Computational Methods in an Algebra of Regular Hedge Expressions

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Abstract. We propose an algebra of regular hedge expressions built on top of regular hedge grammars as a framework for the analysis and manipulation of hedge languages. We show how linear systems of hedge language equations (LS for short) can be used as an intermediate representation on which to perform the computation of quotient, intersection, product derivative, and factor matrix of regular hedge languages. Regular hedge grammars and LSs are shown to be formalisms of same expressive power for the representation of hedge languages, and we give algorithms to convert between these two formalisms.

1 Introduction

Regular hedge languages are a natural generalization of regular word languages to the framework of sequences of trees, also known as *hedges* [8, 9]. They play an important role in theoretical computer science where they are best known as a formal model of XML schema [17, 21].

There are many equivalent ways to define regular hedge languages: by hedge automata [3, 6, 16], regular hedge expressions [16], regular expression types for XML [10, 11], F-regular sets [19], etc. In this paper we consider the characterization of regular hedge languages by regular hedge grammars [15] and investigate computational methods for their quotient, intersection, product derivative, and factor matrix. These notions are natural generalizations of the notions with the same name in the Kleene algebra of regular word languages [7, 13]. Computational methods for the quotient and intersection of regular word languages are well known, whereas algorithms for the computation of product derivative and factor matrix of regular languages are more recent developments. In [18], the product derivative is obtained from the computation of the greatest fixed point of a continuous operator which renders the result as a finite intersection of regular languages. Another approach was proposed in [14], where a characterization of the product derivative as finite intersection of regular languages is obtained from the least fixed point of a monotone operator defined on a finite lattice.

Algorithms for the computation of quotient, intersection, product derivative, and factor matrix of regular hedge languages are less known, although they can be very useful in XML typed programming and type safe XML transformations.

Our study of these notions is carried out in an algebra of regular hedge expressions built on top of regular hedge grammars. We identify a notion of linear system of hedge language equations (LS for short) with 3 important properties: (1) it has a unique solution; (2) the solution is a tuple of regular hedge languages; and (3) there is a simple algorithm that yields a representation of the solution by regular hedge grammars. Our main insights are:

- For every regular hedge grammar G we can compute a system of characteristic equations of the language generated by G, i.e., an LS that has the first element of its solution generated by G.
- Given a system of characteristic equations of L_1 and a system of characteristic equations of L_2, we can compute (1) systems of characteristic equations for quotient $L_1^{-1}L_2$ and intersection $L_1 \cap L_2$, and (2) a finite set $\mathfrak{N}$ of regular hedge grammars such that the product derivative of L_1 and L_2 is the intersection of the languages generated by the grammars in $\mathfrak{N}$.
- The computation of regular hedge grammars for the factor matrix of the language generated by a regular hedge grammar can be achieved through computations of regular hedge grammars for intersection and product derivative of languages generated by regular hedge grammars.

In this way, we reduce the computation of regular hedge grammars for quotient, intersection, product derivative, and factor matrix to the computation and solving of LSs. All computations are carried out in the same framework—that of regular hedge expressions,—and yield regular hedge grammars for the result.

The computation of systems of characteristic equations is attained in a differential calculus of regular hedge expressions produced by carrying over several notions from the theory of regular word expressions, such as partial derivatives and linear forms of regular hedge grammars [1]. We show that several properties of Antimirov's calculus of partial derivatives are preserved by our differential calculus. In this setting, linear systems of hedge language equations are a natural adaptation of the notion of system of linear equations over a Kleene algebra [13].

The paper is structured as follows. In Sect. 2 we recall basic notions and results from the theories of regular languages and regular hedge languages, define an algebra of regular hedge expressions, and introduce the notion of linear system of hedge language equations. In Sect. 3, we show that such a system has a unique solution which is a tuple of regular hedge languages, and give a method to compute a representation of the solution by regular hedge grammars. Section 4 presents the differential calculus for regular hedge expressions. Section 5 describes how the computation of quotient, intersection, and product derivative of two regular hedge languages can be reduced to solving LSs. The computation of regular hedge grammars for the left factors, right factors, and factor matrix of a regular hedge language is described in Sect. 6. Section 7 concludes.

2 Preliminaries

We recall some well known notions and results from the theory of lattices and partial orders that will be used throughout this paper. A *poset* is a pair $(\mathcal{U}, \leq)$ consisting of a set $\mathcal{U}$ and a reflexive partial order $\leq$ on $\mathcal{U}$. A subset $\mathcal{X}$ of $\mathcal{U}$ is *directed* if $\mathcal{X} \neq \emptyset$ and for all $x, y \in \mathcal{X}$ there exists $z \in \mathcal{X}$ such that $x \leq z$ and $y \leq z$. A poset $(\mathcal{U}, \leq)$ is a *complete partial order* (*cpo* for short) if $\mathcal{U}$ has a least element $\bot_{\mathcal{U}}$ and every directed subset $\mathcal{X}$ of $\mathcal{U}$ has a least upper bound $\bigvee \mathcal{X}$. A *complete lattice* is a poset $(\mathcal{U}, \leq)$ in which every subset $\mathcal{X}$ of $\mathcal{U}$ has a least upper bound (supremum) $\bigvee \mathcal{X}$ and a greatest lower bound (infimum) $\bigwedge \mathcal{X}$. Suppose $(\mathcal{U}, \leq)$ is a poset, and $F : \mathcal{U} \to \mathcal{U}$. F is *monotone* if $F(x) \leq F(y)$ whenever $x \leq y$. An element $x \in \mathcal{U}$ is a *fixed point* of F if $F(x) = x$. If F is monotone and $(\mathcal{U}, \leq)$ is a complete lattice, then F has a $\leq$-least fixed point μF, a $\leq$-greatest fixed point νF, and $\mu F = \bigwedge\{x \in \mathcal{U} \mid F(x) \leq x\}$, $\nu F = \bigvee\{x \in \mathcal{U} \mid x \leq F(x)\}$ [20]. If $(\mathcal{U}, \leq)$ is a cpo then F is *continuous* if $F(\bigvee \mathcal{X}) = \bigvee F(\mathcal{X})$ for all directed subsets $\mathcal{X}$ of $\mathcal{U}$. In this case, F has a $\leq$-least fixed point μF, and $\mu F = \bigvee\{F^n(\bot_{\mathcal{U}}) \mid n \in \mathbb{N}\}$.

From now on we assume given a finite set of symbols Σ, called an *alphabet*. We write Σ^* for the set of words over Σ, and ϵ for the empty word. A *hedge* over Σ is an expression of the set $\mathcal{H}(\Sigma)$ generated by the grammar

$$h ::= \epsilon \mid a\langle h\rangle\, h$$

where $a \in \Sigma$. A *tree* is an element of the set $\mathcal{T}(\Sigma) := \{a\langle h\rangle \mid a \in \Sigma,\ h \in \mathcal{H}(\Sigma)\}$.

A *language* (resp. *hedge language*) is an arbitrary set of words (resp. hedges). The *length* $|h|$ of a hedge h is defined by $|\epsilon| := 0$ and $|a\langle h_1\rangle\, h_2| := 1 + |h_2|$. The *size* $\|h\|$ of h is defined by $\|\epsilon\| := 0$ and $\|a\langle h_1\rangle\, h_2\| := 1 + \|h_1\| + \|h_2\|$.

The *sum* of two (hedge) languages R and S is their union $R \cup S$. Their *product* $R.S$ is the set $\{v\, w \mid v \in R, w \in S\}$. The *asterate* R^* is the set $\bigcup_{n\in\mathbb{N}} R^n$ where $R^0 := \{\epsilon\}$ and $R^{n+1} := R.R^n$. Also, if $H \subseteq \mathcal{H}(\Sigma)$ and $a \in \Sigma$, then $a\langle H\rangle$ is the hedge language $\{a\langle h\rangle \mid h \in H\}$.

The *quotient* of H with respect to a hedge h is $h^{-1}H := \{h' \mid h\, h' \in H\}$. Obviously, $\epsilon^{-1}H = H$ and $(a\langle h_1\rangle\, h_2)^{-1}H = h_2^{-1}(a\langle h_1\rangle^{-1}(H))$. Similarly, we define the *right quotient* $H\, h^{-1}$ as the set of hedges $\{h' \mid h'\, h \in H\}$. Both notions can be generalized as follows: if H_1, H_2 are hedge languages then the *left quotient of* H_1 *with respect to* H_2 is $H_2^{-1} H_1 := \{h \mid \exists h'.\ (h' \in H_2 \land h'\, h \in H_1)\}$ and the *right quotient of* H_1 *with respect to* H_2 is $H_1\, H_2^{-1} := \{h \mid \exists h'.\ (h' \in H_2 \land h\, h' \in H_1)\}$. Note that in general we have $H_2^{-1} H_1 = \bigcup_{h_2 \in H_2} h_2^{-1} H_1$ and $H_1\, H_2^{-1} = \bigcup_{h_2\in H_2} H_1\, h_2^{-1}$.

In the rest of this section, $\mathcal{K}$ denotes an element of the set $\{\mathcal{P}(\Sigma^*), \mathcal{P}(\mathcal{H}(\Sigma))\}$. We also consider the set $\mathcal{M}_{m,n}(\mathcal{K})$ of $m \times n$ matrices with elements from $\mathcal{K}$. We define the *sum* and *product* of such matrices as the standard extensions of the sum and product operations on $\mathcal{K}$. The *asterate* $\boldsymbol{L}^*$ of $\boldsymbol{L} \in \mathcal{M}_{n,n}(\mathcal{K})$ is defined by induction on n:

– If $n = 1$ then $\boldsymbol{L}$ is a 1×1 matrix (L) and $\boldsymbol{L}^* := (L^*)$.

– If $n > 1$ then we split $\boldsymbol{L}$ into four sub-matrices such that $\boldsymbol{L} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix}$ with $\boldsymbol{A}$ and $\boldsymbol{D}$ square matrices, and define

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix}^* := \begin{pmatrix} (\boldsymbol{A}^*.\boldsymbol{B}.\boldsymbol{D}^*.\boldsymbol{C})^*.\boldsymbol{A}^* & (\boldsymbol{A}^*.\boldsymbol{B}.\boldsymbol{D}^*.\boldsymbol{C})^*.\boldsymbol{A}^*.\boldsymbol{B}.\boldsymbol{D}^* \\ (\boldsymbol{D}^*.\boldsymbol{C}.\boldsymbol{A}^*.\boldsymbol{B})^*.\boldsymbol{D}^*.\boldsymbol{C}.\boldsymbol{A}^* & (\boldsymbol{D}^*.\boldsymbol{C}.\boldsymbol{A}^*.\boldsymbol{B})^*.\boldsymbol{D}^* \end{pmatrix}.$$

See, e.g., [7, Ch. 3] or [13] for the intuition behind the definition of $\boldsymbol{L}^*$ and why it does not depend on the splitting into four sub-matrices.

Let $\boldsymbol{L} \in \mathcal{M}_{m,n}(\mathcal{K})$, $\boldsymbol{C}, \boldsymbol{D} \in \mathcal{M}_{n,1}(\mathcal{K})$, $i \in \{1, \ldots, m\}$, and $j \in \{1, \ldots, n\}$. Then $\boldsymbol{L}_{i,j}$ denotes the element of $\boldsymbol{L}$ at position (i, j), $\boldsymbol{C}_i$ denotes the element of $\boldsymbol{C}$ at position $(i, 1)$, and $\boldsymbol{C} \subseteq \boldsymbol{D}$ indicates that $\boldsymbol{C}_i \subseteq \boldsymbol{D}_i$ for all $i \in \{1, \ldots, n\}$. We call the matrix $\boldsymbol{L}$ *ϵ-free* if none of its elements contains ϵ, and *linear* if all its elements are sets of words or hedges of length 1. A matrix is *constant* if all its elements are either $\emptyset$ of $\{\epsilon\}$. The distinguished matrices $\mathtt{E}, \mathtt{I} \in \mathcal{M}_{n,n}(\mathcal{K})$ are defined as follows: $\mathtt{E}$ is the matrix with all elements $\emptyset$, whereas $\mathtt{I}$ is the matrix with all elements $\emptyset$, except those on the diagonal, which are $\{\epsilon\}$.

We will make use of the well known fact that the algebras $(\mathcal{K}, \{\cup, ., {}^*, \emptyset, \{\epsilon\}\})$ and $(\mathcal{M}_{n,n}(\mathcal{K}), \{\cup, ., {}^*, \mathtt{E}, \mathtt{I}\})$ are Kleene algebras [12, 13].

Lemma 1. *Let $\boldsymbol{L} \in \mathcal{M}_{n,n}(\mathcal{K})$, and $\boldsymbol{C} \in \mathcal{M}_{n,1}(\mathcal{K})$. If $\boldsymbol{L}$ is ϵ-free then the equation $\boldsymbol{X} = \boldsymbol{C} \cup \boldsymbol{L}.\boldsymbol{X}$ has the unique solution $\boldsymbol{L}^*.\boldsymbol{C}$ in $\mathcal{M}_{n,1}(\mathcal{K})$.*

Proof. In this proof we write $\mathbf{0}$ for the matrix of $\mathcal{M}_{n,1}(\mathcal{K})$ with all elements equal to $\emptyset$, and consider the operator $\mathtt{L} : \mathcal{M}_{n,1}(\mathcal{K}) \to \mathcal{M}_{n,1}(\mathcal{K})$ defined by $\mathtt{L}(\boldsymbol{X}) := \boldsymbol{C} \cup \boldsymbol{L}.\boldsymbol{X}$ for all $\boldsymbol{X} \in \mathcal{M}_{n,1}(\mathcal{K})$. Note that $(\mathcal{M}_{n,1}(\mathcal{K}), \subseteq)$ is a cpo, and the solutions of the system of equations $\boldsymbol{X} = \boldsymbol{C} \cup \boldsymbol{L}.\boldsymbol{X}$ coincide with the fixed points of $\mathtt{L}$. Since

$$\mathtt{L}\left(\bigcup_{\boldsymbol{X} \in \mathcal{I}} \boldsymbol{X}\right) = \boldsymbol{C} \cup \boldsymbol{L}.\left(\bigcup_{\boldsymbol{X} \in \mathcal{I}} \boldsymbol{X}\right) = \bigcup_{\boldsymbol{X} \in \mathcal{I}} (\boldsymbol{C} \cup \boldsymbol{L}.\boldsymbol{X}) = \bigcup_{\boldsymbol{X} \in \mathcal{I}} \mathtt{L}(\boldsymbol{X})$$

whenever $\emptyset \neq \mathcal{I} \subseteq \mathcal{M}_{n,1}(\mathcal{K})$, we learn that $\mathtt{L}$ is continuous and thus it has the $\subseteq$-least fixed point $\bigcup_{i=0}^{\infty} \mathtt{L}^i(\mathbf{0})$. But $(\mathcal{M}_{n,n}(\mathcal{K}), \{\cup, ., {}^*, \mathtt{E}, \mathtt{I}\})$ is a Kleene algebra, therefore $\bigcup_{i=0}^{\infty} \mathtt{L}^i(\mathbf{0}) = (\bigcup_{i=0}^{\infty} \boldsymbol{L}).\boldsymbol{C} = \boldsymbol{L}^*.\boldsymbol{C}$. We prove by contradiction that $\boldsymbol{L}^*.\boldsymbol{C}$ is the only fixed point of $\mathtt{L}$.

