\Pi\Sigma^*$-extension $(\mathbb{E}(t),\sigma)$ of $(\mathbb{E},\sigma)$ with $\sigma(t)=\tau(\alpha)\,t+\tau(\beta)$ by Lemma 23; in particular, we can take the $\mathbb{F}$-isomorphism $\tau'\colon\mathbb{H}(x)\to\mathbb{E}(t)$ with $\tau(x)=t$ and $\tau'(h)=\tau(h)$ for all $h\in\mathbb{H}$. If $(\mathbb{E}(t),\sigma)$ is a reduced $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$, we are done. If not, $\alpha=1$, and for $f:=\tau(\beta)\in\mathbb{E}$ there are $g\in\mathbb{E}$ and $f'\in\mathbb{F}$ such that (7). Note: if $(\mathbb{F},\sigma)$ is RS-computable, we can solve problem RS, and we get such f' and g explicitly. Then by Lemma 22 there is a Σ^*-extension $(\mathbb{E}(t'),\sigma)$ of $(\mathbb{E},\sigma)$ with $\sigma(t')=t'+f'$ together with an $\mathbb{F}$-isomorphism $\tau''\colon\mathbb{E}(t)\to\mathbb{E}(t')$ with $\tau''(t)=c\,t'+g$ and $\tau''(h)=\tau'(h)$ for all $h\in\mathbb{E}$. Note that by construction $(\mathbb{E}(t),\sigma)$ is a reduced $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$, and $\rho:=\tau''\circ\tau'$ is an $\mathbb{F}$-isomorphism from $\mathbb{H}(x)$ to $\mathbb{E}(t')$. In particular, if $\tau\colon\mathbb{H}\to\mathbb{E}$ and g are given explicitly, also $\rho\colon\mathbb{H}(x)\to\mathbb{E}(t')$ can be given explicitly with $\rho(x)=t'+g$ and $\rho(h)=\tau(h)$ for all $h\in\mathbb{H}$. □

As a consequence, we obtain Alg. 1; the correctness follows by the proof of Theorem 24. From the applicational point of view we rely on the following algorithm [40, Algorithm 1]. Namely, due its generic specification, e.g., the following classes of difference fields $(\mathbb{F},\sigma)$ are RS-complete, i.e., Algorithm 1 can be executed in the summation package Sigma [46]: $(\mathbb{F},\sigma)$ is a $\Pi\Sigma^*$-field or it is $\Pi\Sigma^*$-extension over a free difference field [23] or over a difference field containing radicals [24], like $\sqrt{k}$.

Example **25.** Consider the $\Pi\Sigma^*$-field $(\mathbb{Q}(k)(s_1)(s_{1,1})(s_{1,1,1}),\sigma)$ over $\mathbb{Q}$ with

$$k=k+1,\ \sigma(s_1)=s_1+\tfrac{1}{k+1},\ \sigma(s_{1,1})=s_{1,1}+\tfrac{\sigma(s_1)}{k+1},\ \sigma(s_{1,1,1})=s_{1,1,1}+\tfrac{\sigma(s_{1,1})}{k+1}. \tag{16}$$

By Thm. 20 the extension is not reduced: we find, e.g., for $f=\frac{1}{(k+1)^3}$ the solution

$$g=s_1^3-3s_{1,1}\,s_1+3s_{1,1,1} \tag{17}$$

of (5). We transform this extension to a reduced one as follows.

(1) We start with the $\Pi\Sigma^*$-field $(\mathbb{Q}(k),\sigma)$ over $\mathbb{Q}$ with $\sigma(k)=k+1$ and take the $\mathbb{Q}$-isomorphism $\tau\colon\mathbb{Q}(k)\to\mathbb{Q}(k)$ with $\tau(f)=f$ for all $f\in\mathbb{Q}(k)$.
(2) Now we apply our algorithm for problem RS with $f=\frac{1}{k+1}$: since we do not find $f'\in\mathbb{Q}$ and $g\in\mathbb{Q}(k)$ (by executing the implementation of Sigma), it follows that $(\mathbb{Q}(k)(s_1),\sigma)$ is a reduced $\Pi\Sigma^*$-extension of $(\mathbb{Q}(k),\sigma)$. Hence we

keep $(\mathbb{Q}(k)(s_1),\sigma)$ and extend the $\mathbb{Q}$-isomorphism from $\mathbb{Q}(k)$ to $\tau\colon\mathbb{Q}(k)(s_1)\to\mathbb{Q}(k)(s_1)$ with $\tau(s_1)=s_1$, i.e., $\tau(h)=h$ for all $h\in\mathbb{Q}(k)(s_1)$.

(3) We apply our algorithm for problem RS for $f=\frac{\sigma(s_1)}{k+1}$ in $(\mathbb{Q}(k)(s_1),\sigma)$ and find $f'=\frac{-1}{2(k+1)^2}$ and $g=\frac{1}{2}s_1^2$. By Lemma 22 we can take the Σ^*-extension $(\mathbb{Q}(k)(s_1)(s_2),\sigma)$ of $(\mathbb{Q}(k)(s_1),\sigma)$ with $s_2=s_2+\frac{1}{(k+1)^2}$; by construction it follows that $(\mathbb{Q}(k)(s_1)(s_2),\sigma)$ is a reduced extension of $(\mathbb{Q}(k),\sigma)$. Moreover, we can extend the isomorphism τ to $\tau\colon\mathbb{Q}(k)(s_1)(s_{1,1})\to\mathbb{Q}(k)(s_1)(s_2)$ with

$$\tau(s_{1,1})=\frac{1}{2}\left(s_1^2+s_2\right). \tag{18}$$

(4) Finally, we solve RS for $f=\tau(\frac{\sigma(s_{1,1})}{k+1})$ in $(\mathbb{Q}(k)(s_1)(s_2),\sigma)$ and find $f'=\frac{1}{3(k+1)^3}$ and $g=\frac{1}{6}\left(s_1^3+3s_2s_1\right)$. Hence we can define the Σ^*-ext. $(\mathbb{Q}(k)(s_1)(s_2)(s_3),\sigma)$ of $(\mathbb{Q}(k)(s_1)(s_2),\sigma)$ with $\sigma(s_3)=s_3+\frac{1}{(k+1)^3}$; by construction, $(\mathbb{Q}(k)(s_1)(s_2)(s_3),\sigma)$ is a reduced extension of $(\mathbb{Q}(k),\sigma)$, and we can extend our $\mathbb{Q}$-isomorphism to $\tau\colon\mathbb{Q}(k)(s_1)(s_{1,1})(s_{1,1,1})\to\mathbb{Q}(k)(s_1)(s_2)(s_3)$ with

$$\tau(s_{1,1,1})=\frac{1}{6}\left(s_1^3+3s_2s_1+2s_3\right). \tag{19}$$

Note that $h=s_3$ is a solution of $\Delta(h)=\frac{1}{(k+1)^3}$ and consequently, $\tau^{-1}(h)$ (which is nothing else than (17)) is a solution of $\Delta(\tau^{-1}(h))=\tau^{-1}(\frac{1}{(k+1)^3})=\frac{1}{(k+1)^3}$.

Remark **26.** Reinterpreting $s_1,s_{1,1},s_{1,1,1}$ in Ex. 25 as harmonic sums leads to the following identities which are reflected by (18) and (19): for $k\in\mathbb{N}$,

$$S_{1,1}(k)=\frac{1}{2}\left(S_1(k)^2+S_2(k)\right),\quad S_{1,1,1}(k)=\frac{1}{6}\left(S_1(k)^3+3S_2(k)S_1(k)+2S_3(k)\right);$$

these identities occur, e.g., in [10] or in [16, Cor. 3] combined with [26, Prop. 2.1].

We remark that any $\mathbb{F}$-isomorphism is of this shape due to the following lemma; note that the product case is analogous, see [43, Prop. 4.4 and 4.8].

Lemma 27. *Let $(\mathbb{F}(t),\sigma)$ and $(\mathbb{F}(s),\sigma)$ be Σ^*-extensions of $(\mathbb{F},\sigma)$ with $\sigma(t)=t+f$ and $\sigma(s)=s+f'$, and let $\mathbb{K}:=\text{const}_\sigma\mathbb{F}$. If $\tau\colon\mathbb{F}(t)\to\mathbb{F}(s)$ is an $\mathbb{F}$-isomorphism, there are $g\in\mathbb{F}$ and $c\in\mathbb{K}^*$ as in (15) such that $\tau(t)=c\,s+g$.*

Proof. Let $\tau\colon\mathbb{F}(t)\to\mathbb{F}(s)$ be an $\mathbb{F}$-isomorphism. Note: $\Delta(\tau(t))=\tau(\Delta(t))=\tau(f)=f$. By Thm. 10 $\tau(t)=c\,s+g$ for some $g\in\mathbb{F}$ and $c\in\mathbb{K}$, and thus (15). □

Application: Suppose we are given a $\Pi\Sigma^*$-extension $(\mathbb{F}(x_1)\dots(x_e),\sigma)$ of a $\Pi\Sigma^*$-field $(\mathbb{F},\sigma)$ over $\mathbb{K}$, and one has to compute solutions $g\in\mathbb{F}(x_1)\dots(x_e)$ of (5) for various instances of $f\in\mathbb{F}$. Then the following strategy is straightforward. Compute once and for all a reduced $\Pi\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ together with an $\mathbb{F}$-isomorphism $\tau\colon\mathbb{F}(x_1)\dots(x_e)\to\mathbb{F}(t_1)\dots(t_e)$; define S as in (8) and set $f_i:=\Delta(t_i)\in\mathbb{F}$ for $i\in S$. Then for each summand $f\in\mathbb{F}$ we can apply Theorem 10 as follows: it suffices to look for c_i with $i\in S$ and $w\in\mathbb{F}$ such that

$$\Delta(w)=f+\sum_{i\in S}c_i\,f_i;$$

note that this problem (among others) can be solved with Karr's algorithm [21] or our simplified version [44]. Then given such a solution, one gets the solution (10) for (5). Hence with $g':=\tau^{-1}(g)\in\mathbb{F}(x_1)\dots(x_r)$ we get the required solution $\Delta(g')=f$, since $\Delta(g')=\Delta(\tau^{-1}(g))=\tau^{-1}(\Delta(g))=\tau^{-1}(f)=f$.

5. Complete-reduced $\Pi\Sigma^*$-extensions

We refine reduced $\Pi\Sigma^*$-extension to complete-reduced $\Pi\Sigma^*$-extensions as follows.

Definition **28.** A $\Pi\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ is called *complete-reduced over* $\mathbb{F}$ or in short *complete-reduced* if for any Σ^*-extension t_i $(1 \leq i \leq e)$ with $f := \Delta(t_i)$ and r with $f \in \mathbb{F}(t_1)\dots(t_r)\setminus\mathbb{F}(t_1)\dots(t_{r-1})$ the following property holds: there are no $g \in \mathbb{F}(t_1)\dots(t_{i-1})$ and $f' \in \mathbb{F}(t_1)\dots(f_{r-1})$ such that (7).

The proof of the following theorem is analogously to the proof of Theorem 24. The resulting algorithm is just Alg. 1: the only difference is that one always executes line (6) independently whether j is 0 or not.

Theorem 29. *For any $\Pi\Sigma^*$-extension $(\mathbb{H},\sigma)$ of $(\mathbb{F},\sigma)$ there is a complete-reduced $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ and an $\mathbb{F}$-isomorphism $\tau\colon \mathbb{H} \to \mathbb{E}$. Such a $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ and τ can be given explicitly, if $(\mathbb{F},\sigma)$ is RS-computable.*

Example **30. (1)** In Ex. 25 we got for the $\Pi\Sigma^*$-field $(\mathbb{Q}(k)(s_1)(s_{1,1})(s_{1,1,1}),\sigma)$ with (16) the isomorphic $\Pi\Sigma^*$-field $(\mathbb{Q}(k)(s_1)(s_2)(s_3),\sigma)$ with (4). Since in each step we solved problem RS, the resulting extension is complete-reduced.
(2) Take the $\Pi\Sigma^*$-field $(\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(e)(s_{6,1,3}),\sigma)$ with (4) and (6). Solving problem RS for each extension shows that the extension is complete-reduced.

Theorem 20 can be carried over to complete-reduced extensions as follows.

Theorem 31. *Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with $\mathbb{E} := \mathbb{F}(t_1)\dots(t_e)$. Then the following statements are equivalent.*

(1) This extension is complete-reduced.
(2) For any i,j with $1 \leq i \leq j \leq e$, $(\mathbb{F}(t_1)\dots(t_j),\sigma)$ is a reduced $\Pi\Sigma^*$-extension of $(\mathbb{F}(t_1)\dots(t_i),\sigma)$.
(3) For any j $(1 \leq j \leq e)$ with

$$S = S(j) = \{i | j \leq i \leq e \text{ and } \Delta(t_i) \in \mathbb{F}(t_1)\dots(t_{j-1})\} \tag{20}$$

and for any $g \in \mathbb{E}$ with $\Delta(g) \in \mathbb{F}(t_1)\dots(t_{j-1})$ we have (10) for some $c_i \in \mathrm{const}_\sigma\mathbb{F}$ and $w \in \mathbb{F}(t_1)\dots(t_{j-1})$.
(4) For any Σ^*-extension t_i $(1 \leq i \leq e)$ with $f := \Delta(t_i)$ and r s.t. $f \in \mathbb{F}(t_1)\dots(t_r)\setminus \mathbb{F}(t_1)\dots(t_{r-1})$ the following holds: There is no Σ^*-ext. $(\mathbb{F}(t_1)\dots(t_{i-1})(s),\sigma)$ of $(\mathbb{F},\sigma)$ with $\Delta(s) \in \mathbb{F}(t_1)\dots(t_{r-1})$ in which we have g with (5).

Proof. This extension is not complete-reduced if and only if there is a j, $1 \leq j \leq e$, such that for $f := \Delta(t_j)$ with $f \in \mathbb{F}(t_1)\dots(t_r) \setminus \mathbb{F}(t_1)\dots(t_{r-1})$ for some r $(1 \leq r \leq j)$ we have the following property: there are $f' \in \mathbb{F}(t_1)\dots(t_{r-1})$ and $g \in \mathbb{F}(t_1)\dots(t_{j-1})$ with (7). But this is equivalent to the fact that there are r,j with $1 \leq r \leq j \leq e$ such that $(\mathbb{F}(t_1)\dots(t_j),\sigma)$ is not a reduced $\Pi\Sigma^*$-extension of $(\mathbb{F}(t_1)\dots(t_r),\sigma)$. Hence **(1)** is equivalent to **(2)**. The other equivalences are an immediate consequence of Theorem 20. ☐

We emphasize the equivalence (1)⇔(3) of Theorem 31: For any $f \in \mathbb{E}$ we can apply Theorem 10. Namely, let j be minimal such that $f \in \mathbb{F}(t_1)\dots(t_j)$ and define $S = S(j)$ by (20). Then for any solution $g \in \mathbb{E}$ of (5) it follows that (10) for some $w \in \mathbb{F}(t_1)\dots(t_{j-1})$ and $c_i \in \mathrm{const}_\sigma\mathbb{F}$.

6. The depth and reordering of complete-reduced $\Pi\Sigma^*$-extensions

As indicated in the introduction, reducing the nested depth of a given indefinite sum expression, like e.g., d'Alembertian solutions [30, 2, 39] of a linear recurrence, is an important issue in the context of $\Pi\Sigma^*$-fields. In order to measure the nested depth, we introduce the following depth function [47].

Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with the field $\mathbb{E} := \mathbb{F}(t_1)\dots(t_e)$ and with $\sigma(t_i) = a_i\,t_i$ or $\sigma(t_i) = t_i + a_i$ for $1 \leq i \leq e$. The *depth function for elements of* $\mathbb{E}$ *over* $\mathbb{F}$, $\delta_{\mathbb{F}}\colon \mathbb{E} \to \mathbb{N}$, is defined as follows.

(1) For any $g \in \mathbb{F}$, $\delta_{\mathbb{F}}(g) := 0$.
(2) If $\delta_{\mathbb{F}}$ is defined for $(\mathbb{F}(t_1)\dots(t_{i-1}),\sigma)$ with $i > 1$, we define $\delta_{\mathbb{F}}(t_i) := \delta_{\mathbb{F}}(a_i)+1$; for $g = \frac{g_1}{g_2} \in \mathbb{F}(t_1)\dots(t_i)$, with $g_1, g_2 \in \mathbb{F}[t_1,\dots,t_i]$ coprime, we define
$$\delta_{\mathbb{F}}(g) := \max(\{\delta_{\mathbb{F}}(t_j) | 1 \leq j \leq i \text{ and } t_j \text{ occurs in } g_1 \text{ or } g_2\} \cup \{0\}).$$

The *extension depth* of a $\Pi\Sigma^*$-extension $(\mathbb{E}(x_1)\dots(x_r),\sigma)$ of $(\mathbb{E},\sigma)$ is defined by $\max(0,\delta_{\mathbb{F}}(x_1),\dots,\delta_{\mathbb{F}}(x_r))$.

Example **32.** For the $\Pi\Sigma^*$-field $(\mathbb{F},\sigma)$ with $\mathbb{F} = \mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(e)(s_{6,1,3})$ and with (4) and (6) we have
$$\delta_{\mathbb{Q}}(k) = 1,\ \delta_{\mathbb{Q}}(s_1) = \delta_{\mathbb{Q}}(s_2) = \delta_{\mathbb{Q}}(s_3) = 2,\ \delta_{\mathbb{Q}}(s_{1,3}) = \delta_{\mathbb{Q}}(e) = 3,\ \delta_{\mathbb{Q}}(s_{6,1,3}) = 4.$$
The extension depth of the $\Pi\Sigma^*$-extension $(\mathbb{F},\sigma)$ of $(\mathbb{Q},\sigma)$ is 4.

If one wants to simplify the nested depth of sums in a $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$, the following property is crucial: for any $f, g \in \mathbb{E}$ with (5) we have
$$\delta_{\mathbb{F}}(f) \leq \delta_{\mathbb{F}}(g) \leq \delta_{\mathbb{F}}(f) + 1; \tag{21}$$
in other words, if we find a sum representation g for a summand f with (5), the depth of g should be bounded by (21).

Subsequently, we show that property (21) is closely related to reduced and complete-reduced $\Pi\Sigma^*$-extensions. For this task we assume that the $\Pi\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(t_i) = a_i\,t_i + f_i$ for all i with $1 \leq i \leq e$ is $\mathbb{F}$*-ordered*, i.e., the extensions are built in the order of their depths:
$$\delta_{\mathbb{F}}(t_1) \leq \delta_{\mathbb{F}}(t_2) \leq \cdots \leq \delta_{\mathbb{F}}(t_e); \tag{22}$$
we remark that any $\Pi\Sigma^*$-extension can be reordered in this form.

Theorem 33. *Let $(\mathbb{E},\sigma)$ be an $\mathbb{F}$-ordered $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with the tower of $\Pi\Sigma^*$-extensions*
$$\mathbb{F} = \mathbb{F}_0 \leq \mathbb{F}_1 \leq \cdots \leq \mathbb{F}_d = \mathbb{E}, \tag{23}$$
such that for $1 \leq i \leq d$ the following holds: $\mathbb{F}_i = \mathbb{F}_{i-1}(x_1^{(i)})\dots(x_{e_i}^{(i)})$ with $e_i > 0$ and $\delta_{\mathbb{F}}(x_j^{(i)}) = i$ for all $1 \leq j \leq e_i$. Then the following two statements are equivalent:

(1) For $0 \leq i \leq j \leq d$, the $\Pi\Sigma^*$-extension $(\mathbb{F}_j,\sigma)$ of $(\mathbb{F}_i,\sigma)$ is reduced.
(2) For any $f, g \in \mathbb{E}$ as in (5) we have (21).

Proof. Let $(\mathbb{E},\sigma)$ be an $\mathbb{F}$-ordered $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ as claimed above such that statement **(1)** holds. Let $f \in \mathbb{E}$ with $j := \delta_{\mathbb{F}}(f)$ and $g \in \mathbb{E}$ with (5). If $j = d$, (21) clearly holds. Otherwise, let $j < d$. Since the extension $(\mathbb{E},\sigma)$ of $(\mathbb{F}_j,\sigma)$ is reduced, we can apply Theorem 10 and it follows that $g = \sum_{i=1}^{e_{j+1}} c_i x_i^{(j+1)} + w$ where $w \in \mathbb{F}_j$ and $c_i \in \text{const}_\sigma\mathbb{F}$. Since $\delta_{\mathbb{F}}(g) \leq j+1$, statement **(2)** holds.