If $\boldsymbol{Y} \in \mathcal{M}_{n,1}(\mathcal{K})$ were another fixed point, then $\boldsymbol{L}^*.\boldsymbol{C} \subsetneq \boldsymbol{Y}$ and $\boldsymbol{Y} = \boldsymbol{C} \cup \boldsymbol{L}.\boldsymbol{Y}$. Since $\boldsymbol{L}^*.\boldsymbol{C} \subsetneq \boldsymbol{Y}$ there exists a word $h \in \Sigma^*$ (if $\mathcal{K} = \mathcal{P}(\Sigma^*)$) or a hedge $h \in \mathcal{H}(\Sigma)$ (if $\mathcal{K} = \mathcal{P}(\mathcal{H}(\Sigma))$) of minimal length such that $h \in \boldsymbol{Y}_j \setminus (\boldsymbol{L}^*.\boldsymbol{C})_j$ for some $j \in \{1, \ldots, n\}$. Because $\boldsymbol{Y}_j = \boldsymbol{C}_j \cup \bigcup_{k=1}^{n} \boldsymbol{L}_{j,k}.\boldsymbol{Y}_k$ and $\boldsymbol{C}_j \subseteq (\boldsymbol{L}^*.\boldsymbol{C})_j$, we have:

$$h \in \boldsymbol{Y}_j \setminus (\boldsymbol{L}^*.\boldsymbol{C})_j \subseteq (\boldsymbol{C}_j \cup \bigcup_{k=1}^{n} \boldsymbol{L}_{j,k}.\boldsymbol{Y}_k) \setminus \boldsymbol{C}_j \subseteq \bigcup_{k=1}^{n} \boldsymbol{L}_{j,k}.\boldsymbol{Y}_k$$

and therefore there exists $l \in \{1, \ldots, n\}$ such that $h \in \boldsymbol{L}_{j,l}.\boldsymbol{Y}_l$. We learn that there exists $h_1 \in \boldsymbol{L}_{j,l}$ and $h' \in \boldsymbol{Y}_l$ such that $h = h_1\, h'$. Also,

$$h_1\, h' = h \notin (\boldsymbol{L}^*.\boldsymbol{C})_j = \boldsymbol{C}_j \cup \bigcup_{k=1}^{n} \boldsymbol{L}_{j,k}.(\boldsymbol{L}^*.\boldsymbol{C})_k \supseteq \boldsymbol{L}_{j,l}.(\boldsymbol{L}^*.\boldsymbol{C})_l \supseteq \{h_1\}.(\boldsymbol{L}^*.\boldsymbol{C})_l$$

implies $h' \notin (\boldsymbol{L}^*.\boldsymbol{C})_l$. Thus we have found a hedge h' of length $|h| - |h_1| < |h|$ such that $h' \in \boldsymbol{Y}_l \setminus (\boldsymbol{L}^*.\boldsymbol{C})_l$ for some $l \in \{1, \ldots, n\}$, which contradicts our choice of h. We conclude that there is no $\boldsymbol{Y} \in \mathcal{M}_{n,1}(\mathcal{K})$ such that $\boldsymbol{L}^*.\boldsymbol{C} \subsetneq \boldsymbol{Y}$ and $\boldsymbol{Y} = \boldsymbol{C} \cup \boldsymbol{L}.\boldsymbol{Y}$, hence $\boldsymbol{L}^*.\boldsymbol{C}$ is the unique solution of the equation $\boldsymbol{X} = \boldsymbol{C} \cup \boldsymbol{L}.\boldsymbol{X}$ in $\mathcal{M}_{n,1}(\mathcal{K})$. □

2.1 Linear systems of Hedge Language Equations

Let $X_1, \ldots, X_n$ be hedge language variables, and $\Sigma = \{a_1, \ldots, a_m\}$. The set $\mathcal{C}_\Sigma(X_1, \ldots, X_n)$ of *linear coefficients in* $X_1, \ldots, X_n$ *over* Σ is the set of subsets of $\{a_i\langle X_j\rangle \mid 1 \le i \le m, 1 \le j \le n\}$. A *linear system of hedge language equations* (LS for short) is a system of language equations of the form

$$\begin{cases} X_1 = C_1 \cup \bigcup_{j=1}^n L_{1,j}(X_1, \ldots, X_n).X_j \\ \quad\vdots \\ X_n = C_n \cup \bigcup_{j=1}^n L_{n,j}(X_1, \ldots, X_n).X_j \end{cases} \tag{1}$$

where $X_1, \ldots, X_n$ are hedge language variables, $C_i \in \{\emptyset, \{\epsilon\}\}$, $L_{i,j}(X_1, \ldots, X_n) \in \mathcal{C}_\Sigma(X_1, \ldots, X_n)$ for all $i, j \in \{1, \ldots, n\}$. A *solution* of (1) is an n-tuple of hedge languages $(H_1, \ldots, H_n)$ such that $H_i = C_i \cup \bigcup_{j=1}^n L_{i,j}(H_1, \ldots, H_n).H_j$ for all $i \in \{1, \ldots, n\}$.

The matriceal representation of system (1) is the matriceal language equation $\boldsymbol{X} = \boldsymbol{C} \cup \boldsymbol{L}.\boldsymbol{X}$ where

$$\boldsymbol{X} := \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}, \; \boldsymbol{C} := \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}, \; \boldsymbol{L} := \begin{pmatrix} L_{1,1}(X_1, \ldots, X_n) & \cdots & L_{1,n}(X_1, \ldots, X_n) \\ \vdots & \ddots & \vdots \\ L_{n,1}(X_1, \ldots, X_n) & \cdots & L_{n,n}(X_1, \ldots, X_n) \end{pmatrix}.$$

A *system of characteristic equations* of $L \subseteq \mathcal{H}(\Sigma)$ is an LS which has a solution L as first component.

Example 1. For any $a \in \Sigma$, $L \subseteq \mathcal{H}(\Sigma)$ and $n \in \mathbb{N}$, we define $a^0\langle L\rangle := L$ and $a^{n+1}\langle L\rangle := a\langle a^n\langle L\rangle\rangle$. Then the system of language equations

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} \emptyset \\ \{\epsilon\} \\ \{\epsilon\} \end{pmatrix} \cup \begin{pmatrix} \emptyset & \emptyset & b\langle X_1\rangle \cup a\langle X_2\rangle \\ \emptyset & \emptyset & c\langle X_2\rangle \\ \emptyset & \emptyset & \emptyset \end{pmatrix} . \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

is an LS with solution $(L_1, L_2, \{\epsilon\})$ with $L_1 = \{b^n\langle a\langle L_2\rangle\rangle \mid n \ge 0\}$ and $L_2 = \{c^m\langle \epsilon\rangle \mid m \ge 0\}$, because

$$\begin{pmatrix} \emptyset \\ \{\epsilon\} \\ \{\epsilon\} \end{pmatrix} \cup \begin{pmatrix} \emptyset & \emptyset & b\langle L_1\rangle \cup a\langle L_2\rangle \\ \emptyset & \emptyset & c\langle L_2\rangle \\ \emptyset & \emptyset & \emptyset \end{pmatrix} . \begin{pmatrix} L_1 \\ L_2 \\ \{\epsilon\} \end{pmatrix} = \begin{pmatrix} (b\langle L_1\rangle \cup a\langle L_2\rangle).\{\epsilon\} \\ \{\epsilon\} \cup c\langle L_2\rangle.\{\epsilon\} \\ \{\epsilon\} \cup \emptyset.\{\epsilon\} \end{pmatrix} = \begin{pmatrix} L_1 \\ L_2 \\ \{\epsilon\} \end{pmatrix}.$$

Also, we conclude that this LS is a system of characteristic equations of L_1. □

A general method for solving LSs will be presented in Sect. 3.

2.2 Regular Hedge Languages. Main Properties

In this paper we consider the characterization of regular hedge languages by regular hedge grammars [15]. Regular hedge grammars are the generative counterpart of nondeterministic finite hedge automata [6], and the conversion between these two formalisms is straightforward: People more familiar with hedge automata can simply read "automaton" instead of "grammar", "state" instead of "nonterminal", and "transition rule $a\langle s\rangle \to \mathtt{n}$" instead of "production $\mathtt{n} \to a\langle s\rangle$."

Regular hedge grammars make use of the notion of regular expression. The set $\mathcal{T}_{\mathrm{Reg}}(\Sigma,\mathcal{X})$ of *regular expressions* over an alphabet Σ and set of variables $\mathcal{X}$ is defined by the grammar

$$r ::= 0 \mid 1 \mid a \mid x \mid r+r \mid r\cdot r \mid r^{\star}$$

where $a \in \Sigma$ and $x \in \mathcal{X}$. To interpret such expressions, we assume given an *assignment* $\sigma : \mathcal{X} \to \Sigma^*$, and define the homomorphism $[\![\;]\!]_\sigma$ between algebras $(\mathcal{T}_{\mathrm{Reg}}(\Sigma,\mathcal{X}), \{+,\cdot,^\star,0,1\})$ and $(\mathcal{P}(\Sigma^*), \{\cup, ., ^*, \emptyset, \{\epsilon\}\})$ by $[\![0]\!]_\sigma := \emptyset$, $[\![1]\!]_\sigma := \{\epsilon\}$, $[\![a]\!]_\sigma := \{a\}$, $[\![r_1+r_2]\!]_\sigma := [\![r_1]\!]_\sigma \cup [\![r_2]\!]_\sigma$, $[\![r_1\cdot r_2]\!]_\sigma := [\![r_1]\!]_\sigma.[\![r_2]\!]_\sigma$, and $[\![r^\star]\!]_\sigma := ([\![r]\!]_\sigma)^*$. If $\mathcal{X} = \emptyset$ then $[\![\;]\!]_\sigma$ does not depend on σ; In this case we simply write $\mathcal{T}_{\mathrm{Reg}}(\Sigma)$ instead of $\mathcal{T}_{\mathrm{Reg}}(\Sigma,\emptyset)$ and $[\![r]\!]$ instead of $[\![r]\!]_\sigma$ for any $r \in \mathcal{T}_{\mathrm{Reg}}(\Sigma)$.

A language $L \subseteq \Sigma^*$ is *regular* if there exists $r \in \mathcal{T}_{\mathrm{Reg}}(\Sigma)$ such that $L = [\![r]\!]$.

The set of regular subexpressions of $r \in \mathcal{T}_{\mathrm{Reg}}(\Sigma,\mathcal{X})$ is

$$\mathcal{SE}(r) := \begin{cases} \{r\} & \text{if } r \in \Sigma \cup \mathcal{X} \cup \{0,1\},\\ \{r\} \cup \mathcal{SE}(r_1) \cup \mathcal{SE}(r_2) & \text{if } r = r_1 + r_2 \text{ or } r = r_1 \cdot r_2,\\ \{r\} \cup \mathcal{SE}(r_1) & \text{if } r = r_1^\star. \end{cases}$$

and the *size* of a regular expression is defined by: $\|0\| = \|1\| = \|a\| = \|x\| := 1$ for all $a \in \Sigma$ and $x \in \mathcal{X}$; $\|r_1+r_2\| = \|r_1\cdot r_2\| := 1+\|r_1\|+\|r_2\|$; and $\|r^\star\| := 1+\|r\|$.

Definition 1 (Regular hedge grammar). *A* regular hedge grammar *(RHG for short) is a tuple $G = \langle \mathcal{N}, \Sigma, \mathcal{P}, r\rangle$ where Σ is an alphabet, $\mathcal{N}$ is a finite set of nonterminals, $r \in \mathcal{T}_{\mathrm{Reg}}(\mathcal{N})$, and $\mathcal{P}$ is a finite set of productions of the form $\mathtt{n} \to a\langle s\rangle$ with $\mathtt{n} \in \mathcal{N}$, $a \in \Sigma$, and $s \in \mathcal{T}_{\mathrm{Reg}}(\mathcal{N})$, and such that there is at most one such rule in $\mathcal{P}$ for every $\mathtt{n} \in \mathcal{N}$ and $a \in \Sigma$. The regular expression r is called the* initial *of G and is denoted by $\mathtt{ini}(G)$.*

The hedge language *generated* by an RHG $G = \langle \mathcal{N}, \Sigma, \mathcal{P}, r\rangle$ from $s \in \mathcal{T}_{\mathrm{Reg}}(\mathcal{N})$ is $[\![s]\!]_G := \{h \mid G \vdash h \in s\}$ where the relation $G \vdash h \in s$ is defined inductively by

$$\frac{G \vdash w \rightsquigarrow h \quad [w \in [\![s]\!]]}{G \vdash h \in s}$$

$$\frac{}{G \vdash \epsilon \rightsquigarrow \epsilon} \qquad \frac{G \vdash h_1 \in s \quad G \vdash w \rightsquigarrow h_2 \quad [(\mathtt{n} \to a\langle s\rangle) \in \mathcal{P}]}{G \vdash \mathtt{n}\, w \rightsquigarrow a\langle h_1\rangle\, h_2}.$$

The hedge language *generated* by G is $[\![G]\!] := [\![\mathtt{ini}(G)]\!]_G$. A set of hedges is a *regular hedge language* (*RHL* for short) if it is generated by an RHG.

A regular expression $s \in \mathcal{T}_{\mathrm{Reg}}(\mathcal{N})$ is *productive* with respect to G if $[\![s]\!]_G \neq \emptyset$. An RHG $G = (\mathcal{N}, \Sigma, \mathcal{P}, r)$ is *productive* if $[\![G]\!] \neq \emptyset$. G is 0-*normalized* if

$$(\mathcal{SE}(r) \setminus \{r\}) \cup \bigcup_{\mathtt{n} \to a\langle s\rangle \in \mathcal{P}} \mathcal{SE}(s)$$

consists of regular expressions that are productive with respect to G.

Intuitively, the non-productive regular subexpressions in an RHG can be safely replaced with 0. To formalize this intuition, we define a monotone operator that can distinguish between the productive and non-productive nonterminals of an RHG. Let $G = \langle \mathcal{N}, \Sigma, \mathcal{P}, r\rangle$ be an RHG, $\mathcal{M} := \mathcal{N} \cup \bigcup_{\mathtt{n} \to a\langle s\rangle \in \mathcal{P}} \mathcal{SE}(s)$, and $\mathtt{P} : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{M})$ be the monotone operator on the complete lattice $(\mathcal{P}(\mathcal{M}), \subseteq)$, represented compactly by the inference rules

$$\frac{s \quad [\mathtt{n} \to a\langle s\rangle \in \mathcal{P}]}{\mathtt{n}} \qquad \frac{}{1} \qquad \frac{}{s^\star} \qquad \frac{r_1}{r_1 + r_2} \qquad \frac{r_2}{r_1 + r_2} \qquad \frac{r_1 \quad r_2}{r_1 \cdot r_2}$$

where each rule states that if the expressions above the bar are in the input set of $\mathtt{P}$, the conditions within square brackets hold, and the expression below the bar is in $\mathcal{M}$, then the expression below the bar is in the output set of $\mathtt{P}$.

Lemma 2. $\mu\mathtt{P} = \{s \in \mathcal{M} \mid s \textit{ is productive}\}$.

Proof. First, we prove the inclusion $\mu\mathtt{P} \subseteq \{s \in \mathcal{M} \mid s \text{ is productive}\}$. We show that every $s \in \mu\mathtt{P}$ is productive by induction on the structure of s.

If $s = \mathtt{n} \in \mathcal{N}$ then there exists $\mathtt{n} \to a\langle s'\rangle \in \mathcal{P}$ with $s' \in \mu\mathtt{P}$, and we can apply the induction hypothesis to infer $s' \in M$, that is, there exists a derivation $\mathcal{D}$ of $G \vdash h \in s'$ for some $h \in \mathcal{H}(\Sigma)$. Then we can construct the derivation

$$\cfrac{\cfrac{\mathcal{D} \qquad \cfrac{}{G \vdash \epsilon \rightsquigarrow \epsilon}}{G \vdash \mathtt{n} \rightsquigarrow a\langle h\rangle}}{G \vdash a\langle h\rangle \in \mathtt{n}}$$

which shows that $a\langle h\rangle \in [\![\mathtt{n}]\!]_G$, thus $\mathtt{n}$ is productive.

If $s = 1$ or $s = s_1^\star$ then we can construct the derivation

$$\frac{G \vdash \epsilon \rightsquigarrow \epsilon}{G \vdash \epsilon \in s}$$

which indicates that $\epsilon \in [\![s]\!]_G$ and thus s is productive.

If $s = r_1 + r_2 \in \mu\mathtt{P}$ then there exists $r_i \in \mu\mathtt{P}$ for some $r_i \in \{r_1, r_2\}$. Then r_i is productive by induction hypothesis, and thus there exists a derivation $\mathcal{D}$ of $G \vdash h \in r_i$ for some $h \in \mathcal{H}(\Sigma)$. Then $\mathcal{D}$ is of the form

$$\frac{\mathcal{D}'}{G \vdash h \in r_i}$$

for some derivation $\mathcal{D}'$ of $G \vdash w \rightsquigarrow h$ with $w \in [\![r_i]\!]$. Since $[\![r_i]\!] \subseteq [\![s]\!]$, we can show that s is productive by constructing the derivation

$$\frac{\mathcal{D}'}{G \vdash h \in s}$$

If $s = r_1 \cdot r_2 \in \mu\mathtt{P}$ then $r_1, r_2 \in \mu\mathtt{P}$. By induction hypothesis, r_1 and r_2 are productive, thus there exist derivations

$$\frac{\mathcal{D}_i}{G \vdash h_i \in r_i}$$

with $h_i \in \mathcal{H}(\Sigma)$, and

$$\mathcal{D}_i = \frac{\mathcal{D}_i'}{G \vdash w_i \rightsquigarrow h_i}$$

with $w_i \in [\![r_i]\!]$, for $i \in \{1, 2\}$. Then $w_1 w_2 \in [\![r_1]\!].[\![r_2]\!] = [\![s]\!]$ and we can show that s is productive by constructing the derivation

$$\frac{\mathtt{conc}(\mathcal{D}_1, \mathcal{D}_2)}{G \vdash h_1 h_2 \in s}$$

where

$$\mathtt{conc}\left(\frac{}{G \vdash \epsilon \rightsquigarrow \epsilon}, \mathcal{D}\right) := \mathcal{D},$$

$$\mathtt{conc}\left(\frac{\mathcal{D} \quad \mathcal{D}_1}{G \vdash \mathtt{n}\, w \rightsquigarrow a\langle h_0 \rangle\, h}, \frac{\mathcal{D}_2}{G \vdash w' \rightsquigarrow h'}\right) := \frac{\mathcal{D} \quad \mathtt{conc}(\mathcal{D}_1, \mathcal{D}_2)}{G \vdash \mathtt{n}\, w\, w' \rightsquigarrow a\langle h_0 \rangle\, h\, h'}.$$

Next, we prove the inclusion $\{s \in \mathcal{M} \mid s \text{ is productive}\} \subseteq \mu\mathtt{P}$. Let $s \in \mathcal{M}$ be productive. We show that $s \in \mu\mathtt{P}$ by induction on the complexity measure $(l(s), \|s\|)$ of s ordered lexicographically, where $l(s)$ is the size of a shortest possible derivation of a relation $G \vdash h \in s$, and $\|s\|$ is the size of s. If $s = 1$ or $s = s_1^\star$ then $\epsilon \in [\![s]\!]$ and the shortest possible derivation of a relation $G \vdash h \in s$ is

$$\frac{G \vdash \epsilon \rightsquigarrow \epsilon}{G \vdash \epsilon \in s}.$$

Since $\{1, s_1^\star\} \subseteq \mathtt{P}(\emptyset)$, we conclude that $s \in \mu\mathtt{P}$.

If $s = \mathtt{n} \in \mathcal{N}$ then there exists a production $\mathtt{n} \to a\langle s' \rangle \in \mathcal{P}$ and a shortest possible derivation

$$\cfrac{\cfrac{\mathcal{D} \quad \cfrac{}{G \vdash \epsilon \rightsquigarrow \epsilon}}{G \vdash \mathtt{n} \rightsquigarrow a\langle h \rangle}}{G \vdash a\langle h \rangle \in \mathtt{n}}$$

with $\mathcal{D}$ a derivation of $G \vdash h \in s'$. This shows that $s' \in \mathcal{M}$ and $(l(s'), \|s'\|) < (l(\mathtt{n}), \|\mathtt{n}\|)$. By induction hypothesis we have $s' \in \mu\mathtt{P}$, and the first defining inference rule of $\mathtt{P}$ yields $\mathtt{n} \in \mu\mathtt{P}$ too.

If $s = r_1 + r_2$ then a shortest possible derivation is

$$\frac{\mathcal{D}}{G \vdash h \in r_1 + r_2}$$

with $\mathcal{D}$ a shortest derivation of $G \vdash w \rightsquigarrow h$ for some $w \in [\![r_1 + r_2]\!] = [\![r_1]\!] \cup [\![r_2]\!]$. Then $w \in [\![r_i]\!]$ for some $r_i \in \{r_1, r_2\}$, and we can construct the derivation

$$\frac{\mathcal{D}}{G \vdash h \in r_i}$$

which shows that r_i is productive with $l(r_i) \leq l(s)$ and $\|r_i\| < \|s\|$. By induction hypothesis, we get $r_i \in \mu(\mathtt{P})$, hence $r_1 + r_2 \in \mu\mathtt{P}$ too.

If $s = r_1 \cdot r_2$ then a shortest possible derivation is

$$\frac{\dfrac{\mathcal{D}}{G \vdash w \rightsquigarrow h}}{G \vdash h \in r_1 \cdot r_2}$$

with $w \in [\![r_1 \cdot r_2]\!] = [\![r_1]\!].[\![r_2]\!]$, thus $w = w_1 w_2$ with $w_1 \in [\![r_1]\!]$ and $w_2 \in [\![r_2]\!]$. Let k be the length of w_1. Then there exist $\mathtt{n}_1, \ldots, \mathtt{n}_k \in \mathcal{N}$ and $a_1\langle h'_1\rangle, \ldots, a_k\langle h'_k\rangle, h_2 \in \mathcal{H}(\Sigma)$ such that $w_1 = \mathtt{n}_1 \ldots \mathtt{n}_k$, $h = a_1\langle h'_1\rangle \ldots a_k\langle h'_k\rangle\, h_2$, and the derivations

$$\mathcal{D}''_2 = \frac{\mathcal{D}'''_2}{G \vdash w_2 \rightsquigarrow h_2} \quad \text{and} \quad \mathcal{D}_j = \frac{\mathcal{D}'_j}{G \vdash \mathtt{n}_j \rightsquigarrow a_j\langle h'_j\rangle} \qquad (1 \leq j \leq k)$$

are subderivations of

$$\mathcal{D}' = \frac{\mathcal{D}}{G \vdash w \rightsquigarrow h}.$$

Let $h_1 = a_1\langle h'_1\rangle \ldots a_k\langle h'_k\rangle$. We can assemble the derivations $\mathcal{D}_1, \ldots, \mathcal{D}_k$ into a derivation $\mathcal{D}''_1$ of $G \vdash w_1 \rightsquigarrow h_1$ such that the length of $\mathcal{D}''_1$ is smaller than or equal to the length of $\mathcal{D}'$. Since $\|r_1\| < \|s\|$ and

$$\frac{\mathcal{D}''_1}{G \vdash h_1 \in r_1}$$

is a derivation of length not bigger than the length of

$$\frac{\mathcal{D}'}{G \vdash h \in r_1 \cdot r_2},$$

we learn by induction hypothesis that $r_1 \in \mu\mathtt{P}$. Also,

$$\frac{\mathcal{D}''_2}{G \vdash h_2 \in r_2}$$

is of length not bigger than the length of

$$\frac{\mathcal{D}'}{G \vdash h \in r_1 \cdot r_2},$$

and therefore we learn by induction hypothesis that $r_2 \in \mu\mathtt{P}$. Thus both $r_1, r_2 \in \mu\mathtt{P}$, hence $r_1 \cdot r_2 \in \mu\mathtt{P}$ too. □

Note that the set $\mu\mathtt{P}$ is decidable because the domain of the monotone operator $\mathtt{P}$ is a finite set. By Lemma 2, we can assume the existence of an algorithm that computes the set of non-productive nonterminals of G. We will make use of this algorithm in the proof of Lemma 3, which indicates that non-productive nonterminals and non-productive subexpressions can be eliminated from G.

Lemma 3. *For every RHG G there exists a 0-normalized RHG G' such that $[\![G]\!] = [\![G']\!]$.*

Proof. Let $G = \langle \mathcal{N}, \Sigma, \mathcal{P}, r\rangle$, and $\mathfrak{R}$ be the confluent and terminating TRS

$$\begin{aligned}\mathfrak{R} :=& \mathfrak{R}_0 \cup \{\mathtt{n} \to 0 \mid \mathtt{n} \in \mathcal{N} \text{ is non-productive}\}, \text{where}\\ \mathfrak{R}_0 :=& \{0 + x \to x,\ x + 0 \to x,\ 0 \cdot x \to 0,\ x \cdot 0 \to 0,\ 0^\star \to 1\}.\end{aligned}$$

We write $s{\downarrow_\mathfrak{R}}$ for the normal form of a regular expression $s \in \mathcal{T}_{\text{Reg}}(\mathcal{N})$ with respect to $\mathfrak{R}$, and define the RHG $G' = \langle \mathcal{N}', \Sigma, \mathcal{P}', r'\rangle$ where

- $\mathcal{N}' = \{\mathtt{n} \in \mathcal{N} \mid \mathtt{n} \text{ is productive}\}$,
- $\mathcal{P}' = \{\mathtt{n} \to a\langle s{\downarrow_\mathfrak{R}}\rangle \mid (\mathtt{n} \to a\langle s\rangle) \in \mathcal{P} \wedge \mathtt{n} \in \mathcal{N}' \wedge s{\downarrow_\mathfrak{R}} \neq 0\}$,
- $r' = r{\downarrow_\mathfrak{R}}$.

Then $[\![G']\!] = [\![G]\!]$ because:

1. the replacement with 0 of non-productive nonterminals in the production rules and initial regular expression of an RHG is language preserving, and
2. the rewrite rules of $\mathfrak{R}$ applied to the production rules and the initial regular expression of an RHG are language preserving too.

Moreover, normal forms with respect to $\mathfrak{R}$ can not contain 0 as proper subexpression, and for every $\mathtt{n} \in \mathcal{N}'$ there exists at least one production $\mathtt{n} \to a\langle s\rangle \in \mathcal{P}'$ because $\mathtt{n}$ is productive. Thus, G' is 0-normalized. □

Example 2. Let $G = \langle\{\mathtt{n}_1, \mathtt{n}_2, \mathtt{n}_3\}, \{a, b, c\}, \mathcal{P}, (\mathtt{n}_1 + \mathtt{n}_2)^\star \cdot (\mathtt{n}_1 + \mathtt{n}_3)\rangle$ with

$$\mathcal{P} = \{\mathtt{n}_1 \to a\langle \mathtt{n}_2 \cdot \mathtt{n}_3^\star + \mathtt{n}_1^\star\rangle, \mathtt{n}_1 \to c\langle \mathtt{n}_3 \cdot \mathtt{n}_3\rangle, \mathtt{n}_2 \to b\langle \mathtt{n}_3\rangle, \mathtt{n}_3 \to c\langle \mathtt{n}_2 + \mathtt{n}_3\rangle\}.$$

In this case, the least fixed point of $\mathtt{P}$ is $\mu\mathtt{P} = \{\mathtt{n}_3^\star, \mathtt{n}_1^\star, \mathtt{n}_2 \cdot \mathtt{n}_3^\star + \mathtt{n}_1^\star, \mathtt{n}_1\}$, thus the only productive nonterminal is $\mathtt{n}_1$. We consider the confluent and terminating TRS $\mathfrak{R} := \mathfrak{R}_0 \cup \{\mathtt{n}_2 \to 0, \mathtt{n}_3 \to 0\}$ and compute

$$\begin{aligned}\mathcal{N}' :=& \{\mathtt{n}_1\},\\ \mathcal{P}' :=& \{\mathtt{n}_1 \to a\langle(\mathtt{n}_2 \cdot \mathtt{n}_3^\star + \mathtt{n}_1^\star){\downarrow_\mathfrak{R}}\rangle\} = \{\mathtt{n}_1 \to a\langle \mathtt{n}_1^\star\rangle\},\\ r' :=& ((\mathtt{n}_1 + \mathtt{n}_2)^\star \cdot (\mathtt{n}_1 + \mathtt{n}_3)){\downarrow_\mathfrak{R}} = \mathtt{n}_1^\star \cdot \mathtt{n}_1.\end{aligned}$$

Then $G' := \langle \mathcal{N}', \{a, b, c\}, \mathcal{P}', r'\rangle$ is a 0-normalized RHG and $[\![G']\!] = [\![G]\!]$. □

Because of Lemma 3, we can always assume that an RHL is generated by a 0-normalized RHG. From now on we denote by $\mathcal{G}(\mathcal{N}, \Sigma)$ the set of 0-normalized RHGs with nonterminals from $\mathcal{N}$ over alphabet Σ, and represent RHLs by 0-normalized RHGs.