Conversely, let $(\mathbb{E},\sigma)$ be an $\mathbb{F}$-ordered $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ such that statement **(1)** does not hold. Then there are $l,r\geq 1$ such that $(\mathbb{F}_r,\sigma)$ is not a reduced $\Pi\Sigma^*$-extension of $(\mathbb{F}_l,\sigma)$. In particular, there is a Σ^*-extension $x_u^{(v)}$ for some $l<v\leq r$ and $1\leq u\leq e_v$ with $f:=\Delta(x_u^{(v)})\notin\mathbb{F}_l$ s.t. the following property holds: there are $f'\in\mathbb{F}_l$ and $g\in\mathbb{F}_{v-1}(x_1^{(v)})\dots(x_{u-1}^{(v)})$ such that (7). Note that $\delta_{\mathbb{F}}(f')<\delta_{\mathbb{F}}(f)$. Hence for $h:=x_u^{(l)}-g$, $\Delta(h)=f-\Delta(g)=f'$ and $\delta_{\mathbb{F}}(h)>\delta_{\mathbb{F}}(f)>\delta_{\mathbb{F}}(f')$. Thus, $\delta_{\mathbb{F}}(h)>\delta_{\mathbb{F}}(f')+1$, and **(2)** does not hold. □

$\mathbb{F}$-ordered complete-reduced $\Pi\Sigma^*$-extensions are covered by $\mathbb{F}$-ordered $\Pi\Sigma^*$-extensions of the form (23) for which statement (2) of Thm. 33 holds. Hence we get

Corollary 34. *Let $(\mathbb{E},\sigma)$ be an $\mathbb{F}$-ordered $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$. If the extension is complete-reduced, then for any $f,g\in\mathbb{E}$ with* (5) *we have* (21).

Example **35.** As pointed out in Ex. 30.2 the $\mathbb{Q}$-ordered $\Pi\Sigma^*$-extension $(\mathbb{F},\sigma)$ of $(\mathbb{Q},\sigma)$ with $\mathbb{F}=\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(e)(s_{6,1,3})$ and with (4) and (6) is complete-reduced. Hence we can apply Corollary 34: for any $f,g\in\mathbb{F}$ with (5) we have (21). E.g., if $\delta_{\mathbb{F}}(f)\geq 2$, i.e., $f\in\mathbb{F}$, then (12). If $\delta_{\mathbb{F}}(f)=1$, i.e., $f\in\mathbb{Q}(k)$, then (11).

Example **36.** The $\Pi\Sigma^*$-field from Ex. 13 is not reduced. Hence, as predicted in Theorem 33 we could find f and g in this field with (5) such that $\delta_{\mathbb{F}}(g)>\delta_{\mathbb{F}}(f)+1$.

In order to exploit Corollary 34 in full generality, it is necessary to transform a $\Pi\Sigma^*$-extension to an $\mathbb{F}$-ordered complete-reduced extension. It turns out that this task is not straightforward[5]. We start with the following illustrative example.

Example **37.** Given $(\mathbb{F},\sigma)$ as in Ex. 35, we consider the Σ^*-extension $(\mathbb{F}(s_{2,1,3}),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(s_{2,1,3})=s_{2,1,3}+\frac{\sigma(s_{1,3})}{(k+1)^2}$. Subsequently, we try to transform this extension such that it is again a $\mathbb{Q}$-ordered completed-reduced extension of $(\mathbb{Q},\sigma)$. First, we verify that $s_{2,1,3}$ is not a complete-reduced extension: by solving problem RS with $f=\frac{\sigma(s_{1,3})}{(k+1)^2}$ we arrive at $f'=\frac{1}{2(k+1)^6}$ and $g=\frac{1}{2}\left(s_3^2-2e+2s_{1,3}\,s_2\right)$. Hence we can construct the Σ^*-extension $(\mathbb{F}(s_6),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(s_6)=s_6+\frac{1}{(k+1)^6}$. In particular, we get

$$\Delta(\frac{1}{2}\left(s_3^2-2e+2s_{1,3}\,s_2+s_6\right))=\frac{\sigma(s_{1,3})}{(k+1)^2}. \tag{24}$$

Next, we rearrange the $\Pi\Sigma^*$-field $(\mathbb{F}(s_6),\sigma)$ and obtain the $\mathbb{Q}$-ordered $\Pi\Sigma^*$-extension $(\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_6)(s_{1,3})(e)(s_{6,1,3}),\sigma)$ of $(\mathbb{Q},\sigma)$. In addition, we find the $\mathbb{Q}$-isomorphism

$$\rho\colon\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(e)(s_{6,1,3})(s_{2,1,3})\to\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_6)(s_{1,3})(e)(s_{6,1,3})$$

by keeping all variables fixed except

$$\rho(s_{2,1,3})=\frac{1}{2}\left(s_3^2-2e+2s_{1,3}\,s_2+s_6\right). \tag{25}$$

Due to this change, we have to check if the extensions $s_{1,3},e,s_{6,1,3}$ on top of s_6 are still complete-reduced. It turns out that $s_{6,1,3}$ is not complete-reduced. Similarly as above, we obtain the Σ^*-extension $(\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_6)(s_{1,3})(e)(E),\sigma)$ of

[5] In Section 7.1 we shall propose another solution by embedding a $\Pi\Sigma^*$-extension into a depth-optimal $\Pi\Sigma^*$-extension; see also Ex. 45 which is related to Ex. 37.

$(\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_6)(s_{1,3})(e), \sigma)$ with $\sigma(E) = E + \frac{\sigma(s_3)\big(\sigma(s_6)(k+1)^6-1\big)}{(k+1)^7})$ such that

$$\Delta(s_{1,3}s_6 - E) = \frac{\sigma(s_{1,3})}{(k+1)^6}. \tag{26}$$

In particular, we get the $\mathbb{Q}$-isomorphism

$$\mu\colon \mathbb{Q}(k)(s_1)(s_2)(s_3)(s_6)(s_{1,3})(e)(s_{6,1,3}) \to \mathbb{Q}(k)(s_1)(s_2)(s_3)(s_6)(s_{1,3})(e)(E)$$

by keeping all variables fixed except

$$\mu(s_{6,1,3}) = s_{1,3}s_6 - E. \tag{27}$$

To sum up, we managed to transform the $\Pi\Sigma^*$-field $(\mathbb{F}, \sigma)$ to the $\mathbb{Q}$-ordered complete-reduced $\Pi\Sigma^*$-extension $(\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_6)(s_{1,3})(e)(E), \sigma)$ of $(\mathbb{Q}, \sigma)$ with

$$\begin{aligned} &\sigma(k) = k+1, && \sigma(s_1) = s_1 + \tfrac{1}{k+1}, && \sigma(s_2) = s_2 + \tfrac{1}{(k+1)^2},\\ &\sigma(s_3) = s_3 + \tfrac{1}{(k+1)^3}, && \sigma(s_6) = s_6 + \tfrac{1}{(k+1)^6} && \sigma(s_{1,3}) = s_{1,3} + \tfrac{\sigma(s_3)}{k+1},\\ &\sigma(e) = e + \tfrac{\sigma(s_2)\,\sigma(s_3)}{k+1}, && \sigma(E) = E + \tfrac{\sigma(s_3)\big(\sigma(s_6)(k+1)^6-1\big)}{(k+1)^7} \end{aligned} \tag{28}$$

together with the $\mathbb{Q}$-isomorphism $\tau := \mu \circ \rho$ with

$$\tau\colon \mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(e)(s_{6,1,3})(s_{2,1,3}) \to \mathbb{Q}(k)(s_1)(s_2)(s_3)(s_6)(s_{1,3})(e)(E); \tag{29}$$

here all variables are fixed except

$$\tau(s_{2,1,3}) = \frac{1}{2}\left(s_3^2 - 2e + 2s_{1,3}\,s_2 + s_6\right) \quad\text{and}\quad \tau(s_{6,1,3}) = s_{1,3}s_6 - E. \tag{30}$$

Remark **38.** Reinterpreting the variables of the previous example as indefinite sums yields the following identities (which are reflected by (30)): for all $k \in \mathbb{N}$,

$$\begin{aligned} S_{2,1,3}(k) &= \frac{1}{2}S_3(k)^2 - \sum_{i=1}^{k} \frac{S_2(i)S_3(i)}{i} + S_{1,3}(k)\,S_2(k) + \frac{1}{2}S_6(k),\\ S_{6,1,3}(k) &= S_{1,3}(k)S_6(k) - \sum_{i=1}^{k} \frac{S_3(i)\left(S_6(i)i^6-1\right)}{i^7}. \end{aligned} \tag{31}$$

Subsequently, we will make this transformation more precise. In order to deal with Π-extensions (see case 2 in the proof of Thm. 40), we need the following lemma.