2.3 Regular Hedge Expressions

Assume $\mathcal{X}$ is a set of variables, disjoint from Σ. The set $\mathcal{T}_{\text{HReg}}(\Sigma, \mathcal{X})$ of *regular hedge expressions* over Σ and a $\mathcal{X}$ is defined recursively by the grammar

$$\mathtt{H} ::= 0 \mid 1 \mid x \mid G \mid a\langle \mathtt{H} \rangle \mid \mathtt{H} + \mathtt{H} \mid \mathtt{H} \cdot \mathtt{H} \mid \mathtt{H}^\star$$

where $x \in \mathcal{X}$ and $G \in \mathcal{G}(\mathcal{N}, \Sigma)$ for some $\mathcal{N}$. An *assignment* is a mapping $\sigma : \mathcal{X} \to \mathcal{P}(\mathcal{H}(\Sigma))$. The hedge language $[\![\mathtt{H}]\!]_\sigma$ denoted by $\mathtt{H} \in \mathcal{T}_{\text{HReg}}(\Sigma, \mathcal{X})$ under assignment σ is defined by $[\![0]\!]_\sigma := \emptyset$, $[\![1]\!]_\sigma := \{\epsilon\}$, $[\![x]\!]_\sigma := \sigma(x)$, $[\![a\langle \mathtt{H} \rangle]\!]_\sigma := a\langle [\![\mathtt{H}]\!]_\sigma \rangle$, $[\![\mathtt{H}_1 \cdot \mathtt{H}_2]\!]_\sigma := [\![\mathtt{H}_1]\!]_\sigma . [\![\mathtt{H}_2]\!]_\sigma$, $[\![\mathtt{H}_1 + \mathtt{H}_2]\!]_\sigma := [\![\mathtt{H}_1]\!]_\sigma \cup [\![\mathtt{H}_2]\!]_\sigma$, and $[\![\mathtt{H}^\star]\!]_\sigma := [\![\mathtt{H}]\!]_\sigma^*$. This definition renders $[\![\ ]\!]_\sigma : \mathcal{T}_{\text{HReg}}(\Sigma, \mathcal{X}) \to \mathcal{P}(\mathcal{H}(A))$ a homomorphism of algebras $(\mathcal{T}_{\text{HReg}}(\Sigma), \{+, \cdot, ^\star, 0, 1\} \cup \{a\langle\ \rangle \mid a \in \Sigma\})$ and $\mathcal{A}_{\text{HReg}}(\Sigma) := (\mathcal{P}(\mathcal{H}(\Sigma)), \{\cup, ., ^*, \emptyset, \{\epsilon\}\} \cup \{a\langle\ \rangle \mid a \in \Sigma\})$ for any assignment σ. We write $\mathtt{H}_1 \doteq_\sigma \mathtt{H}_2$ if $[\![\mathtt{H}_1]\!]_\sigma = [\![\mathtt{H}_2]\!]_\sigma$. We say that $\mathtt{H}_1, \mathtt{H}_2 \in \mathcal{T}_{\text{HReg}}(\Sigma, \mathcal{X})$ are *equivalent*, and write $\mathtt{H}_1 \doteq \mathtt{H}_2$, if $\mathtt{H}_1 \doteq_\sigma \mathtt{H}_2$ for every assignment σ. If $\mathcal{X} = \emptyset$ then $[\![\mathtt{H}]\!]_\sigma$ does not depend on σ. In this case we simply write $[\![\mathtt{H}]\!]$ instead of $[\![\mathtt{H}]\!]_\sigma$, and $\mathcal{T}_{\text{HReg}}(\Sigma)$ instead of $\mathcal{T}_{\text{HReg}}(\Sigma, \emptyset)$.

Lemma 4. $[\![\mathtt{H}]\!]$ *is an RHL for every* $\mathtt{H} \in \mathcal{T}_{\text{HReg}}(\Sigma)$.

Proof. We prove by induction on the structure of $\mathtt{H}$ that there exists $G \in \mathcal{G}(\mathcal{N}, \Sigma)$ for some $\mathcal{N}$ such that $G \doteq \mathtt{H}$. If $\mathtt{H} \in \{0, 1\}$ then we can choose $G = \langle \emptyset, \Sigma, \emptyset, \mathtt{H} \rangle$.

If $\mathtt{H} = a\langle \mathtt{H}_1 \rangle$ then by induction hypothesis there exists $G_1 = \langle \mathcal{N}, \Sigma, \mathcal{P}, r_1 \rangle$ with $G_1 \doteq \mathtt{H}_1$. Let $G = \langle \mathcal{N}, \Sigma \cup \{\mathtt{n}\}, \mathcal{P} \cup \{\mathtt{n} \to a\langle r_1 \rangle\}, \mathtt{n} \rangle$ where $\mathtt{n}$ is a fresh symbol. Then $G \in \mathcal{G}(\mathcal{N} \cup \{\mathtt{n}\}, \Sigma)$ and $G \doteq \mathtt{H}$.

If $\mathtt{H} = \mathtt{H}_1 \circ \mathtt{H}_2$ with $\circ \in \{+, \cdot\}$ then by induction hypothesis there exist $\langle \mathcal{N}, \Sigma_i, \mathcal{P}_i, r_i \rangle \in \mathcal{G}(\mathcal{N}_i, \Sigma)$ such that $G_i \doteq \mathtt{H}_i$ for $i \in \{1, 2\}$. Let $G_2' = \langle \mathcal{N}_2', \Sigma, \mathcal{P}_2', r_2' \rangle$ be the result of renaming the nonterminals of G_2 with fresh nonterminals. Then $\mathcal{N}_1 \cap \mathcal{N}_2' = \emptyset$, $[\![G_2']\!] = [\![G_2]\!]$, and we can construct $G := \langle \mathcal{N}_1 \cup \mathcal{N}_2', \Sigma, \mathcal{P}_1 \cup \mathcal{P}_2', (r_1 \circ r_2')\downarrow_{\mathfrak{R}_0} \rangle \in \mathcal{G}(\mathcal{N}_1 \cup \mathcal{N}_2', \Sigma)$ such that $G \doteq \mathtt{H}$.

It is worth noticing that if $\mathtt{n} \to a\langle s \rangle \in \mathcal{P}_1$ implies $\mathtt{n} \to a\langle s \rangle \in \mathcal{P}_2$ for all $\mathtt{n} \in \mathcal{N}_1 \cap \mathcal{N}_2$, then we can construct directly $G := \langle \mathcal{N}_1 \cup \mathcal{N}_2, \Sigma, \mathcal{P}_1 \cup \mathcal{P}_2, (r_1 \circ r_2)\downarrow_{\mathfrak{R}_0} \rangle$ such that $G \doteq \mathtt{H}_1 \circ \mathtt{H}_2$.

If $\mathtt{H} = \mathtt{H}_1^\star$ then we can assume by induction hypothesis that $\mathtt{H}_1 \doteq G_1$ for some $G_1 = \langle \mathcal{N}, \Sigma, \mathcal{P}, r_1 \rangle$. Then $G = \langle \mathcal{N}, \Sigma, \mathcal{P}, r_1^\star \rangle \in \mathcal{G}(\mathcal{N}, \Sigma)$ and $G \doteq \mathtt{H}$. □

For $\mathcal{R} \in \{\mathcal{T}_{\text{Reg}}(\Sigma, \mathcal{X}), \mathcal{T}_{\text{HReg}}(\Sigma, \mathcal{X})\}$ we also consider the set $\mathcal{M}_{m,n}(\mathcal{R})$ of $m \times n$ matrices with elements from $\mathcal{R}$, on which we define the operations "+" and "·" as the standard extensions of the operations "+" and "·" on $\mathcal{R}$. The asterate $\boldsymbol{L}^\star$ of $\boldsymbol{L} \in \mathcal{M}_{n,n}(\mathcal{R})$ is defined like in the case when $\boldsymbol{L} \in \mathcal{M}_{n,n}(\mathcal{K})$ by using "+" instead of "∪" and "·" instead of ".". For $\boldsymbol{L} \in \mathcal{M}_{m,n}(\mathcal{R})$ we define the language matrix $[\![\boldsymbol{L}]\!]$ by $[\![\boldsymbol{L}]\!]_{i,j} := [\![\boldsymbol{L}_{i,j}]\!]$ for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

3 Solving Linear Systems of Hedge Language Equations

In this section we prove one of our main results: Every linear system of hedge language equations has a unique solution, and the solution is a tuple of regular hedge languages. Let $\boldsymbol{X} = \boldsymbol{C} \cup \boldsymbol{L}.\boldsymbol{X}$ with

$$\boldsymbol{X} := \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}, \ \boldsymbol{C} := \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}, \ \boldsymbol{L} := \begin{pmatrix} L_{1,1}(X_1,\ldots,X_n) & \cdots & L_{1,n}(X_1,\ldots,X_n) \\ \vdots & \ddots & \vdots \\ L_{n,1}(X_1,\ldots,X_n) & \cdots & L_{n,n}(X_1,\ldots,X_n) \end{pmatrix}.$$

be an LS over alphabet $\Sigma = \{a_1, \ldots, a_m\}$, and suppose $(H_1, \ldots, H_n)$ is a solution of the LS. This means that if we define

$$\boldsymbol{H} := \begin{pmatrix} H_1 \\ \vdots \\ H_n \end{pmatrix}, \ \boldsymbol{C} := \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}, \ \boldsymbol{S} := \begin{pmatrix} L_{1,1}(H_1,\ldots,H_n) & \cdots & L_{1,n}(H_1,\ldots,H_n) \\ \vdots & \ddots & \vdots \\ L_{n,1}(H_1,\ldots,H_n) & \cdots & L_{n,n}(H_1,\ldots,H_n) \end{pmatrix}$$

then $\boldsymbol{H} = \boldsymbol{C} \cup \boldsymbol{S}.\boldsymbol{H}$.

We consider the sets $\mathcal{N} := \{\mathtt{n}_{k,l} \mid k \in \{1,\ldots,m\}, l \in \{1,\ldots,n\}\}$ and $\mathcal{X} := \{x_j \mid 1 \le j \le n\}$ of distinct fresh symbols, and define $\mathsf{S} \in \mathcal{M}_{n,n}(\mathcal{T}_{\mathrm{Reg}}(\emptyset,\mathcal{N}))$, $\mathsf{C} \in \mathcal{M}_{n,1}(\mathcal{T}_{\mathrm{Reg}}(\emptyset))$, $\mathsf{H} \in \mathcal{M}_{n,1}(\mathcal{T}_{\mathrm{Reg}}(\emptyset,\mathcal{X}))$, and $\sigma : \mathcal{N} \cup \mathcal{X} \to \mathcal{P}(\mathcal{H}(\Sigma))$ with

$$\begin{aligned}
\mathsf{S}_{i,j} &:= \sum_{a_k\langle X_l\rangle \in L_{i,j}(X_1,\ldots,X_n)} \mathtt{n}_{k,l} && (1 \le i,j \le n)\\
\mathsf{C}_i &:= 0 \text{ if } C_i = \emptyset \text{ and } 1 \text{ if } C_i = \{\epsilon\} && (1 \le i \le n)\\
\mathsf{H}_i &:= x_i && (1 \le i \le n)\\
\sigma(\mathtt{n}_{k,l}) &:= a_k\langle H_l\rangle && (1 \le k \le m,\ 1 \le l \le n)\\
\sigma(x_j) &:= H_j && (1 \le j \le n).
\end{aligned}$$

We call S and C the *regular matrix* and *constant vector* of the LS, respectively. Then $[\![\ ]\!]_\sigma : \mathcal{T}_{\mathrm{Reg}}(\emptyset,\mathcal{N}\cup\mathcal{X}) \to \mathcal{P}(\mathcal{H}(\Sigma))$ is a morphism of algebras such that $[\![\mathsf{C}]\!]_\sigma = \boldsymbol{C}$, $[\![\mathsf{S}]\!]_\sigma = \boldsymbol{S}$, and $[\![\mathsf{H}]\!]_\sigma = \boldsymbol{H}$. Therefore

$$[\![\mathsf{H}]\!]_\sigma = \boldsymbol{H} = \boldsymbol{C} \cup \boldsymbol{S}.\boldsymbol{H} = [\![\mathsf{C}]\!]_\sigma \cup [\![\mathsf{S}]\!]_\sigma.[\![\mathsf{H}]\!]_\sigma = [\![\mathsf{C} + \mathsf{S}\cdot\mathsf{H}]\!]_\sigma,$$

and we learn that both $[\![\mathsf{H}]\!]_\sigma$ and $[\![\mathsf{S}^\star \cdot \mathsf{C}]\!]_\sigma$ are solutions of the matriceal equation $\boldsymbol{X} = \boldsymbol{C} \cup \boldsymbol{S}.\boldsymbol{X}$. Note that $\boldsymbol{S}$ is ϵ-free because all its elements are subsets of $\bigcup_{k=1}^m \bigcup_{l=1}^n a_k\langle H_l\rangle$ which is a set of trees. By Lemma 1, we get $\mathsf{H} \doteq_\sigma \mathsf{S}^\star \cdot \mathsf{C}$. Let $r_i := (\mathsf{S}^\star \cdot \mathsf{C})_i$ for $1 \le i \le n$. Then the following relations hold:

1. $H_i = [\![r_i]\!]_\sigma$ for all $i \in \{1,\ldots,n\}$,
2. $\mathtt{n}_{k,l} \doteq_\sigma a_k\langle r_l\rangle$ for all $\mathtt{n}_{k,l} \in \mathcal{N}$.

These relations indicate that $H_i = [\![G_i]\!]$ where $G_i = \langle \mathcal{N}, \Sigma, \mathcal{P}, r_i\rangle$ with

$$\mathcal{P} := \{\mathtt{n}_{k,l} \to a_k\langle r_l\rangle \mid k \in \{1,\ldots,m\}, l \in \{1,\ldots,n\}\}$$

for all $i \in \{1,\ldots,n\}$. We have ended up with a construction of RHGs for all the elements of a solution of the LS. This shows that the solution is a tuple of RHLs. Moreover, since the RHGs G_i $(1 \le i \le n)$ are uniquely defined, we conclude that the solution of the LS is unique.