Lemma 39. *Let $(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{F}(t_1)\dots(t_e)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ and let $f \in \mathbb{E}$; moreover let $(\mathbb{E}(x), \sigma)$ be a Π-extension of $(\mathbb{E}, \sigma)$ with $\frac{\sigma(x)}{x} \in \mathbb{F}$. If there are $f' \in \mathbb{F}(x)$ and $g \in \mathbb{E}(x)$ s.t. (7), then there are $g \in \mathbb{E}$ and $f' \in \mathbb{F}$ s.t. (7).*

Proof. Let $f \in \mathbb{E}$, $g \in \mathbb{E}(x)$ and $f' \in \mathbb{F}(x)$ as claimed above. For convenience, denote by $\mathbb{E}(x)^{(\text{prop})}$ (resp. by $\mathbb{F}(x)^{(\text{prop})}$) all proper rational functions from $\mathbb{E}(x)$ (resp. from $\mathbb{F}(x)$), i.e., for each element the degree of the numerator (w.r.t. x) is smaller than the degree of the denominator. By polynomial division we can write $g = p_1 + q_1$ and $f' = p_2 + q_2$ such that $p_1 \in \mathbb{E}[x]$, $q_1 \in \mathbb{E}(x)^{(\text{prop})}$ and $p_2 \in \mathbb{F}[x]$, $q_2 \in \mathbb{F}(x)^{(\text{prop})}$. Since $\frac{\sigma(x)}{x} \in \mathbb{F}$, it is immediate that $\sigma(p_1) \in \mathbb{E}[x]$, and consequently, $\Delta(p_1) \in \mathbb{E}[x]$. Moreover, since $\sigma(q_1) \in \mathbb{E}(x)^{(\text{prop})}$ (the degrees of polynomials in x do not change under the action of σ), $\Delta(q_1) \in \mathbb{E}(x)^{(\text{prop})}$. Analogously, $\Delta(p_2) \in \mathbb{F}[x]$ and $\Delta(q_2) \in \mathbb{F}(x)^{(\text{prop})}$. Since $\mathbb{E}(x) = \mathbb{E}[x] \oplus \mathbb{E}(x)^{(\text{prop})}$

forms a direct sum (as vector spaces over $\mathbb{E}$), (7) implies $\Delta(p_1) + p_2 = f$ and $\Delta(q_1) + q_2 = 0$. Now let $\gamma \in \mathbb{E}$ and $\phi' \in \mathbb{F}$ be the constants of the polynomials p_1 and p_2, respectively. Then by coefficient comparison in $\Delta(p_1) + p_2 - f = 0$, $\Delta(\gamma) + \phi - f = 0$; this completes the lemma. □

Theorem 40. *For any $\Pi\Sigma^*$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ there is a complete-reduced $\mathbb{F}$-ordered $\Pi\Sigma^*$-extension $(\mathbb{E}', \sigma)$ of $(\mathbb{F}, \sigma)$ together with an $\mathbb{F}$-isomorphism $\tau\colon \mathbb{E} \to \mathbb{E}'$; in particular,*

$$\delta_{\mathbb{F}}(\tau(h)) \leq \delta_{\mathbb{F}}(h) \tag{32}$$

for all $h \in \mathbb{H}$. Such $(\mathbb{E}', \sigma)$ and τ can be given explicitly, if $(\mathbb{F}, \sigma)$ is RS-computable.

Proof. Let $(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{F}(t_1) \dots (t_e)$ be a $\Pi\Sigma^*$-ext. of $(\mathbb{F}, \sigma)$. We show the theorem by induction on the depth. If $\delta_{\mathbb{F}}(t_1) = \dots \delta_{\mathbb{F}}(t_e) = 1$, the claim follows by Lemma 9. Now suppose that we have shown the assumption for any extension whose extension depth is $\leq d+1$ and $r \geq 0$ or less extensions have depth $d+1$. Now consider our $\Pi\Sigma^*$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ with extension depth $d+1$ where exactly $r+1$ extensions have depth $d+1$. W.l.o.g. we may assume that this extension is $\mathbb{F}$-ordered, i.e., $\delta_{\mathbb{F}}(t_e) = d+1$. By our assumption we get an $\mathbb{F}$-ordered complete-reduced $\Pi\Sigma^*$-ext. $(\mathbb{F}(x_1) \dots (x_{e-1}), \sigma)$ of $(\mathbb{F}, \sigma)$ with an $\mathbb{F}$-isomorphism τ from $\mathbb{F}(t_1) \dots (t_{e-1})$ to $\mathbb{F}(x_1) \dots (x_{e-1})$ such that (32) for all $h \in \mathbb{F}(t_1) \dots (t_{e-1})$.

Case 1: If t_e is a Π-ext., define $\alpha := \tau(\frac{\sigma(t_e)}{t_e})$, and take the Π-ext. $(\mathbb{F}(x_1) \dots (x_e), \sigma)$ of $(\mathbb{F}(x_1) \dots (x_{e-1}), \sigma)$ with $\sigma(x_e) = \alpha\, x_e$, and extend the $\mathbb{F}$-isomorphism τ with $\tau(t_e) = x_e$; this is possible by Lemma 23. Note that $\delta_{\mathbb{F}}(x_e) \leq \delta_{\mathbb{F}}(\tau(x_e)) = \delta_{\mathbb{F}}(t_e) = d+1$ by (32); in particular, (32) for all $h \in \mathbb{F}(x_1) \dots (x_e)$.

Case 1.1: If $\delta_{\mathbb{F}}(x_e) = d+1$, $(\mathbb{F}(x_1) \dots (x_e), \sigma)$ forms an $\mathbb{F}$-ordered complete-reduced $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$, and we are done.

Case 1.2: Otherwise bring it to an $\mathbb{F}$-ordered form: for some l with $0 \leq l < e$, we obtain[6] $(\mathbb{F}(x_1) \dots (x_l)(x_e)(x_{l+1}) \dots (x_{e-1}), \sigma)$. Suppose x_i $(i > l)$ is not complete-reduced; let j be minimal s.t. $f := \Delta(x_i) \in \mathbb{F}(x_1) \dots (x_l)(x_e)(x_{l+1}) \dots (x_j)$. Then there are $g \in \mathbb{F}(x_1) \dots (x_{i-1})(x_e)$ and $f' \in \mathbb{F}(x_1) \dots (x_{j-1})(x_e)$ s.t. (7). Hence by Lemma 39 we find such f' and g which are free of x_e, and thus $(\mathbb{F}(x_1) \dots (x_j), \sigma)$ is not a complete-reduced extension of $(\mathbb{F}, \sigma)$; a contradiction to the assumption. This completes this part of the proof.

Case 2: Suppose that x_e is a Σ^*-extension with $f := \Delta(x_e)$. Let j be minimal such that there are $f' \in \mathbb{F}(x_1) \dots (x_j)$ and $g \in \mathbb{F}(x_1) \dots (x_{e-1})$ as in (7). Note: If $(\mathbb{F}, \sigma)$ is RS-computable, such f' and g can be computed explicitly. Then there is a Σ^*-extension $(\mathbb{H}, \sigma)$ of $(\mathbb{F}(x_1) \dots (x_{e-1}), \sigma)$ with $\mathbb{H} = \mathbb{F}(x_1) \dots (x_{e-1})(s)$ and $\sigma(s) = s + f'$; in particular, there is the $\mathbb{F}$-isomorphism $\rho\colon \mathbb{F}(x_1) \dots (x_e) \to \mathbb{H}$ with $\rho(h) = h$ for all $h \in \mathbb{F}(x_1) \dots (x_{e-1})$ and $\rho(x_e) = s + g$ by Lemma 22. Clearly, $\delta_{\mathbb{F}}(\rho(h)) \leq \delta_{\mathbb{F}}(h)$ for all $h \in \mathbb{F}(x_1) \dots (x_e)$. By construction, $(\mathbb{H}, \sigma)$ is a complete-reduced $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$.

Case 2.1: If $\delta_{\mathbb{F}}(f') = \delta_{\mathbb{F}}(f) = d$, then $(\mathbb{H}, \sigma)$ is an $\mathbb{F}$-ordered complete-reduced $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$. Finally, with $\tau' = \rho \circ \tau$ we get an $\mathbb{F}$-isomorphism from $\mathbb{F}(t_1) \dots (t_e)$ to $\mathbb{F}(x_1) \dots (x_e)$ such that $\delta_{\mathbb{F}}(\tau'(h)) \leq \delta_{\mathbb{F}}(h)$ for all $h \in \mathbb{F}(t_1) \dots (t_e)$; this completes this part of the induction.

Case 2.2: If $\delta_{\mathbb{F}}(f') < \delta_{\mathbb{F}}(f)$, rearrange the extension $(\mathbb{H}, \sigma)$ to an $\mathbb{F}$-ordered $\Pi\Sigma^*$-extension $(\mathbb{H}', \sigma)$ of $(\mathbb{F}, \sigma)$ with $\mathbb{H}' = \mathbb{F}(x_1) \dots\dots (x_l)(s)(x_{l+1}) \dots (x_{e-1})$ for some

[6]Note that the extensions below of x_{l+1} are $\mathbb{F}$-ordered and complete-reduced; this fact will be exploited in Alg. 2.

Algorithm 2 `ToCompleteReducedOrderedField`$((\mathbb{E},\sigma),k)$

In: A $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ with $\mathbb{E}=\mathbb{F}(t_1)\dots(t_e)$ s.t. $(\mathbb{F},\sigma)$ is RS-computable; $k\in\mathbb{N}$ s.t. $(\mathbb{F}(t_1)\dots(t_k),\sigma)$ is an $\mathbb{F}$-ordered complete-reduced extension of $(\mathbb{F},\sigma)$.
Out: An $\mathbb{F}$-ordered complete-reduced $\Pi\Sigma^*$-extension $(\mathbb{E}',\sigma)$ of $(\mathbb{F},\sigma)$ together with an $\mathbb{F}$-isomorphism $\tau\colon\mathbb{E}\to\mathbb{E}'$.

IF $k\geq e$, THEN RETURN $((\mathbb{E},\sigma),\mathrm{id}_{\mathbb{E}})$ FI
$((\mathbb{F}(x_1)\dots(x_{e-1}),\sigma),\tau)$:=`ToCompleteReducedOrderedField`$((\mathbb{F}(t_1)\dots(t_{e-1}),\sigma),k)$;
IF t_e is a Π-extension, i.e., $\alpha:=\tau(\frac{\sigma(t_e)}{t_e})\in\mathbb{F}(x_1)\dots(x_{e-1})$ THEN
    Take the Π-ext. $(\mathbb{F}(x_1)\dots(x_e),\sigma)$ of $(\mathbb{F}(x_1)\dots(x_{e-1}),\sigma)$ with $\sigma(x_e)=\alpha\,x_e$; bring it to an $\mathbb{F}$-ordered form with $\mathbb{E}':=\mathbb{F}(x_1)\dots(x_l)(x_e)(x_{l+1})\dots(x_{e-1})$ for some l with $0\leq l<e$. Take the $\mathbb{F}$-isomorphism $\tau'\colon\mathbb{F}(t_1)\dots(t_e)\to\mathbb{E}'$ with $\tau'(h)=\tau(h)$ for all $h\in\mathbb{F}(t_1)\dots(t_{e-1})$ and $\tau'(t_e)=x_e$. RETURN $((\mathbb{E}',\sigma),\tau')$.
FI
Let $f'\in\mathbb{F}(x_1)\dots(x_j)\setminus\mathbb{F}(x_1)\dots(x_{j-1})$ and $g\in\mathbb{F}(x_1)\dots(x_{e-1})$ be the result of problem RS for $f:=\tau(\Delta(t_e))$ and $(\mathbb{F}(x_1)\dots(x_{e-1}),\sigma)$.
Define the Σ^*-extension $(\mathbb{H},\sigma)$ of $(\mathbb{F}(x_1)\dots(x_{e-1}),\sigma)$ with $\mathbb{H}:=\mathbb{F}(x_1)\dots(t_{e-1})(s)$ and $\sigma(s)=s+f'$ together with the $\mathbb{F}$-isomorphism $\rho\colon\mathbb{F}(x_1)\dots(x_e)\to\mathbb{H}$ with $\rho(h)=h$ for all $h\in\mathbb{F}(x_1)\dots(x_{e-1})$ and $\rho(x_e)=s+g$.
IF $\delta_{\mathbb{F}}(f')=\delta_{\mathbb{F}}(f)$ THEN RETURN $((\mathbb{H},\sigma),\rho\circ\tau)$ FI
Bring $(\mathbb{H},\sigma)$ to an $\mathbb{F}$-ordered ext. $(\mathbb{H}',\sigma)$ with $\mathbb{H}'=\mathbb{F}(x_1)\dots(x_l)(s)(x_{l+1})\dots(x_{e-1})$ for some $l>j$. As pointed out in Footnote 6 we can execute $((\mathbb{E}',\sigma),\mu)$:=`ToCompleteReducedOrderedField`$((\mathbb{H}',\sigma),l+1)$.
RETURN $((\mathbb{E}',\sigma),\mu\circ\rho\circ\tau)$.

$l>j$ (see again footnote 6). Note that in this case the number of extensions with depth $d+1$ have been reduced by 1. Consequently, we can apply our induction assumption: we can transform $(\mathbb{H}',\sigma)$ to an $\mathbb{F}$-ordered complete-reduced extension $(\mathbb{E}',\sigma)$ of $(\mathbb{F},\sigma)$ with $\mathbb{E}'=\mathbb{F}(x'_1)\dots(x'_e)$ together with an $\mathbb{F}$-isomorphism $\mu\colon\mathbb{H}'\to\mathbb{E}'$ such that $\delta_{\mathbb{F}}(\mu(h))\leq\delta_{\mathbb{F}}(h)$ for all $h\in\mathbb{H}'$. Hence with $\tau':=\mu\circ\rho\circ\tau$ we get an $\mathbb{F}$-isomorphism $\tau'\colon\mathbb{F}(t_1)\dots(t_e)\to\mathbb{E}'$ with $\delta_{\mathbb{F}}(\tau'(h))\leq\delta_{\mathbb{F}}(h)$ for all $h\in\mathbb{F}(t_1)\dots(t_e)$. This finishes the induction step. □

Extracting the reduction steps of the inductive proof of Theorem 40 and taking into account Footnote 6 lead to Algorithm 2. For instance in Example 37 the algorithm is carried out for the input $((\mathbb{F}(s_{2,1,3}),\sigma),7)$. In particular, given a $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ one computes with the input $((\mathbb{E},\sigma),1)$ an isomorphic $\mathbb{F}$-ordered complete-reduced extension.

Remark **41.** Note that we could proceed differently. **Step 1:** Bring a $\Pi\Sigma^*$-extension to the form (23) such that statement (2) in Theorem 33 holds; then we are already in the position to exploit property (1) given in Theorem 33.
Step 2: The computation of an $\mathbb{F}$-ordered complete-reduced extension is immediate: just apply the underlying algorithm of Theorem 29 (it is easy to see that the depth of the extensions cannot be reduced further, and hence the output is an $\mathbb{F}$-ordered complete-reduced extension). However, in order to perform Step 1, our arguments lead to the same algorithm as given in Algorithm 2; only subproblem RS can be slightly modified/simplified. Since these modifications do not lead to any substantial improvement, we just presented Algorithm 2, and we set aside a detailed presentation of the variation sketched in this remark.

7. Depth-optimal $\Pi\Sigma^*$-extensions and refined structural Theorems

In [41] $\Pi\Sigma^*$-extensions have been elaborated to depth-optimal $\Pi\Sigma^*$-extensions. As it turns out, such extensions are closely related to reduced and complete-reduced $\Pi\Sigma^*$-extensions. But, there are also major differences: depth-optimal $\Pi\Sigma^*$-extensions satisfy in general additional properties that are highly relevant in the field of symbolic summation; see [47, 49]. Subsequently, we present in detail how the derived properties of reduced and complete-reduced $\Pi\Sigma^*$-extensions can be carried over to depth-optimal $\Pi\Sigma^*$-extensions. Besides this, we work out their crucial differences in the context of symbolic summation. As a spin off we obtain refined structural theorems that are preferable, e.g., to Theorems 10 and 17.

In the context of reduced $\Pi\Sigma^*$-extensions depth-optimal $\Pi\Sigma^*$-extensions can be introduced as follows. Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with $\mathbb{E}=\mathbb{F}(x_1)\dots(x_l)$. Then by Theorem 20 there is the following alternative characterization for a reduced $\Pi\Sigma^*$-extension $(\mathbb{E}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{E},\sigma)$: for any Σ^*-extension t_i with $f:=\Delta(t_i)\in\mathbb{E}$ $(1\leq i\leq e)$ there is no Σ^*-extension $(\mathbb{E}(s),\sigma)$ of $(\mathbb{E},\sigma)$ in which we have $g\in\mathbb{E}(s)$ with (5). Now suppose in addition the following ordering:

$$\max(\delta_{\mathbb{F}}(x_1),\dots,\delta_{\mathbb{F}}(x_l))+1=\delta_{\mathbb{F}}(t_1)=\delta_{\mathbb{F}}(t_2)=\dots=\delta_{\mathbb{F}}(t_e).$$

Then the following property holds: For for any Σ^*-extension $f:=\Delta(t_i)$ there does not exist a single-nested Σ^*-extension $\mathbb{E}(s)$ with $\delta_{\mathbb{F}}(s)\leq\delta_{\mathbb{F}}(f)$ which provides us with a solution $g\in\mathbb{E}(s)$ for (5).

Essentially, depth-optimal $\Pi\Sigma^*$-extension follow up this construction with the constrained that there does not exist a *tower* of Σ^*-extensions $\mathbb{S}=\mathbb{E}(s_1)\dots(s_r)$ with $\delta_{\mathbb{F}}(s_i)\leq\delta_{\mathbb{F}}(f)$ for $1\leq i\leq r$ which provides us with a solution $g\in\mathbb{S}$ for (5). To be more precise, we introduce depth-optimal $\Pi\Sigma^*$-extensions as follows.

Definition **42.** Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$. A difference field extension $(\mathbb{E}(s),\sigma)$ of $(\mathbb{E},\sigma)$ with $\sigma(s)=s+f$ is called *depth-optimal* Σ^*-extension, in short Σ^δ-extension, if there is no Σ^*-extension $(\mathbb{S},\sigma)$ of $(\mathbb{E},\sigma)$ with extension[7] depth $\leq\delta_{\mathbb{F}}(f)$ and $g\in\mathbb{S}$ such that (5). A $\Pi\Sigma^*$-extension $(\mathbb{E}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{E},\sigma)$ is depth-optimal, in short a $\Pi\Sigma^\delta$-extension, if all Σ^*-extensions[8] are depth-optimal. A $\Pi\Sigma^\delta$-field is a $\Pi\Sigma^*$-field which consists of Π- and Σ^δ-extensions.

Then $\Pi\Sigma^\delta$-extensions can be related to reduced extensions in the following way.

Lemma 43. *Let $(\mathbb{E},\sigma)$ be an $\mathbb{F}$-ordered $\Pi\Sigma^\delta$-extension of $(\mathbb{F},\sigma)$ with (23) s.t. for $1\leq i\leq d$ we have $\mathbb{F}_i=\mathbb{F}_{i-1}(x_1^{(i)})\dots(x_{e_i}^{(i)})$ with $e_i>0$ and $\delta_{\mathbb{F}}(x_j^{(i)})=i$ for all $1\leq j\leq e_i$. Then for $0\leq i\leq j\leq d$, the $\Pi\Sigma^\delta$-extension $(\mathbb{F}_j,\sigma)$ of $(\mathbb{F}_i,\sigma)$ is reduced.*

Proof. Suppose that the lemma holds with depth $d\geq 0$ and consider a $\Pi\Sigma^\delta$-extension $(\mathbb{F}_{d+1},\sigma)$ of $(\mathbb{F}_d,\sigma)$ with $\mathbb{F}_{d+1}=\mathbb{F}_d(t_1)\dots(t_e)$ and $\delta_{\mathbb{F}}(t_i)=d+1$ for $1\leq i\leq e$. Clearly, $(\mathbb{F}_{d+1},\sigma)$ is a reduced extension of $(\mathbb{F}_d,\sigma)$ by Lemma 9. For any j $(1\leq j\leq e)$ with $f_j:=\Delta(t_j)\in\mathbb{F}_d$ and for any r $(0\leq r<d)$ we conclude as follows. Since t_j is a Σ^δ-ext., there is no Σ^*-ext. $(\mathbb{F}_d(t_1)\dots(t_{j-1})(s),\sigma)$ of $(\mathbb{F}_d(t_1)\dots(t_{j-1}),\sigma)$ with $\Delta(s)\in\mathbb{F}_r$ s.t. $\Delta(g)=f_j$ for some $g\in\mathbb{F}_d(t_1)\dots(t_{j-1})(s)$. By Thm. 20 ((1)$\Leftrightarrow$(3)),$(\mathbb{F}_d(t_1)\dots(t_e),\sigma)$ is a reduced extension of $(\mathbb{F}_r,\sigma)$. This completes the induction. $\square$

[7]Note that $\Pi\Sigma^\delta$-extensions are defined relatively to the ground field $(\mathbb{F},\sigma)$ over which the depth-function $\delta_{\mathbb{F}}$ is defined. Throughout this section we assume that this ground field is $\mathbb{F}$.

[8]In addition, note that Σ^δ-extensions belong to the class of Σ^*-extensions by Theorem 5.