Example 3. Consider the LS

$$X = a_1\langle X\rangle.X \cup a_2\langle X\rangle.X.$$

In this case we construct the sets $\mathcal{N} := \{\mathtt{n}_{1,1}, \mathtt{n}_{2,1}\}$, $\mathcal{X} := \{x_1\}$. The regular matrix and constant vector of this LS are

$$\mathsf{S} = (\mathtt{n}_{1,1} + \mathtt{n}_{2,1}) \in \mathcal{M}_{1,1}(\mathcal{T}_{\mathrm{Reg}}(\emptyset, \mathcal{N})) \text{ and } \mathsf{C} = (0) \in \mathcal{M}_{1,1}(\mathcal{T}_{\mathrm{Reg}}(\emptyset)).$$

Since $\mathsf{S}^\star \cdot \mathsf{C} = (0)$, we obtain the RHG

$$G_1 = \langle \{\mathtt{n}_{1,1}, \mathtt{n}_{2,1}\}, \{a_1, a_2\}, \{\mathtt{n}_{1,1} \to a_1\langle 0\rangle, \mathtt{n}_{2,1} \to a_2\langle 0\rangle\}, 0\rangle$$

that generates the first (and only) component of the solution. Note that this RHG is not 0-normalized. If we 0-normalize it as indicated in the proof of Lemma 3, we obtain the equivalent RHG $\langle \emptyset, \{a_1, a_2\}, \emptyset, 0\rangle$. □

Example 4. Consider the LS over alphabet $\Sigma = \{a_1, a_2, a_3\}$:

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} \{\epsilon\} \\ \emptyset \\ \emptyset \end{pmatrix} \cup \begin{pmatrix} a_1\langle X_1\rangle \cup a_3\langle X_1\rangle & a_2\langle X_3\rangle & \emptyset \\ \emptyset & \emptyset & a_2\langle X_1\rangle \\ a_2\langle X_1\rangle & a_3\langle X_1\rangle & \emptyset \end{pmatrix} . \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}. \tag{2}$$

In this case we construct the sets $\mathcal{N} = \{\mathtt{n}_{k,l} \mid 1 \le k, l \le 3\}$, $\mathcal{X} = \{x_1, x_2, x_3\}$. The regular matrix and constant vector of this LS are

$$\mathsf{S} = \begin{pmatrix} \mathtt{n}_{1,1} + \mathtt{n}_{3,1} & \mathtt{n}_{2,3} & 0 \\ 0 & 0 & \mathtt{n}_{2,1} \\ \mathtt{n}_{2,1} & \mathtt{n}_{3,1} & 0 \end{pmatrix} \in \mathcal{M}_{3,3}(\mathcal{T}_{\mathrm{Reg}}(\emptyset, \mathcal{N})),\ \mathsf{C} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathcal{M}_{3,1}(\mathcal{T}_{\mathrm{Reg}}(\emptyset)).$$

The initials r_1, r_2, r_3 of the RHGs for the solution are defined by

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} := \mathsf{S}^\star \cdot \mathsf{C}.$$

This computation yields

$$\begin{cases} r_1 := ((\mathtt{n}_{1,1} + \mathtt{n}_{3,1})^\star \cdot \mathtt{n}_{2,3} \cdot (\mathtt{n}_{2,1} \cdot \mathtt{n}_{3,1})^\star \cdot \mathtt{n}_{2,1} \cdot \mathtt{n}_{2,1})^\star \cdot (\mathtt{n}_{1,1} + \mathtt{n}_{3,1})^\star, \\ r_2 := ((\mathtt{n}_{2,1} \cdot \mathtt{n}_{3,1})^\star \cdot \mathtt{n}_{2,1} \cdot \mathtt{n}_{2,1} \cdot (\mathtt{n}_{1,1} + \mathtt{n}_{3,1})^\star \cdot \mathtt{n}_{2,3})^\star \\ \qquad \cdot (\mathtt{n}_{2,1} \cdot \mathtt{n}_{3,1})^\star \cdot \mathtt{n}_{2,1} \cdot \mathtt{n}_{2,1} \cdot (\mathtt{n}_{1,1} + \mathtt{n}_{3,1})^\star, \\ r_3 := (\mathtt{n}_{3,1} \cdot \mathtt{n}_{2,1})^\star \cdot \mathtt{n}_{2,1} \cdot (\mathtt{n}_{1,1} + \mathtt{n}_{3,1})^\star + (\mathtt{n}_{3,1} \cdot \mathtt{n}_{2,1})^\star \cdot \mathtt{n}_{2,1} \cdot (\mathtt{n}_{1,1} + \mathtt{n}_{3,1})^\star \\ \qquad \cdot \mathtt{n}_{2,3} \cdot ((\mathtt{n}_{2,1} \cdot \mathtt{n}_{3,1})^\star \cdot \mathtt{n}_{2,1} \cdot \mathtt{n}_{2,1} \cdot (\mathtt{n}_{1,1} + \mathtt{n}_{3,1})^\star \cdot \mathtt{n}_{2,3})^\star \\ \qquad \cdot (\mathtt{n}_{2,1} \cdot \mathtt{n}_{3,1})^\star \cdot \mathtt{n}_{2,1} \cdot \mathtt{n}_{2,1} \cdot (\mathtt{n}_{1,1} + \mathtt{n}_{3,1})^\star. \end{cases}$$

We conclude that (2) has the unique solution $([\![G_1]\!], [\![G_2]\!], [\![G_3]\!])$ where $G_i = \langle \mathcal{N}, \Sigma, \mathcal{P}, r_i\rangle$ for $1 \le i \le 3$ with $\mathcal{P} := \{\mathtt{n}_{k,l} \to a_k\langle r_l\rangle \mid 1 \le k, l \le 3\}$. These RHGs are 0-normalized but there are some productive nonterminals that do not contribute to the computation of X_i. It is not hard to see that we can safely remove the nonterminals $\mathtt{n}_{k,l}$ for which there are no occurrences of $a_k\langle X_l\rangle$ in the LS. By doing so, we obtain the equivalent but smaller 0-normalized RHGs $G'_i = \langle \{\mathtt{n}_{1,1}, \mathtt{n}_{3,1}, \mathtt{n}_{2,3}, \mathtt{n}_{2,1}\}, \Sigma, \mathcal{P}', r_i\rangle$ $(1 \le i \le 3)$ with

$$\mathcal{P}' := \{\mathtt{n}_{1,1} \to a_1\langle r_1\rangle,\ \mathtt{n}_{3,1} \to a_3\langle r_1\rangle,\ \mathtt{n}_{2,3} \to a_2\langle r_3\rangle,\ \mathtt{n}_{2,1} \to a_2\langle r_1\rangle\}.$$ □

3.1 LS as Generative Systems of Regular Hedge Languages

It is interesting to note that LSs can also be viewed as a particular kind of grammar for regular hedge languages.

Definition 2 (Nest grammar). *A* nest grammar *is a tuple* $\langle \mathcal{X}, \Sigma, \mathcal{P}, X_1 \rangle$ *where* $\mathcal{X}$ *is a finite set of nonterminals,* $X_1 \in \mathcal{X}$ *is the start symbol, and* $\mathcal{P}$ *is a finite set of productions of the form*

$$X \to a\langle Y \rangle\, Z \text{ or } X \to \epsilon$$

with $X, Y, Z \in \mathcal{X}$ *and* $a \in \Sigma$.

The *hedge language generated by a nest grammar* $G = \langle \mathcal{X}, \Sigma, \mathcal{P}, X_1 \rangle$ *from a nonterminal* $X \in \mathcal{X}$ is the set of hedges $h \in \mathcal{H}(\Sigma)$ that can be generated from X with the productions of $\mathcal{P}$. We denote this language by $[\![X]\!]_G$. The *language of* G is defined to be $[\![X_1]\!]_G$.

The conversion between nest grammar and LS is straightforward:

- If

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} \cup \boldsymbol{L}. \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$$

  is an LS then the corresponding nest grammar is $\langle \mathcal{X}, \Sigma, \mathcal{P}, X_1 \rangle$ with

$$\mathcal{P} = \bigcup_{i=1}^{n} \Big(\{X_i \to \epsilon \mid C_i = \{\epsilon\}\} \cup \bigcup_{j=1}^{n} \{X_i \to a\langle Y \rangle X_j \mid a\langle Y \rangle \text{ occurs in } \boldsymbol{L}_{i,j}\} \Big),$$

- If $\langle \{X_1, \ldots, X_n\}, \Sigma, \mathcal{P}, X_1 \rangle$ is a nest grammar then the corresponding LS is

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \boldsymbol{C} \cup \boldsymbol{L}. \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$$

$$\text{with } \boldsymbol{C}_i := \begin{cases} \{\epsilon\} & \text{if } X_i \to \epsilon \in \mathcal{P}, \\ \emptyset & \text{otherwise} \end{cases} \quad \text{and} \quad \boldsymbol{L}_{i,j} := \bigcup_{X_i \to a\langle Y \rangle X_j \in \mathcal{P}} a\langle Y \rangle$$

  for all $i, j \in \{1, \ldots, n\}$,

and it is easy to see that if a nest grammar G and an LS are in such a correspondence, then the solution of the LS is $([\![X_1]\!]_G, \ldots, [\![X_n]\!]_G)$. Since the solution of an LS consists of RHLs, we conclude that the languages of nest grammars are RHLs. In the next section we will show that every RHL is the first component of the solution of an LS. Therefore, the class of RHLs over Σ coincides with the class of hedge languages of nest grammars over Σ.

Example 5. The nest grammar corresponding to the LS from Example 4 is

$$G = \langle \{X_1, X_2, X_3\}, \{a_1, a_2, a_3\}, \mathcal{P}, X_1 \rangle \text{ with}$$
$$\mathcal{P} := \{X_1 \to \epsilon, X_1 \to a_1\langle X_1\rangle X_1, X_1 \to a_3\langle X_1\rangle X_1, X_1 \to a_2\langle X_3\rangle X_2,$$
$$X_2 \to a_2\langle X_1\rangle X_3, X_3 \to a_2\langle X_1\rangle X_1, X_3 \to a_3\langle X_1\rangle X_2\}.$$
□

Our notion of nest grammar is strongly related to the kind of context free grammar that characterizes the notion of *nest set* studied by Takahashi [19] in connection with F-regular sets. Nest sets are languages defined over a paired alphabet $\acute{\Sigma} \cup \grave{\Sigma}$ by context free grammars with productions of the form $X \to \acute{a} Y \grave{a} Z$ and $X \to \epsilon$ where X, Y, Z are nonterminal symbols, and $\acute{a}$ and $\grave{a}$ are paired terminal symbols. This kind of grammars are balanced, that is, they generate strings with balanced parentheses that can be parsed into hedges. We note that:

- The class of F-regular sets over Σ coincides with the class of RHLs over Σ.
- Our notion of nest grammar corresponds to the parsed version of context free grammar used in [19] as specification of a nest set.

The context free grammars for nest sets have several remarkable properties that can be carried over to nest grammars, and thus to regular hedge languages. Among others, nest sets over a nested alphabet form a Boolean algebra having the Dyck set as the universe. This result can be lifted to show that regular hedge languages form a Boolean algebra too.

The study of properties of hedge languages via special classes of balanced grammars is an active area of research [2, 4] where hedge languages are studied via their trace, which is a balanced language. In this paper we do not pursue this line of research and prefer to stay in a framework where RHLs are represented by regular hedge expressions. We will explore the representation of RHLs by LSs and show how to use this representation to compute RHGs for all notions mentioned in the introduction.

4 A Differential Calculus for Regular Hedge Expressions

In this section we generalize the differential calculus of regular expressions [1, 5] to a differential calculus of regular hedge expressions. First, we recall some theoretical results about regular expressions. Let $\mathcal{P}_{\text{fin}}(A)$ denote the set of finite subsets of a set A. It is folk knowledge [1, 5] the existence of two computational methods:

- the *constant part* $o : \mathcal{T}_{\text{Reg}}(\Sigma) \to \{0, 1\}$ of complexity $O(n)$, and
- the *linear form* $\mathtt{lf} : \mathcal{T}_{\text{Reg}}(\Sigma) \to \mathcal{P}_{\text{fin}}(\Sigma \times \mathcal{T}_{\text{Reg}}(\Sigma))$ of complexity $O(n^2)$

where n is the size of the input regular expression, such that

- $o(r) = 1$ if $\epsilon \in [\![r]\!]$ and $o(r) = 0$ otherwise.
- $[\![r]\!] = [\![o(r) + \sum_{\langle a,s\rangle \in \mathtt{lf}(r)} a \cdot s]\!]$.

The elements of the set $\partial_a(r) := \{s \in \mathcal{T}_{\mathrm{Reg}}(\Sigma)\backslash\{0\} \mid \langle a, s\rangle \in \mathtt{lf}(r)\}$ are called the *partial derivatives* of r with respect to a. Partial derivatives can be generalized for words and sets of words too, as follows:

$$\partial_\epsilon(r) := r, \quad \partial_{aw}(r) := \bigcup_{s\in\partial_w(r)} \partial_a(s) \qquad \partial_W(r) := \bigcup_{w\in W} \partial_w(r)$$

where $a \in \Sigma$, $w \in \Sigma^*$ and $W \subseteq \Sigma^*$. A remarkable fact is that $\partial_{\Sigma^*}(r)$ is a finite set with at most $\|r\| + 1$ elements [1].

In the remainder of this section we show how the notions of linear form and partial derivative can be carried over to regular hedge expressions in a natural way. To simplify our presentation, we introduce a useful convention of notation: If G is an RHG with nonterminals from $\mathcal{N}$ and $s \in \mathcal{T}_{\mathrm{Reg}}(\mathcal{N})$, then $G(s)$ denotes the RHG obtained from G by replacing the initial of G with s.