7.1. **Embeddings of $\Pi\Sigma^*$-extensions into $\Pi\Sigma^\delta$-extensions.** Similarly to reduced and complete-reduced $\Pi\Sigma^*$-extensions, we can apply Lemmata 22 and 23 iteratively in order to translate a $\Pi\Sigma^*$-extension into a $\Pi\Sigma^\delta$-extension. In particular, this construction can be given explicitly, if one can solve the following problem.

Problem DOS (*D*epth *O*ptimal *S*ummation): **Given** a $\Pi\Sigma^\delta$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$, and given $f\in\mathbb{E}$; compute, if possible, a Σ^δ-extension $(\mathbb{E}(x_1)\dots(x_r),\sigma)$ of $(\mathbb{E},\sigma)$ with extension depth$\leq\delta_{\mathbb{F}}(f)$ s.t. there is a $g\in\mathbb{E}(x_1)\dots(x_r)$ with (5).

Namely, assume that the difference field $(\mathbb{F},\sigma)$ is *DOS-computable*, i.e., for any $\Pi\Sigma^\delta$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ and any $f\in\mathbb{E}$ one can solve problem DOS algorithmically. E.g., due to [47, Algorithm 1] implemented in Sigma any $\Pi\Sigma^*$-field is DOS-computable. In fact a difference field is RS-computable if and only if it is DOS-computable; for further difference field examples see page 10.

Then the embedding mechanism works as follows. Suppose we are given a $\Pi\Sigma^*$-extension $(\mathbb{H},\sigma)$ of $(\mathbb{F},\sigma)$ which we managed to embed into a $\Pi\Sigma^\delta$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ with $\tau\colon\mathbb{H}\to\mathbb{E}$. Now consider the Σ^*-extension $(\mathbb{H}(t),\sigma)$ of $(\mathbb{H},\sigma)$ with $\sigma(t)=t+f$. Then one can either find a Σ^δ-extension $(\mathbb{E}',\sigma)$ of $(\mathbb{E},\sigma)$ with $g\in\mathbb{E}'$ such that $\Delta(g)=\tau(f)$ (by solving problem DOS). In this case, one can embed $(\mathbb{H}(t),\sigma)$ into $(\mathbb{E}',\sigma)$ by extending the $\mathbb{F}$-monomorphism τ to $\tau\colon\mathbb{H}(t)\to\mathbb{E}'$ with $\tau(t)=g$; the correctness follow by $\sigma(\tau(t))=\sigma(g)=g+\tau(f)=\tau(t+f)=\tau(\sigma(t))$. Otherwise, if there is no solution for problem DOS, we can adjoin the Σ^δ-extension $(\mathbb{E}(s),\sigma)$ of $(\mathbb{E},\sigma)$ with $\sigma(s)=s+\tau(f)$ and we can extend the $\mathbb{F}$-monomorphism τ to $\tau\colon\mathbb{H}(t)\to\mathbb{E}(s)$ by $\tau(t)=s$. Similarly, one can treat a Π-extension $\sigma(t)=a\,t$ for some $a\in\mathbb{H}^*$. Summarizing, we arrive at the following result.

Theorem 44. *For any $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ there is a $\Pi\Sigma^\delta$-extension $(\mathbb{E}',\sigma)$ of $(\mathbb{F},\sigma)$ and an $\mathbb{F}$-monomorphism $\tau\colon\mathbb{E}\to\mathbb{E}'$. Such $(\mathbb{E}',\sigma)$ and τ can be constructed explicitly if $(\mathbb{F},\sigma)$ is DOS-computable.*

Example **45.** We take the $\Pi\Sigma^*$-field $(\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(e)(s_{6,1,3})(s_{2,1,3}),\sigma)$ with (4), (6) and $\sigma(s_{2,1,3})=s_{2,1,3}+\frac{\sigma(s_{1,3})}{(k+1)^2}$ from Example 37 and embed it into a $\Pi\Sigma^\delta$-field. It is easy to see that $(\mathbb{F},\sigma)$ with $\mathbb{F}=\mathbb{Q}(k)(s_1)(s_2)(s_3)$ is already a $\Pi\Sigma^\delta$-field; see also [47, Prop. 17]. We continue as follows.

(1) We apply our algorithms implemented in Sigma and verify that there is no Σ^*-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ with extension depth≤ 2 in which we find $g\in\mathbb{E}$ with $\Delta(g)=\frac{\sigma(s_3)}{k+1}$. Hence the Σ^*-extension $(\mathbb{F}(s_{1,3}),\sigma)$ of $(\mathbb{F},\sigma)$ is depth-optimal.
(2) Similarly, we check that $(\mathbb{F}(s_{1,3})(e),\sigma)$ is a Σ^δ-extension of $(\mathbb{F}(s_{1,3}),\sigma)$.
(3) Now, we check $s_{6,1,3}$ by looking at problem DOS with $f=\frac{\sigma(s_{1,3})}{(k+1)^6}$: we find the Σ^δ-extension $(\mathbb{F}(s_{1,3})(e)(s_6)(E),\sigma)$ of $(\mathbb{F}(s_{1,3}),\sigma)$ with
$$\sigma(s_6)=s_6+\frac{1}{(k+1)^6}\quad\text{and}\quad E=E+\frac{\sigma(s_3)\left(\sigma(s_6)(k+1)^6+1\right)}{(k+1)^7}$$
with $\delta_{\mathbb{F}}(s_3),\delta_{\mathbb{F}}(E)\leq 3$ s.t. (26); the $\mathbb{Q}$-monomorphism $\mu\colon\mathbb{F}(s_{1,3})(e)(s_{6,1,3})\to\mathbb{F}(s_{1,3})(e)(s_6)(E)$ can be defined by $\mu(h)=h$ for all $h\in\mathbb{F}(s_{1,3})(e)$ and (27).
(4) We treat $s_{2,1,3}$ by solving problem DOS for $f=\frac{\sigma(s_{1,3})}{(k+1)^2}$. This time no extension is needed, since we find (24); we can extend the $\mathbb{Q}$-monomorphism as in (25).

Summarizing, we arrive at the $\Pi\Sigma^\delta$-field $(\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(e)(s_6)(E),\sigma)$ with (28) together with the $\mathbb{Q}$-isomorphism (29) given by (30).

Usually, one obtains difference field monomorphisms where the transcendental degree of the embedding extension is larger than the embedded extension. For instance, in step 3 of Ex. 45 we embedded a $\mathbb{Q}$-ordered complete-reduced extension with degree 7 into a depth-optimal extension with degree 8.

Remark **46.** Note that in Ex. 45 we rediscovered identity (31): we simplified the sum $S_{6,1,3}(k)$ of depth 4 to a sum expression with depth 3 by introducing the tower of sum extensions $S_6(k)$ and $\sum_{i=1}^{k} i^{-7} S_3(i)\left(S_6(i)i^6 - 1\right)$. In a nutshell, in ordered complete-reduced $\Pi\Sigma^*$-fields like $(\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(e)(s_{6,1,3}), \sigma)$ from the Examples 35 and 45 one might fail to produce sum representations with smallest possible depth. But, transformations of $\Pi\Sigma^*$-fields to $\Pi\Sigma^\delta$-fields lead always to sum representations with optimal nested depth; see [49].

7.2. **Structural theorems.** Comparing reduced and complete-reduced $\Pi\Sigma^*$-extensions with depth-optimal $\Pi\Sigma^*$-extensions, the following theorem[9] summarizes one of the decisive differences.

Theorem 47. *[*[47, Result 2]*] Let $(\mathbb{E}, \sigma)$ be a $\Pi\Sigma^\delta$-extension of $(\mathbb{F}, \sigma)$. Any possible reordering (as a $\Pi\Sigma^*$-extension) is again a $\Pi\Sigma^\delta$-extension.*

Namely, if one adjoins a $\Pi\Sigma^\delta$-extension t on top of a $\Pi\Sigma^\delta$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ and if one reorganizes, e.g., this extension to an $\mathbb{F}$-ordered version, then this $\mathbb{F}$-ordered extension is again depth-optimal. This flexibility is completely different to reduced and complete-reduced $\Pi\Sigma^*$-extensions: as worked out in Algorithm 2 and illustrated in Example 37, one has to reorganize the whole difference field in order to get back an $\mathbb{F}$-ordered complete-reduced $\Pi\Sigma^*$-extension.

Example **48.** The $\Pi\Sigma^\delta$-extension $(\mathbb{Q}(k)(s_1)(s_3)(s_{1,3})(e)(s_6)(E)(s_2), \sigma)$ of $(\mathbb{Q}, \sigma)$ with (28) (see Example 45) can be rearranged, e.g., to the $\mathbb{Q}$-ordered $\Pi\Sigma^*$-extension $(\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(e)(s_6)(E), \sigma)$ of $(\mathbb{Q}, \sigma)$, which we constructed already in Example 37. Then due to Theorem 47 this extension is again a $\Pi\Sigma^\delta$-extension.

As an immediate consequence, we end up at structural properties which do not depend on the order of the extensions; compare, e.g., Corollary 34.

Theorem 49. *[$\Pi\Sigma^\delta$-structural theorem] Let $(\mathbb{E}, \sigma)$ be a $\Pi\Sigma^\delta$-extension of $(\mathbb{F}, \sigma)$. Then for any $f, g \in \mathbb{E}$ with* (5) *we have* (21)*. In particular, if $\mathbb{E} = \mathbb{F}(t_1)\dots(t_e)$ and*

$$S = \{1 \leq i \leq e | \delta_{\mathbb{F}}(t_i) = \delta_{\mathbb{F}}(f) + 1 \text{ and } t_i \text{ is a } \Sigma^*\text{-extension}\},$$

then (10) *for some $c, c_i \in \mathbb{K}$ and $w \in \mathbb{E}$ with $\delta_{\mathbb{F}}(w) \leq \delta_{\mathbb{F}}(f)$.*

Proof. Take any $\Pi\Sigma^\delta$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$. Then by Theorem 47 we can bring this extension to an $\mathbb{F}$-ordered $\Pi\Sigma^\delta$-extension of the form (23). By Lemma 43 the $\Pi\Sigma^*$-extension $(\mathbb{F}_j, \sigma)$ of $(\mathbb{F}_i, \sigma)$ is reduced for any $0 \leq i \leq j \leq d$. Hence by Theorem 33 the first part follows. The second part follows by Theorem 10. □

Example **50.** Take the $\Pi\Sigma^\delta$-field $(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{Q}(k)(s_1)(s_3)(s_{1,3})(e)(s_6)(E)(s_2)$ and with (28), and let $f \in \mathbb{E}$ with $\delta_{\mathbb{Q}}(f) = 2$. Then for any $g \in \mathbb{E}$ with (5) we have

$$g = w + c_1\, s_{1,3} + c_2\, e + c_3\, E \quad \text{for some } w \in \mathbb{Q}(k, s_1, s_3, s_2) \text{ and } c_1, c_2, c_3 \in \mathbb{Q}.$$

Combining this result with Theorem 15 we end up at the following refinement.

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[9]The proof of Thm. 47 relies on additional properties of $\Pi\Sigma^\delta$-extensions elaborated in [47].

Theorem 51. *[Refined $\Pi\Sigma^{\delta}$-structural theorem] Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma^{\delta}$-extension of $(\mathbb{F},\sigma)$ with $f\in\mathbb{E}$; suppose*[10] *that $\mathbb{E}=\mathbb{F}(s_1)\dots(s_u)(t_1)\dots(t_e)$ such that $\delta_{\mathbb{F}}(s_i)\leq\delta_{\mathbb{F}}(f)+1$ for all $1\leq i\leq u$ and such that $\delta_{\mathbb{F}}(t_i)>\delta_{\mathbb{F}}(f)+1$ for all $1\leq i\leq e$; let $\{x_1,\dots,x_r\}=\mathit{InnerNode}_{\mathbb{F}\leq\mathbb{F}(s_1)\dots(s_u)}(f)$. If there is a $g\in\mathbb{E}$ with* (5)*, then*

$$g=\sum_{a\in \mathit{SumLeaf}_{\mathbb{F}\leq\mathbb{F}(s_1)\dots(s_u)}(f)}c_a a+w\quad \text{for some } c_a\in\text{const}_{\sigma}\mathbb{F} \text{ and } w\in\mathbb{F}(x_1,\dots,x_r).$$

Example **52.** Take again the $\Pi\Sigma^{\delta}$-field $(\mathbb{E},\sigma)$ as in Example 50, and take on top the Σ^{δ}-extension $(\mathbb{E}(t),\sigma)$ of $(\mathbb{E},\sigma)$ with $\sigma(t)=t+\frac{\sigma(s_{1,3})\sigma(s_2)}{k+1}$; let

$$f=\frac{k^2\left(k^2(s_2+k(s_3+k(s_3(s_2+2s_6+3)+1)))-1\right)-2s_3}{k^7}$$

with $\delta_{\mathbb{Q}}(f)=2$. Following Theorem 51, we reorder the $\Pi\Sigma^{\delta}$-field to the $\mathbb{Q}$-ordered $\Pi\Sigma^{\delta}$-field $(\mathbb{D}(t),\sigma)$ with $\mathbb{D}=\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_6)(s_{1,3})(e)(E)$. Then we get $\text{InnerNode}_{\mathbb{Q}\leq\mathbb{D}}(f)=\{k,s_2,s_3\}$ and $\text{SumLeaf}_{\mathbb{F}\leq\mathbb{D}}(f)=\{s_1,s_{1,3},e,E\}$. Hence,

$$g=w+c_1\,s_1+c_2\,s_{1,3}+c_3\,e+c_4\,E\quad\text{for some } c_1,c_2,c_3,c_4\in\mathbb{Q} \text{ and } w\in\mathbb{Q}(k,s_2,s_3);$$

note that we could exclude t. Indeed, we find

$$g=s_1+3s_{1,3}+e+2\,E+\frac{s_2s_3k^7-(s_3(s_2+2s_6+3)+1)k^6-s_3k^5-s_2k^4+k^2+2s_3}{k^7}.$$

Note that these results lead to fine-tuned telescoping algorithms that enables one to handle efficiently a tower of up to 100 Σ^{δ}-extensions in the summation package Sigma; for an example from particle physics see [9]. Besides this, we emphasize

Theorem 53. *[*[47, Result 6]*] Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma^{\delta}$-ext. of $(\mathbb{F},\sigma)$; let $f\in\mathbb{E}$. If there is a $\Pi\Sigma^*$-extension $(\mathbb{H},\sigma)$ of $(\mathbb{F},\sigma)$ with $g\in\mathbb{H}$ s.t.* (5)*, then there is a Σ^{δ}-extension $(\mathbb{E}',\sigma)$ of $(\mathbb{F},\sigma)$ with a solution $g'\in\mathbb{E}'$ of* (5) *s.t. $\delta_{\mathbb{F}}(g')\leq\delta_{\mathbb{F}}(g)$.*

In short, Π-extensions are not needed to find a telescoping solution with optimal depth. This result is connected to Liouville's Theorem 1 where exponential extensions can be excluded if one looks for a solution of the integration problem.

Finally, we work out alternative characterizations as given in Theorems 20 and 31 for reduced and complete-reduced $\Pi\Sigma^*$-extensions. Here we need

Lemma 54. *Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with $f\in\mathbb{E}$. If there is a Σ^*-extension $(\mathbb{S},\sigma)$ of $(\mathbb{E},\sigma)$ with extension depth$\leq d$ such that there is a $g\in\mathbb{S}\setminus\mathbb{E}$ with* (5)*, then there is a Σ^*-extension $(\mathbb{S}'(s),\sigma)$ of $(\mathbb{E},\sigma)$ with extension depth$\leq d$ and with $\mathit{SumLeaf}_{\mathbb{E}\leq\mathbb{S}'(s)}=\{s\}$ such that there is a $w\in\mathbb{S}'$ with $\Delta(s+w)=f$.*

Proof. Subsequently we construct the desired extension from the given extension $(\mathbb{S},\sigma)$ of $(\mathbb{E},\sigma)$. Let $\text{SumLeaf}_{\mathbb{E}\leq\mathbb{S}}=\{s_1,\dots,s_r\}$. Then we can reorder the difference field $(\mathbb{S},\sigma)$ to $(\mathbb{E}(x_1)\dots(x_l)(s_1)\dots(s_r),\sigma)$ such that this is a Σ^*-extension of $(\mathbb{E},\sigma)$. W.l.o.g. we may assume that $g\notin\mathbb{E}(x_1)\dots(x_r)$: otherwise, we neglect the leaf extensions s_i and repeat the construction from above. If $r=1$, we are done. Otherwise, we continue as follows. Since $\Delta(s_i)\in\mathbb{E}(x_1)\dots(x_l)$ for $1\leq i\leq r$, $(\mathbb{E}(x_1)\dots(x_l)(s_1)\dots(s_r),\sigma)$ is a reduced Σ^*-extension of $(\mathbb{E}(x_1)\dots(x_l),\sigma)$ by Lemma 9. Applying Theorem 10 it follows that $g=w+\sum_{i=1}^{r}c_is_i$ for $c_i\in\text{const}_{\sigma}\mathbb{F}$ and $w\in\mathbb{E}(x_1)\dots(x_l)$; w.l.o.g. we may assume that $c_r\neq 0$, otherwise we reorder the extensions s_i accordingly. Define $\phi:=\sum_{i=1}^{r}c_i\Delta(s_i)\in\mathbb{E}(x_1)\dots(x_l)$.

[10] W.l.o.g. any extension can be brought to this form by Theorem 47.

Then observe that there is no $\gamma\in\mathbb{E}(x_1)\dots(x_l)$ such that $\Delta(\gamma)=\phi$. Otherwise, for $h:=(\gamma-\sum_{i=1}^{r-1}c_is_i)/c_r\in\mathbb{E}(x_1)\dots(x_l)(s_1)\dots(s_{r-1})$ we get $\Delta(h)=\Delta(s_r)$, and thus s_r is not a Σ^*-extension by Theorem 5; a contradiction. Consequently, we can apply Theorem 5 and construct the Σ^*-extension $(\mathbb{E}(x_1)\dots(x_l)(s),\sigma)$ of $(\mathbb{E}(x_1)\dots(x_l),\sigma)$ with $\sigma(s)=s+\phi$ and $\delta_{\mathbb{F}}(s)\leq d$. Note that for $h:=w+s\in\mathbb{E}(s_1)\dots(s_k)(s)$ we have $\Delta(h)=\Delta(g)=f$. If $\mathrm{SumLeaf}_{\mathbb{E}\leq\mathbb{E}(x_1)\dots(x_l)(s)}$ contains only s, we are done. Otherwise we repeat the construction from above. □

Theorem 55. *Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with $\mathbb{E}=\mathbb{F}(t_1)\dots(t_e)$. Then the following statements are equivalent:*

(1) This extension is depth-optimal.
(2) For any Σ^*-extension t_i with $f:=\Delta(t_i)\in\mathbb{F}(t_1)\dots(t_{i-1})$ and $1\leq i\leq e$ there does not exist a $\Pi\Sigma^*$-extension $(\mathbb{H},\sigma)$ of $(\mathbb{F}(t_1)\dots(t_{i-1}),\sigma)$ with extension depth $\leq\delta_{\mathbb{F}}(f)$ in which we find $g\in\mathbb{H}$ such that (5).
(3) For any Σ^*-extension $(\$,\sigma)$ of $(\mathbb{E},\sigma)$ with extension depth $\mathfrak{d}$ the following holds:

$$\forall f,g\in\$:\ \Delta g=f\ \wedge\ \delta_{\mathbb{F}}(f)\geq\mathfrak{d}\ \Rightarrow\ \delta_{\mathbb{F}}(g)\leq\delta_{\mathbb{F}}(f)+1. \tag{33}$$