Definition 3. *Let $G = \langle \mathcal{N}, \Sigma, \mathcal{P}, r\rangle \in \mathcal{G}(\mathcal{N}, \Sigma)$.*

1. *The set of* children *of G is* $\mathtt{Cld}(G) := \{G(s) \mid \exists\,(\mathtt{n} \to a\langle s\rangle) \in \mathcal{P}\}$.
2. *The* linear form *of G is*

$$\mathtt{lf}(G) := \{\langle a\langle G(r_1)\rangle, G(r_2)\rangle \mid \langle \mathtt{n}, r_2\rangle \in \mathtt{lf}(r) \wedge (\mathtt{n} \to a\langle r_1\rangle) \in \mathcal{P}\}.$$

3. *The* affine forms *of G are the regular hedge expressions*

$$\mathtt{af}_1(G) := o(\mathtt{ini}(G)) + \sum_{\langle \mathtt{n},s\rangle\in\mathtt{lf}(\mathtt{ini}(G))} G(\mathtt{n}) \cdot G(s) \in \mathcal{T}_{\mathrm{HReg}}(\Sigma),$$

$$\mathtt{af}_2(G) := o(\mathtt{ini}(G)) + \sum_{\langle a\langle G_1\rangle, G_2\rangle\in\mathtt{lf}(G)} a\langle G_1\rangle \cdot G_2 \in \mathcal{T}_{\mathrm{HReg}}(\Sigma).$$

4. *The set of* partial derivatives *of G w.r.t. $a\langle h\rangle \in \mathcal{T}(\Sigma)$ is the finite set*

$$\partial_{a\langle h\rangle}(G) := \{G' \mid \langle a\langle G_1\rangle, G'\rangle \in \mathtt{lf}(G) \wedge h \in [\![G_1]\!]\}.$$

   This definition has a straightforward extension to arbitrary hedges: We define $\partial_\epsilon(G) := \{G\}$ and $\partial_{a\langle h_1\rangle\, h_2}(G) := \bigcup_{G'\in\partial_{a\langle h_1\rangle}(G)} \partial_{h_2}(G')$.
   If H is a hedge language and $M \in \mathcal{P}(\mathcal{G}(\mathcal{N}, \Sigma))$ then we also define

$$\partial_H(G) := \bigcup_{h\in H} \partial_h(G), \quad \partial_h(M) := \bigcup_{G\in M} \partial_h(G), \quad \partial_H(M) := \bigcup_{h\in H} \partial_h(M).$$

Note that the inclusion $\partial_{\mathcal{H}(\Sigma)}(G) \subseteq \{G(s) \mid s \in \partial_{\mathcal{N}^*}(\mathtt{ini}(G))\}$ holds for any RHG G; if G is 0-normalized then $\partial_{\mathcal{H}(\Sigma)}(G) = \{G(s) \mid s \in \partial_{\mathcal{N}^*}(\mathtt{ini}(G))\}$ and all partial derivatives of G are productive and 0-normalized. Therefore $\partial_{\mathcal{H}(\Sigma)}(G)$ has at most $|\partial_{\mathcal{N}^*}(\mathtt{ini}(G))|$ elements, which is at most $1 + \|\mathtt{ini}(G)\|$ [1].

The following proposition is the hedge counterpart of [1, Prop. 2.5].

Proposition 1. *If $G \in \mathcal{G}(\mathcal{N}, \Sigma)$ then $G \doteq \mathtt{af}_1(G) \doteq \mathtt{af}_2(G)$.*

Proof. Let $r := \mathtt{ini}(G)$. By [1, Prop. 2.5] we have $r \doteq o(r) + \sum_{\langle \mathtt{n}, s\rangle \in \mathtt{lf}(r)} \mathtt{n} \cdot s$, hence

$$\begin{aligned}
[\![G]\!] = [\![r]\!]_G &= [\![o(r)]\!] \cup \bigcup_{\langle \mathtt{n}, r_2\rangle \in \mathtt{lf}(r)} [\![\mathtt{n} \cdot r_2]\!]_G \\
&= [\![o(r)]\!] \cup \bigcup_{\langle \mathtt{n}, r_2\rangle \in \mathtt{lf}(r)} [\![\mathtt{n}]\!]_G.[\![r_2]\!]_G = [\![\mathtt{af}_1(G)]\!] \\
&= [\![o(r)]\!] \cup \bigcup_{\langle \mathtt{n}, r_2\rangle \in \mathtt{lf}(r)} \bigcup_{(\mathtt{n} \to a\langle r_1\rangle) \in \mathcal{P}} (a\langle [\![r_1]\!]_G\rangle.[\![r_2]\!]_G) \\
&= [\![o(\mathtt{ini}(G))]\!] \cup \bigcup_{\langle a\langle G_1\rangle, G_2\rangle \in \mathtt{lf}(G)} (a\langle [\![G_1]\!]\rangle.[\![G_2]\!]) = [\![\mathtt{af}_2(G)]\!]. \qquad \square
\end{aligned}$$

Corollary 1. *For every $G = \langle \mathcal{N}, \Sigma, \mathcal{P}, r_1\rangle \in \mathcal{G}(\mathcal{N}, \Sigma)$ we can compute a system of characteristic equations of $[\![G]\!]$.*

Proof. Let $\{r_1, \ldots, r_n\} = \partial_{\mathcal{N}^*}(\{r_1\} \cup \{r \mid \mathtt{n} \to a\langle r\rangle \in \mathcal{P}\})$. Then

$$[\![G(r_i)]\!] = [\![\mathtt{af}_2(G(r_i))]\!] = [\![o(r_i)]\!] \cup \bigcup_{j=1}^{n} L_{i,j}.[\![G(r_j)]\!] \qquad (1 \le i \le n)$$

where $L_{i,j}$ are unions of hedge languages of the form $a\langle [\![G(r_k)]\!]\rangle$ with $a \in \Sigma$ and $1 \le k \le n$. By replacing every $[\![G(r_k)]\!]$ with X_k in this system of n hedge language equations, we obtain a system of characteristic equations of $[\![G]\!]$. $\square$

The following lemma establishes a semantic connection between partial hedge derivatives and quotients of RHLs.

Lemma 5. $\bigcup_{G' \in \partial_h(G)} [\![G']\!] = h^{-1}[\![G]\!]$ *holds for all $G \in \mathcal{G}(\mathcal{N}, \Sigma)$ and $h \in \mathcal{H}(\Sigma)$.*

Proof. By induction on the length of h. If $h = \epsilon$ then $\partial_h(G) = \{G\}$ and thus $\bigcup_{G' \in \partial_h(G)} [\![G']\!] = [\![G]\!] = \epsilon^{-1}[\![G]\!] = h^{-1}[\![G]\!]$. If $h = a\langle h_1\rangle$ then

$$\begin{aligned}
\bigcup_{G' \in \partial_h(G)} [\![G']\!] &= \bigcup_{\substack{\langle a\langle G_1\rangle, G_2\rangle \in \mathtt{lf}(G) \\ h_1 \in [\![G_1]\!]}} (a\langle [\![G_1]\!]\rangle.[\![G_2]\!]) \\
&= a\langle h_1\rangle^{-1} \bigcup_{\langle b\langle G_1\rangle, G_2\rangle \in \mathtt{lf}(G)} (b\langle [\![G_1]\!]\rangle.[\![G_2]\!]) \\
&= a\langle h_1\rangle^{-1} [\![\mathtt{af}_2(G)]\!] = h^{-1}[\![G]\!].
\end{aligned}$$

Otherwise, $h = a\langle h_1\rangle h_2$ and

$$\bigcup_{G' \in \partial_{a\langle h_1\rangle h_2}(G)} [\![G']\!] = \bigcup_{G' \in \partial_{a\langle h_1\rangle}(G)} \bigcup_{G'' \in \partial_{h_2}(G')} [\![G'']\!].$$

Since $|h_2| < |h|$, we have by induction hypothesis that $\bigcup_{G'' \in \partial_{h_2}(G')} [\![G'']\!] = h_2^{-1}[\![G']\!]$ for all $G' \in \partial_{a\langle h_1\rangle}(G)$, therefore

$$\begin{aligned}
\bigcup_{G' \in \partial_{a\langle h_1\rangle\, h_2}(G)} [\![G']\!] &= \bigcup_{G' \in \partial_{a\langle h_1\rangle}(G)} h_2^{-1}[\![G']\!] = h_2^{-1}\Big(\bigcup_{G' \in \partial_{a\langle h_1\rangle}(G)} [\![G']\!]\Big) \\
&= h_2^{-1}(a\langle h_1\rangle^{-1}[\![G]\!]) = (a\langle h_1\rangle\, h_2)^{-1}[\![G]\!] = h^{-1}[\![G]\!]. \qquad \square
\end{aligned}$$

5 Applications

We will illustrate how the theory of LSs enables elegant methods to prove that RHLs are closed under quotient, intersection, and product derivative, and how to compute RHGs for these hedge languages. The computational methods presented in this section share the following characteristics:

1. Their inputs and output are regular hedge grammars.
2. They compute and manipulate systems of characteristic equations for the languages generated by inputs in order to produce a finite number of LSs whose solutions can be used to compute an RHG for the output.

Before presenting these methods, we introduce a few more auxiliary notions.

The *symmetric* h^{s} of a hedge h is defined recursively by $\epsilon^{\mathsf{s}} := \epsilon$, $(a\langle h_1\rangle\, h_2)^{\mathsf{s}} := h_2^{\mathsf{s}}\, a\langle h_1\rangle$. The symmetric of a hedge language H is $H^{\mathsf{s}} := \{h^{\mathsf{s}} \mid h \in H\}$. The *symmetric* r^{s} of $r \in \mathcal{T}_{\text{Reg}}(\mathcal{N})$ is defined recursively by: $0^{\mathsf{s}} := 0$; $1^{\mathsf{s}} := 1$; $\mathtt{n}^{\mathsf{s}} := \mathtt{n}$ if $\mathtt{n} \in \mathcal{N}$; $(r_1 \cdot r_2)^{\mathsf{s}} := r_2^{\mathsf{s}} \cdot r_1^{\mathsf{s}}$; $(r_1 + r_2)^{\mathsf{s}} := r_1^{\mathsf{s}} + r_2^{\mathsf{s}}$; and $(r^\star)^{\mathsf{s}} := (r^{\mathsf{s}})^\star$. The *symmetric* of $G \in \mathcal{G}(\mathcal{N}, \Sigma)$ is $G^{\mathsf{s}} := G(r^{\mathsf{s}})$.

The following facts are easy to check:

- If $r \in \mathcal{T}_{\text{Reg}}(\mathcal{N})$ then $r^{\mathsf{s}} \in \mathcal{T}_{\text{Reg}}(\mathcal{N})$ and $(r^{\mathsf{s}})^{\mathsf{s}} = r$.
- If $G \in \mathcal{G}(\mathcal{N}, \Sigma)$ then $G^{\mathsf{s}} \in \mathcal{G}(\mathcal{N}, \Sigma)$ and $[\![G^{\mathsf{s}}]\!] = [\![G]\!]^{\mathsf{s}}$.

5.1 Quotient of Regular Hedge Languages

In this section we show that the quotient of two RHLs is an RHL and indicate a method to compute a regular hedge grammar for the quotient language.

First, we note that if $S \in \mathcal{G}(\mathcal{N}_1, \Sigma)$ and $G \in \mathcal{G}(\mathcal{N}_2, \Sigma)$ then $[\![S]\!]^{-1}[\![G]\!] = ([\![G^{\mathsf{s}}]\!][\![S^{\mathsf{s}}]\!]^{-1})^{\mathsf{s}}$. Therefore, if we have a method to compute an RHG for $[\![S]\!]^{-1}[\![G]\!]$ then we also have a method to compute an RHG for $[\![G]\!][\![S]\!]^{-1}$ and vice versa. Thus, it suffices to indicate a method to solve the problem

Given: $\Sigma = \{a_1, \ldots, a_m\}$, $S \in \mathcal{G}(\mathcal{N}_1, \Sigma)$, and $G \in \mathcal{G}(\mathcal{N}_2, \Sigma)$
Compute: RHG G' such that $[\![G']\!] = [\![G]\!][\![S]\!]^{-1}$.

Let $S = \langle \mathcal{N}_1, \Sigma, \mathcal{P}_1, s_1\rangle$, $G = \langle \mathcal{N}_2, \Sigma, \mathcal{P}_2, r_1\rangle$, and $\{r_1, \ldots, r_n\} := \partial_{\mathcal{N}_2^*}(r_1)$. Then

$$G(r_i) \doteq \mathtt{af}_1(G(r_i)) \doteq o(r_i) + \sum_{j=1}^{n} A_{i,j} \cdot G(r_j) \qquad (1 \le i \le n) \tag{3}$$

is a system of n hedge language equations where the coefficients $A_{i,j}$ are sums of grammars of the form $G(\mathtt{n})$ with $\mathtt{n} \in \mathcal{N}_2$. We define

1. The assignment $\sigma : \mathcal{N}_2 \to \mathcal{P}(\mathcal{H}(\Sigma))$ by $\sigma(\mathtt{n}) := [\![\mathtt{n}]\!]_G$ for all $\mathtt{n} \in \mathcal{N}_2$.
2. $\mathsf{B} \in \mathcal{M}_{n,n}(\mathcal{T}_{\text{Reg}}(\emptyset, \mathcal{N}_2))$ with $\mathsf{B}_{i,j}$:= the result of replacing in $A_{i,j}$ the occurrences of $G(\mathtt{n})$ with $\mathtt{n}$ for all $\mathtt{n} \in \mathcal{N}_2$ for all $i, j \in \{1, \ldots, n\}$.

Then (3) implies

$$\begin{pmatrix} [\![G(r_1)]\!] \\ \vdots \\ [\![G(r_n)]\!] \end{pmatrix} = \begin{pmatrix} \{\epsilon \mid \epsilon \in [\![G(r_1)]\!]\} \\ \vdots \\ \{\epsilon \mid \epsilon \in [\![G(r_n)]\!]\} \end{pmatrix} \cup [\![\mathsf{B}]\!]_\sigma . \begin{pmatrix} [\![G(r_1)]\!] \\ \vdots \\ [\![G(r_n)]\!] \end{pmatrix}.$$

Let $\mathsf{C} \in \mathcal{M}_{n,1}(\mathcal{T}_{\mathrm{Reg}}(\emptyset))$ be defined by $\mathsf{C}_i := 1$ if $\epsilon \in [\![G(r_i)]\!][\![S]\!]^{-1}$ and $\mathsf{C}_i := 0$ otherwise. Then

$$\begin{pmatrix} [\![G(r_1)]\!][\![S]\!]^{-1} \\ \vdots \\ [\![G(r_n)]\!][\![S]\!]^{-1} \end{pmatrix} = [\![\mathsf{C}]\!] \cup [\![\mathsf{B}]\!]_\sigma . \begin{pmatrix} [\![G(r_1)]\!][\![S]\!]^{-1} \\ \vdots \\ [\![G(r_n)]\!][\![S]\!]^{-1} \end{pmatrix}.$$

The matrix $[\![\mathsf{B}]\!]_\sigma \in \mathcal{M}_{n,n}(\mathcal{P}(\mathcal{H}(\Sigma)))$ is ϵ-free because $[\![\mathsf{B}_{i,j}]\!]_\sigma \subseteq \bigcup_{\mathtt{n} \in \mathcal{N}_2} \sigma(\mathtt{n}) = \bigcup_{\mathtt{n} \in \mathcal{N}_2} [\![\mathtt{n}]\!]_G \subseteq \mathcal{T}(\Sigma)$ for all $i, j \in \{1, \ldots, n\}$. Therefore, the matriceal equation $\boldsymbol{X} = [\![\mathsf{C}]\!] \cup [\![\mathsf{B}]\!]_\sigma . \boldsymbol{X}$ has the unique solution $[\![\mathsf{B}^\star \cdot \mathsf{C}]\!]_\sigma$, which implies

$$[\![G]\!][\![S]\!]^{-1} = [\![G(r_1)]\!][\![S]\!]^{-1} = [\![(\mathsf{B}^\star \cdot \mathsf{C})_1]\!]_\sigma .$$

Let $r := (\mathsf{B}^\star \cdot \mathsf{C})_1 \in \mathcal{T}_{\mathrm{Reg}}(\mathcal{N}_2)$. Then the following relations hold:

1. $[\![G]\!][\![S^{-1}]\!] = [\![r]\!]_\sigma$, and
2. $[\![\mathtt{n}]\!]_\sigma = \bigcup_{\mathtt{n} \to a\langle u \rangle \in \mathcal{P}_2} a\langle [\![u]\!]_\sigma \rangle$ for all $\mathtt{n} \in \mathcal{N}_2$.

These relations indicate that $[\![G]\!][\![S^{-1}]\!] = [\![G(r)]\!]$.

It remains to figure out how to compute the column vector C. For this purpose, we consider the sets $\mathcal{K}_1 := \partial_{\mathcal{H}(\Sigma)}(\{G\} \cup \mathtt{Cld}(G))$, $\mathcal{K}_2 := \partial_{\mathcal{H}(\Sigma)}(\{S\} \cup \mathtt{Cld}(S))$, and the monotone operator $\mathtt{F} : \mathcal{P}(\mathcal{K}_1 \times \mathcal{K}_2) \to \mathcal{P}(\mathcal{K}_1 \times \mathcal{K}_2)$ on the complete lattice $(\mathcal{P}(\mathcal{K}_1 \times \mathcal{K}_2), \subseteq)$ represented compactly by the inference rules

$$\frac{[o(\mathtt{ini}(G')) = 1 \land o(\mathtt{ini}(S')) = 1]}{\langle G', S' \rangle}$$

$$\frac{\langle G'_1, S'_1 \rangle \quad \langle G'_2, S'_2 \rangle \quad [\langle a\langle G'_1 \rangle, G'_2 \rangle \in \mathtt{lf}(G') \land \langle a\langle S'_1 \rangle, S'_2 \rangle \in \mathtt{lf}(S')]}{\langle G', S' \rangle}$$

where each rule states that if the expressions above the bar are in the input set of $\mathtt{F}$, the conditions within square brackets hold, and the expression below the bar is in $\mathcal{K}_1 \times \mathcal{K}_2$, then the expression below the bar is in the output set of $\mathtt{F}$.

Since $\mathcal{K}_1 \times \mathcal{K}_2$ is a finite set, the least fixed point $\mu\mathtt{F} = \bigcap\{X \subseteq \mathcal{K}_1 \times \mathcal{K}_2 \mid \mathtt{F}(X) \subseteq X\}$ is a finite set that can be computed effectively, thus $\mu\mathtt{F}$ is decidable.

Lemma 6. $\{\langle G', S' \rangle \in \mathcal{K}_1 \times \mathcal{K}_2 \mid \epsilon \in [\![G']\!][\![S']\!]^{-1}\} = \mu\mathtt{F}$.

Proof. Let $M := \{\langle G', S' \rangle \in \mathcal{K}_1 \times \mathcal{K}_2 \mid \epsilon \in [\![G']\!][\![S']\!]^{-1}\}$. We prove $\mu\mathtt{F} \subseteq M$ by induction of the length of the inference derivation.

If $\langle G'_1, G'_2 \rangle \in \mu\mathtt{F}$ was deduced by the inference rule

$$\frac{[o(\mathtt{ini}(G')) = 1 \land o(\mathtt{ini}(S')) = 1]}{\langle G', S' \rangle}$$

then $\epsilon \in [\![G']\!] \cap [\![S']\!]$ and thus $\epsilon \in [\![G']\!][\![S']\!]^{-1}$, therefore $\langle G', S'\rangle \in M$.

If $\langle G', S'\rangle \in \mu\mathtt{F}$ was deduced by a derivation with the last inference step

$$\frac{\langle G'_1, S'_1\rangle \quad \langle G'_2, S'_2\rangle \quad [\langle a\langle G'_1\rangle, G'_2\rangle \in \mathtt{lf}(G') \wedge \langle a\langle S'_1\rangle, S'_2\rangle \in \mathtt{lf}(S')]}{\langle G', S'\rangle}$$

then $\langle G'_1, S'_1\rangle, \langle G'_2, S'_2\rangle \in \mu\mathtt{F}$ and thus $\langle G'_1, S'_1\rangle, \langle G'_2, S'_2\rangle \in M$ by induction hypothesis. This means that $\epsilon \in [\![a\langle G'_1\rangle]\!][\![a\langle S'_1\rangle]\!]^{-1} \cap [\![G'_2]\!][\![S'_2]\!]^{-1}$. It follows that $\epsilon \in [\![a\langle G'_1\rangle \cdot S'_1]\!][\![a\langle G'_2\rangle \cdot S'_2]\!]^{-1} \subseteq [\![\mathtt{af}_2(G')]\!][\![\mathtt{af}_2(S')]\!]^{-1} = [\![G']\!][\![S']\!]^{-1}$, and thus $\langle G', S'\rangle \in M$.

We note that $\langle G', S'\rangle \in M$ iff $\epsilon \in [\![G']\!][\![S']\!]^{-1}$ iff $[\![G']\!] \cap [\![S']\!] \neq \emptyset$. Therefore, for every pair $\langle G', S'\rangle \in M$ we can define the complexity measure $|\langle G', S'\rangle|$ as the minimal size of a hedge of $[\![G']\!] \cap [\![S']\!]$. We prove $M \subseteq \mu\mathtt{F}$ by induction on the complexity measure of elements of M.

If $\langle G', S'\rangle \in M$ and $|\langle G', S'\rangle| = 0$ then $\epsilon \in [\![G']\!] \cap [\![S']\!]$. Thus $o(\mathtt{ini}(G')) = 1$, $o(\mathtt{ini}(S')) = 1$, and we can perform the derivation

$$\frac{[o(\mathtt{ini}(G')) = 1 \wedge o(\mathtt{ini}(S')) = 1]}{\langle G', S'\rangle}$$

to deduce that $\langle G', S'\rangle \in \mu\mathtt{F}$.

If $\langle G', S'\rangle \in M$ and $|\langle G', S'\rangle| = p > 0$ then $\epsilon \notin [\![G']\!] \cap [\![S']\!]$ and there exists $h = a\langle h_1\rangle\, h_2 \in [\![G']\!] \cap [\![S']\!]$ of length p. From $a\langle h_1\rangle\, h_2 \in [\![G']\!] \cap [\![S']\!] = ([\![\mathtt{ini}(G')]\!] \cup \bigcup_{\langle b\langle G'_1\rangle, G'_2\rangle \in \mathtt{lf}(G')} [\![b\langle G'_1\rangle]\!].[\![G'_2]\!]) \cap ([\![\mathtt{ini}(S')]\!] \cup \bigcup_{\langle b\langle S'_1\rangle, S'_2\rangle \in \mathtt{lf}(S')} [\![b\langle S'_1\rangle]\!].[\![S'_2]\!])$, we learn that there exist pairs $\langle a\langle G'_1\rangle, G'_2\rangle \in \mathtt{lf}(G')$ and $\langle a\langle S'_1\rangle, S'_2\rangle \in \mathtt{lf}(S')$ such that $h_i \in [\![G'_i]\!] \cap [\![S'_i]\!]$ for $i \in \{1, 2\}$. Thus $\langle G'_i, S'_i\rangle \in M$ with $|\langle G'_i, S'_i\rangle| \leq \|h_i\| < \|h\|$, and we can apply the induction hypothesis to conclude that $\langle G'_i, S'_i\rangle \in \mu\mathtt{F}$ for $i \in \{1, 2\}$. Finally, we can apply the inference rule

$$\frac{\langle G'_1, S'_1\rangle \quad \langle G'_2, S'_2\rangle \quad [\langle a\langle G'_1\rangle, G'_2\rangle \in \mathtt{lf}(G') \wedge \langle a\langle S'_1\rangle, S'_2\rangle \in \mathtt{lf}(S')]}{\langle G', S'\rangle}$$

to conclude that $\langle G'_1, G'_2\rangle \in \mu\mathtt{F}$. □

Corollary 2. *For every $i \in \{1, \ldots, n\}$ we have $\mathsf{C}_i = 1$ iff $\langle G_i, S\rangle \in \mu\mathtt{F}$.*

Since $\mu\mathtt{F}$ is a decidable set, the components of C can be computed effectively.

Example 6. Let $G = \langle\{\mathtt{n}_1, \mathtt{n}_2\}, \Sigma, \mathcal{P}_1, \mathtt{n}_1^\star \cdot \mathtt{n}_2^\star\rangle$ and $S = \langle\Sigma, \{\mathtt{n}\}, \mathcal{P}_2, \mathtt{n}\rangle$ with $\Sigma = \{a_1, a_2, a_3\}$ and

$$\mathcal{P}_1 = \{\mathtt{n}_1 \to a_1\langle\mathtt{n}_2 + \mathtt{n}_3\rangle, \mathtt{n}_2 \to a_2\langle\mathtt{n}_1 + \mathtt{n}_3\rangle, \mathtt{n}_3 \to a_3\langle 1\rangle\},$$
$$\mathcal{P}_2 = \{\mathtt{n} \to a_1\langle\mathtt{n}^\star\rangle, \mathtt{n} \to a_2\langle\mathtt{n}\rangle, \mathtt{n} \to a_3\langle\mathtt{n}^\star\rangle\},$$

and let's compute an RHG for $[\![G]\!][\![S]\!]^{-1}$. In this case we have

$$\mathtt{af}_1(G) = \mathtt{af}_1(G(\mathtt{n}_1^\star \cdot \mathtt{n}_2^\star)) = 1 + G(\mathtt{n}_1) \cdot G + G(\mathtt{n}_2) \cdot G(\mathtt{n}_2^\star),$$
$$\mathtt{af}_1(G(\mathtt{n}_2^\star)) = 1 + G(\mathtt{n}_2) \cdot G(\mathtt{n}_2^\star)$$

and $[\![G]\!][\![S]\!]^{-1} = [\![G((\mathsf{B}^\star \cdot \mathsf{C})_1)]\!]$ where

$$\mathsf{B}^\star = \begin{pmatrix} \mathtt{n}_1 & \mathtt{n}_2 \\ 0 & \mathtt{n}_2 \end{pmatrix}^\star = \begin{pmatrix} \mathtt{n}_1^\star & \mathtt{n}_1^\star \cdot \mathtt{n}_2 \cdot \mathtt{n}_2^\star \\ 0 & \mathtt{n}_2^\star \end{pmatrix} \text{ and } \begin{cases} \mathsf{C}_1 := 1 \text{ iff } \langle G, S\rangle \in \mu\mathtt{F}, \\ \mathsf{C}_2 := 1 \text{ iff } \langle G(n_2^\star), S\rangle \in \mu\mathtt{F}. \end{cases}$$

In this example, $\mathtt{F}$ is defined on $\mathcal{P}(\mathcal{K}_1 \times \mathcal{K}_2)$ where

$$\mathcal{K}_1 = \{G, G(\mathtt{n}_2^\star), G(\mathtt{n}_2 + \mathtt{n}_3), G(\mathtt{n}_1 + \mathtt{n}_3), G(1)\},\ \mathcal{K}_2 = \{S, S(1), S(\mathtt{n}^\star)\}.$$

$\mathtt{F}$ is represented compactly by the following 20 inference rules:

$$\frac{}{\langle G, S(1)\rangle} \quad \frac{}{\langle G, S(\mathtt{n}^\star)\rangle} \quad \frac{}{\langle G(\mathtt{n}_2^\star), S(1)\rangle} \quad \frac{}{\langle G(\mathtt{n}_2^\star), S(\mathtt{n}^\star)\rangle} \quad \frac{}{\langle G(1), S(1)\rangle}$$

$$\frac{}{\langle G(1), S(\mathtt{n}^\star)\rangle} \quad \frac{\langle G(\mathtt{n}_2+\mathtt{n}_3), S(\mathtt{n}^\star)\rangle \ \langle G, S(1)\rangle}{\langle G, S\rangle} \quad \frac{\langle G(\mathtt{n}_2+\mathtt{n}_3), S(\mathtt{n}^\star)\rangle \ \langle G, S(\mathtt{n}^\star)\rangle}{\langle G, S(\mathtt{n}^\star)\rangle}$$

$$\frac{\langle G(\mathtt{n}_1+\mathtt{n}_3), S\rangle \quad \langle G(\mathtt{n}_2^\star), S(1)\rangle}{\langle G, S\rangle} \quad \frac{\langle G(\mathtt{n}_1+\mathtt{n}_3), S\rangle \quad \langle G(\mathtt{n}_2^\star), S(\mathtt{n}^\star)\rangle}{\langle G, S(\mathtt{n}^\star)\rangle}$$

$$\frac{\langle G(\mathtt{n}_1+\mathtt{n}_3), S\rangle \quad \langle G(\mathtt{n}_2^\star), S(1)\rangle}{\langle G(\mathtt{n}_2^\star), S\rangle} \quad \frac{\langle G(\mathtt{n}_1+\mathtt{n}_3), S\rangle \quad \langle G(\mathtt{n}_2^\star), S(\mathtt{n}^\star)\rangle}{\langle G(\mathtt{n}_2^\star), S(\mathtt{n}^\star)\rangle}$$

$$\frac{\langle G(\mathtt{n}_1+\mathtt{n}_3), S\rangle \quad \langle G(1), S(1)\rangle}{\langle G(\mathtt{n}_2+\mathtt{n}_3), S\rangle} \quad \frac{\langle G(\mathtt{n}_1+\mathtt{n}_3), S\rangle \quad \langle G(1), S(\mathtt{n}^\star)\rangle}{\langle G(\mathtt{n}_2+\mathtt{n}_3), S(\mathtt{n}^\star)\rangle}$$

$$\frac{\langle G(1), S(\mathtt{n}^\star)\rangle \quad \langle G(1), S(1)\rangle}{\langle G(\mathtt{n}_2+\mathtt{n}_3), S\rangle} \quad \frac{\langle G(1), S(\mathtt{n}^\star)\rangle \quad \langle G(1), S(\mathtt{n}^\star)\rangle}{\langle G(\mathtt{n}_2+\mathtt{n}_3), S(\mathtt{n}^\star)\rangle}$$

$$\frac{\langle G(\mathtt{n}_2+\mathtt{n}_3), S(\mathtt{n}^\star)\rangle \quad \langle G(1), S(1)\rangle}{\langle G(\mathtt{n}_1+\mathtt{n}_3), S\rangle} \quad \frac{\langle G(\mathtt{n}_2+\mathtt{n}_3), S(\mathtt{n}^\star)\rangle \quad \langle G(1), S(\mathtt{n}^\star)\rangle}{\langle G(\mathtt{n}_1+\mathtt{n}_3), S(\mathtt{n}^\star)\rangle}$$

$$\frac{\langle G(1), S(\mathtt{n}^\star)\rangle \quad \langle G(1), S(1)\rangle}{\langle G(\mathtt{n}_1+\mathtt{n}_3), S\rangle} \quad \frac{\langle G(1), S(\mathtt{n}^\star)\rangle \quad \langle G(1), S(\mathtt{n}^\star)\rangle}{\langle G(\mathtt{n}_1+\mathtt{n}_3), S(\mathtt{n}^\star)\rangle}$$

Since $\langle G, S\rangle \in \mathtt{F}^3(\emptyset) \subseteq \mu\mathtt{F}$ and $\langle G(\mathtt{n}_2^\star), S\rangle \in \mathtt{F}^3(\emptyset) \subseteq \mu\mathtt{F}$, we conclude that $\mathsf{C}_1 = \mathsf{C}_2 = 1$, hence $[\![G]\!][\![S]\!]^{-1} = [\![G(\mathtt{n}_1^\star + \mathtt{n}_1^\star \cdot \mathtt{n}_2 \cdot \mathtt{n}_2^\star)]\!]$ because

$$(\mathsf{B}^\star \cdot \mathsf{C})_1 = \left(\begin{pmatrix} \mathtt{n}_1^\star & \mathtt{n}_1^\star \cdot \mathtt{n}_2 \cdot \mathtt{n}_2^\star \\ 0 & \mathtt{n}_2^\star \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)_1 = \mathtt{n}_1^\star + \mathtt{n}_1^\star \cdot \mathtt{n}_2 \cdot \mathtt{n}_2^\star. \qquad \square$$

5.2 Intersection of Regular Hedge Languages

In this section we show that the intersection of RHLs is an RHL, and indicate a method to solve the problem

Given: $\Sigma = \{a_1, \ldots, a_m\}$, $S \in \mathcal{G}(\mathcal{N}_1, \Sigma)$, and $G \in \mathcal{G}(\mathcal{N}_2, \Sigma)$
Compute: RHG G' such that $[\![G']\!] = [\![S]\!] \cap [\![G]\!]$.