Proof. **(1)⇔(2)** follows by Theorem 53. We show the implication **(1)⇒(3)**. Consider a Σ^*-extension $(\$,\sigma)$ of $(\mathbb{E},\sigma)$ with $\$=\mathbb{E}(s_1)\dots(s_r)$ such that $\delta_{\mathbb{F}}(s_i)\leq\mathfrak{d}$ for $1\leq i\leq r$; let $f,g\in\$$ with (5) and $\delta_{\mathbb{F}}(f)\geq\mathfrak{d}$. By Theorem 47 we may suppose that the $\Pi\Sigma^\delta$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ is ordered with $\mathbb{E}=\mathbb{H}(t_1)\dots(t_e)$ where $\delta_{\mathbb{F}}(\mathbb{H})=\mathfrak{d}$ and $\mathfrak{d}<\delta_{\mathbb{F}}(t_1)\leq\dots\leq\delta_{\mathbb{F}}(t_e)$; note that $f\in\mathbb{H}$. If $e=0$, nothing has to be shown. Otherwise, by reordering we get the $\Pi\Sigma^*$-extension $(\mathbb{H}(s_1)\dots(s_r)(t_1)\dots(t_e),\sigma)$ of $(\mathbb{H},\sigma)$. Now suppose that a Σ^*-extension t_l for some $1\leq l\leq e$ is not depth-optimal; set $\phi:=\Delta(t_l)$. Then there is a Σ^*-extension $(\mathbb{H}(s_1)\dots(s_r)(t_1)\dots(t_{l-1})(x_1)\dots(x_u),\sigma)$ of $\mathbb{H}(s_1)\dots(s_r)(t_1)\dots(t_{l-1})$ with $\delta_{\mathbb{F}}(x_i)\leq\delta_{\mathbb{F}}(\phi)$ for $1\leq i\leq u$ and $\gamma\in\mathbb{H}(s_1)\dots(s_r)(t_1)\dots(t_{l-1})(x_1)\dots(x_u)$ such that $\Delta(\gamma)=\phi$. Since $\delta_{\mathbb{F}}(t_e)>\mathfrak{d}$, we have $\delta_{\mathbb{F}}(\phi)\geq\mathfrak{d}$, and thus it follows that $(\mathbb{H}(t_1)\dots(t_{l-1})(s_1)\dots(s_r)(x_1)\dots(x_u),\sigma)$ is a Σ^*-extension of $(\mathbb{H}(t_1)\dots(t_{l-1}),\sigma)$ with extension depth $\leq\delta_{\mathbb{F}}(\phi)$. Hence $(\mathbb{H}(t_1)\dots(t_l),\sigma)$ is not a Σ^δ-extension of $(\mathbb{H}(t_1)\dots(t_{l-1}),\sigma)$, a contradiction. We conclude that $(\mathbb{H}(s_1)\dots(s_r)(t_1)\dots(t_e),\sigma)$ is a $\Pi\Sigma^\delta$-extension of $(\mathbb{H}(s_1)\dots(s_r),\sigma)$. In particular, it is a reduced extension of $(\mathbb{H},\sigma)$ by Lemma. 43. Hence by Thm. 10 g depends only on those t_i with $\Delta(t_i)\in\mathbb{H}$, i.e., $\delta_{\mathbb{F}}(t_i)\leq\mathfrak{d}+1$. Thus $\delta_{\mathbb{F}}(g)\leq\mathfrak{d}+1$.
Finally, we show the implication **(3)⇒(1)**. Suppose that the $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ with $\mathbb{E}=\mathbb{F}(t_1)\dots(t_e)$ is not depth-optimal. We may suppose that $\mathbb{E}$ is ordered, i.e., $\delta_{\mathbb{F}}(t_i)\leq\delta_{\mathbb{F}}(t_{i+1})$ for all i. Then there is a Σ^*-extension t_u with $f:=\Delta(t_u)$ and $\mathfrak{d}:=\delta_{\mathbb{F}}(f)$ with the following property: there is a Σ^*-extension $(\mathbb{F}(t_1)\dots(t_{u-1})(s_1)\dots(s_r),\sigma)$ of $(\mathbb{F}(t_1)\dots(t_{u-1}),\sigma)$ with $\delta_{\mathbb{F}}(s_i)\leq\mathfrak{d}$ and $f_i:=\Delta(s_i)$ for all i s.t. there is a $g\in\mathbb{F}(t_1)\dots(t_{u-1})(s_1)\dots(s_r)$ with (5); w.l.o.g. we may assume that $\delta_{\mathbb{F}}(s_1)\leq\dots\leq\delta_{\mathbb{F}}(s_r)$. Suppose we can adjoin all s_i as Σ^*-extensions to $\mathbb{F}(t_1)\dots(t_u)$: by reordering we get the Σ^*-ext. $(\mathbb{F}(t_1)\dots(t_{u-1})(s_1)\dots(s_r)(t_u),\sigma)$ of $(\mathbb{F},\sigma)$; since $g\in\mathbb{F}(t_1)\dots(t_{u-1})(s_1)\dots(s_r)$ with (5), t_u is not a Σ^*-extension by Theorem 5; a contradiction. Consequently there is a j with $1\leq j\leq r$ such that we can construct the Σ^*-extension $(\mathbb{E}(s_1)\dots(s_{j-1}),\sigma)$ of $(\mathbb{E},\sigma)$ with $\Delta(s_i)=f_i$ for $1\leq i<j$, but we fail to construct the Σ^*-extension s_j with $f_j=\Delta(s_j)$ on top. By Lemma 54 we can assume that $\delta_{\mathbb{F}}(s_1)\leq\dots\leq\delta_{\mathbb{F}}(s_{j-2})<\delta_{\mathbb{F}}(s_{j-1})\leq\mathfrak{d}$. Define $\mathfrak{d}':=\delta_{\mathbb{F}}(s_{j-1})$; note that $\delta_{\mathbb{F}}(f_j)=\mathfrak{d}'$. By the choice of j it follows with Thm. 5

that there is a $g' \in \mathbb{E}(s_1)\dots(s_{j-1})$ such that

$$\Delta(g') = f_j.$$

Since $(\mathbb{F}(t_1)\dots(t_{u-1})(s_1)\dots(s_j),\sigma)$ is a Σ^*-extension of $(\mathbb{F}(t_1)\dots(t_{u-1}),\sigma)$, $g' \notin \mathbb{F}(t_1)\dots(t_{u-1})(s_1)\dots(s_{j-1})$, i.e., g' depends on a t_i with $i \geq u$. Thus, $\delta_{\mathbb{F}}(g') \geq \delta_{\mathbb{F}}(t_u) = \delta_{\mathbb{F}}(f) + 1 > \mathfrak{d} \geq \delta_{\mathbb{F}}(s_j) = \delta_{\mathbb{F}}(f_j) + 1$. Hence, (33) does not hold. □

To sum up, the structural properties given in Theorems 49 and 51 are valid, even if one adjoins Σ^*-extensions (up to a certain depth) which are not depth-optimal (**(1)**⇒**(3)**). Conversely, exactly property **(3)** characterizes $\Pi\Sigma^\delta$-extensions in contrast to reduced and complete-reduced extensions; see Theorems 20 and 31.

8. Conclusion

Starting with Karr's structural theorem, we obtained various refined versions for reduced, complete-reduced and depth-optimal $\Pi\Sigma^*$-extensions. In particular we worked out one essential draw back of Karr's version of reduced $\Pi\Sigma^*$-extensions if one wants to reduce, e.g., the nested depth of sum expressions: his optimality depends on the order how the elements are adjoined in the field. In particular, if one reorders the tower of extensions w.r.t. the nested depth given by the shift-operator, Karr's structural theorem usually cannot be applied: only if the difference field is reorganized by expensive transformations, one gets back a reduced $\Pi\Sigma^*$-extension of the desired ordered shape; compare Theorem 40. In contrast to that, in the recently defined depth-optimal $\Pi\Sigma^*$-fields any possible reordering (as a $\Pi\Sigma^*$-field) gives again a depth-optimal $\Pi\Sigma^*$-field. As a consequence we could show structural properties that are independent of the extension order.

We emphasize that the presented theorems for the telescoping problem (1) can be immediately carried over to Zeilberger's creative telescoping paradigm [54] used for definite summation; for more details in the setting of $\Pi\Sigma^*$-fields we refer to [46]. More generally, we obtain structural results for parameterized telescoping. For illustrative purposes we rephrase Theorems 10 and 57 explicitly.

Theorem 56. *[Karr's structural theorem for parameterized telescoping] Let* $(\mathbb{E},\sigma)$ *be a reduced* $\Pi\Sigma^*$*-extension of* $(\mathbb{F},\sigma)$ *with* $\mathbb{E} = \mathbb{F}(t_1)\dots(t_e)$ *where* $\sigma(t_i) = a_i\,t_i + f_i$*, and define* S *by* (8)*; let* $f_1,\dots,f_n \in \mathbb{F}$*. If there are* $c_1,\dots,c_n \in \text{const}_\sigma\mathbb{F}$ *and* $g \in \mathbb{E}$ *such that the parameterized telescoping equation*

$$\Delta(g) = c_1\,f_1 + \dots + c_n\,f_n \tag{34}$$

holds, then there are $w \in \mathbb{F}$ *and* $c_i \in \text{const}_\sigma\mathbb{F}$ *such that* (9)*; in particular, for any such* g *there is some* $c \in \text{const}_\sigma\mathbb{F}$ *such that* (10)*.*

Theorem 57. *[*$\Pi\Sigma^\delta$*-structural theorem for parameterized telescoping] Let* $(\mathbb{E},\sigma)$ *be a* $\Pi\Sigma^\delta$*-extension of* $(\mathbb{F},\sigma)$*; let* $f_1,\dots,f_n$ *with* $d := \max(\delta_{\mathbb{F}}(f_1),\dots,\delta_{\mathbb{F}}(f_n))$*. Then for* $g \in \mathbb{E}$ *with* (34) *we have* $\delta_{\mathbb{F}}(g) \leq d+1$*. In particular, if* $\mathbb{E} = \mathbb{F}(t_1)\dots(t_e)$ *and*

$$S = \{1 \leq i \leq e | \delta_{\mathbb{F}}(t_i) = d+1 \text{ and } t_i \text{ is a } \Sigma^*\text{-extension}\},$$

then (10) *for some* $c, c_i \in \mathbb{K}$ *and* $w \in \mathbb{E}$ *with* $\delta_{\mathbb{F}}(w) \leq d$*.*

By concluding, we remark once more that Karr's structural theorem in [21, 22] (Theorem 10) is closely related to Liouville's Theorem (Theorem 1) and Rosenlicht's algebraic proof [38] in the language of differential fields. A natural question is how our new results can be carried over to the differential field case. A positive answer should throw new light on the differential theory of elementary extensions.

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