Suppose $S = \langle \mathcal{N}_1, \Sigma, \mathcal{P}_1, s_1\rangle$ $G = \langle \mathcal{N}_2, \Sigma, \mathcal{P}_2, r_1\rangle$, and

$$\{s_1, \ldots, s_n\} = \partial_{\mathcal{N}_1^*}(s_1) \cup \bigcup_{\mathtt{n}\to a\langle s\rangle \in \mathcal{P}_1} \partial_{\mathcal{N}_1^*}(s)$$

$$\{r_1, \ldots, r_p\} = \partial_{\mathcal{N}_2^*}(r_1) \cup \bigcup_{\mathtt{n}\to a\langle r\rangle \in \mathcal{P}_2} \partial_{\mathcal{N}_2^*}(r)$$

Then the equations

$$S(s_i) \doteq \mathtt{af}_2(S(s_i)) \doteq o(s_i) + \sum_{k=1}^{n} A_{i,k} \cdot S(s_k) \qquad (1 \le i \le n)$$

are an instance of an LS with solution $(S(s_1), \ldots, S(s_n))$, where the coefficients $A_{i,k}$ are sums of regular hedge expressions from the set

$$\{a_j\langle S(s_l)\rangle \mid 1 \le j \le m, 1 \le l \le n\}.$$

Similarly, the equations

$$G(r_j) \doteq \mathtt{af}_2(G(r_j)) \doteq o(r_j) + \sum_{l=1}^{p} B_{j,l} \cdot G(r_j) \qquad (1 \le j \le p)$$

are an instance of an LS with solution $(G(r_1), \ldots, G(r_p))$, where the coefficients $B_{j,l}$ are sums of regular hedge expressions from the set

$$\{a_i\langle G(r_k)\rangle \mid 1 \le i \le m, 1 \le k \le p\}.$$

Let $\mathcal{X} = \{x_1, \ldots, x_n\}$ and $\mathcal{Y} = \{y_1, \ldots, y_p\}$ be sets of distinct fresh symbols. We define

1. The assignments $\sigma_1 : \mathcal{X} \to \mathcal{P}(\mathcal{H}(\Sigma))$ such that $\sigma_1(x_i) := [\![s_i]\!]_S$ for $1 \le i \le n$, and $\sigma_2 : \mathcal{Y} \to \mathcal{P}(\mathcal{H}(\Sigma))$ such that $\sigma_2(y_j) := [\![r_j]\!]_G$ for $1 \le j \le p$.
2. The matrix $\mathsf{A} \in \mathcal{M}_{n,n}(\mathcal{T}_{\mathrm{HReg}}(\Sigma, \mathcal{X}))$ such that $\mathsf{A}_{i,j}$ is the result of replacing in $A_{i,j}$ the occurrences of $S(s_k)$ with x_k for all $1 \le i, j, k \le n$.
3. The vectors $\mathsf{X}, \mathsf{C} \in \mathcal{M}_{n,1}(\mathcal{T}_{\mathrm{HReg}}(\Sigma, \mathcal{X}))$ with $\mathsf{X}_i := x_i$ and $\mathsf{C}_i := o(s_i)$ for $1 \le i \le n$.
4. The matrix $\mathsf{B} \in \mathcal{M}_{p,p}(\mathcal{T}_{\mathrm{HReg}}(\Sigma, \mathcal{Y}))$ such that $\mathsf{B}_{i,j}$ is the result of replacing in $B_{i,j}$ the occurrences of $G(r_k)$ with y_k for all $1 \le i, j, k \le p$.
5. The vectors $\mathsf{Y}, \mathsf{D} \in \mathcal{M}_{n,1}(\mathcal{T}_{\mathrm{HReg}}(\Sigma, \mathcal{Y}))$ with $\mathsf{Y}_i := y_i$ and $\mathsf{D}_i := o(r_i)$ for $1 \le i \le p$.

By construction, we have $\mathsf{X} \doteq_{\sigma_1} \mathsf{C} + \mathsf{A} \cdot \mathsf{X}$ and $\mathsf{Y} \doteq_{\sigma_2} \mathsf{D} + \mathsf{B} \cdot \mathsf{Y}$. Our intention is to generate an LS which has solution the $n \times p$ tuple of languages

$$\begin{array}{l}([\![\mathsf{X}_1]\!]_{\sigma_1} \cap [\![\mathsf{Y}_1]\!]_{\sigma_2}, \ldots, [\![\mathsf{X}_1]\!]_{\sigma_1} \cap [\![\mathsf{Y}_p]\!]_{\sigma_2}, \\ \quad \ldots, \\ \;[\![\mathsf{X}_n]\!]_{\sigma_1} \cap [\![\mathsf{Y}_1]\!]_{\sigma_2}, \ldots, [\![\mathsf{X}_n]\!]_{\sigma_1} \cap [\![\mathsf{Y}_p]\!]_{\sigma_2})\end{array}$$

and then solve it in order to find an RHG for the first component $[\![\mathsf{X}_1]\!]_{\sigma_1} \cap [\![\mathsf{Y}_1]\!]_{\sigma_2} = [\![S]\!] \cap [\![G]\!]$. Every equation of the new LS is obtained by intersecting an equation of $\mathsf{X} \doteq_{\sigma_1} \mathsf{C} + \mathsf{A} \cdot \mathsf{X}$ with an equation of $\mathsf{Y} \doteq_{\sigma_2} \mathsf{D} + \mathsf{B} \cdot \mathsf{Y}$, and introducing a fresh variable $z_{i,j}$ to denote the language $[\![x_i]\!]_{\sigma_1} \cap [\![y_j]\!]_{\sigma_2}$ for every $i \in \{1, \ldots, n\}$ and $j \in \{1 \ldots, p\}$. More formally, we proceed as follows:

1. We consider a set $\mathcal{Z} := \{z_{i,j} \mid 1 \le i \le n, 1 \le j \le p\}$ of distinct fresh variables, and define the assignment $\sigma : \mathcal{Z} \to \mathcal{P}(\mathcal{H}(\Sigma))$ by $\sigma(z_{i,j}) := \sigma_1(x_i) \cap \sigma_2(y_j)$ for all $i \in \{1, \ldots, n\}$ and $j \in \{1 \ldots, p\}$.

2. For every $i \in \{1, \ldots, n\}$ and $j \in \{1 \ldots, p\}$, we consider the i-th equation of the matriceal system $\mathsf{X} \doteq_{\sigma_1} \mathsf{C} + \mathsf{A} \cdot \mathsf{X}$:

$$x_i \doteq_{\sigma_1} \mathsf{C}_i + \sum_{k=1}^{n} \mathsf{A}_{i,k} \cdot x_k \tag{4}$$

and the j-the equation of the matriceal system $\mathsf{Y} \doteq_{\sigma_2} \mathsf{D} + \mathsf{B} \cdot \mathsf{Y}$:

$$y_j \doteq_{\sigma_2} \mathsf{D}_j + \sum_{l=1}^{p} \mathsf{B}_{j,l} \cdot y_l \tag{5}$$

The intersection of equations (4) and (5) yields the equation

$$z_{i,j} \doteq_{\sigma} \min\{\mathsf{C}_i, \mathsf{D}_j\} + \sum_{k=1}^{n} \sum_{l=1}^{p} P_{i,j,k,l} \cdot z_{k,l} \tag{6}$$

where $z_{i,j}$ is obtained by replacing the intersection $x_i \cap y_j$ of the left hand sides with $z_{i,j}$, and $\min\{\mathsf{C}_i, \mathsf{D}_j\} + \sum_{k=1}^{p} \sum_{l=1}^{n} P_{i,j,k,l} \cdot z_{k,l}$ is obtained from the intersection of the right hand sides

$$\left(\mathsf{C}_i + \sum_{k=1}^{n} \mathsf{A}_{i,k} \cdot x_k\right) \cap \left(\mathsf{D}_j + \sum_{l=1}^{p} \mathsf{B}_{j,l} \cdot y_l\right)$$

by applying distributivity of "∩" and "·" over "+", removing the summands of the form $(a\langle x_{k_1}\rangle \cdot x_{k_2}) \cap (b\langle y_{l_1}\rangle \cdot y_{l_2})$ with $a \neq b$, and replacing $\mathsf{C}_i \cap \mathsf{D}_j$ with $\min\{\mathsf{C}_i, \mathsf{D}_j\}$ and $(a\langle x_{k_1}\rangle \cdot x_{k_2}) \cap (a\langle y_{l_1}\rangle \cdot y_{l_2})$ with $a\langle z_{k_1,l_1}\rangle \cdot z_{k_2,l_2}$.

For example, the intersection of the relations

$$\begin{aligned} x_1 &\doteq_{\sigma_1} 0 + (a_1\langle x_2\rangle + a_3\langle x_4\rangle) \cdot x_1 + a_3\langle x_1\rangle \cdot x_2 \\ y_1 &\doteq_{\sigma_2} 1 + (a_1\langle y_1\rangle + a_2\langle y_2\rangle) \cdot y_1 + a_3\langle y_2\rangle \cdot y_2 \end{aligned}$$

yields $z_{1,1} \doteq_{\sigma} 0 + a_1\langle z_{2,1}\rangle \cdot z_{1,1} + a_3\langle z_{4,2}\rangle \cdot z_{1,2} + a_3\langle z_{1,2}\rangle \cdot z_{2,2}$.

Next, we put equations (6) in matriceal form. For this purpose we define

- $\mathsf{E} \in \mathcal{M}_{n \times p,1}(\mathcal{T}_{\text{HReg}}(\emptyset))$ with $\mathsf{E}_{(i-1)\cdot p+j} := \min\{\mathsf{C}_i, \mathsf{D}_j\}$ for all $i \in \{1, \ldots, n\}$ and $j \in \{1 \ldots, p\}$.
- $\mathsf{U} \in \mathcal{M}_{n \times p,1}(\mathcal{T}_{\text{HReg}}(\emptyset, \mathcal{Z}))$ with $\mathsf{U}_{(i-1)\cdot p+j} := z_{i,j}$ for all $i \in \{1, \ldots, n\}$ and $j \in \{1 \ldots, p\}$.
- $\mathsf{Q} \in \mathcal{M}_{n \times p, n \times p}(\mathcal{T}_{\text{HReg}}(\Sigma, \mathcal{Z}))$ with $\mathsf{Q}_{(i-1)\cdot p+j,(k-1)\cdot p+l} := P_{i,j,k,l}$ for all $i, k \in \{1, \ldots, n\}$ and $j, l \in \{1, \ldots, p\}$.

Then relations (6) are equivalent to the matriceal relation $\mathsf{U} \doteq_{\sigma} \mathsf{E} + \mathsf{Q} \cdot \mathsf{U}$. This is the instance of an LS with solution $([\![\mathsf{U}_1]\!]_{\sigma}, \ldots, [\![\mathsf{U}_{n \times p}]\!]_{\sigma})$, and we can compute RHGs for its components by using the constructive method described in Sect. 3. In this particular setting, the construction proceeds as follows:

1. We consider a set of new symbols $\mathcal{N} := \{\mathtt{n}_{k,l} \mid 1 \leq k \leq m, 1 \leq l \leq n \times p\}$ as placeholders for $a_k\langle \mathsf{U}_l \rangle$ in Q, and define the matrix $\mathsf{R} \in \mathcal{M}_{n\times p, n\times p}(\mathcal{T}_{\mathrm{Reg}}(\emptyset, \mathcal{N}))$ where every element $\mathsf{R}_{i,j}$ is obtained from $\mathsf{Q}_{i,j}$ by replacing the occurrences of $a_k\langle \mathsf{U}_l \rangle$ with $\mathtt{n}_{k,l}$.
2. We compute the regular expressions $r'_1, \ldots, r'_{n\times p} \in \mathcal{T}_{\mathrm{Reg}}(\emptyset, \mathcal{N})$ such that

$$\begin{pmatrix} r'_1 \\ \vdots \\ r'_{n\times p} \end{pmatrix} := \mathsf{R}^\star \cdot \mathsf{E}$$

   and conclude that $\mathsf{U}_i \doteq_\sigma G'_i$ where $G'_i := \langle \mathcal{N}, \Sigma, \mathcal{P}, r'_i \rangle$ for all $1 \leq i \leq n \times p$, where $\mathcal{P} := \{\mathtt{n}_{k,l} \to a_k\langle r'_l \rangle \mid 1 \leq k \leq m, 1 \leq l \leq n \times p\}$.

In particular, we have $[\![S]\!] \cap [\![G]\!] = [\![x_1]\!]_{\sigma_1} \cap [\![y_1]\!]_{\sigma_2} = [\![z_{1,1}]\!]_\sigma = [\![\mathsf{U}_1]\!]_\sigma = [\![G'_1]\!]$.

Example 7. Let $\Sigma = \{a_1, a_2\}$, $S = \langle \mathcal{N}_1, \Sigma, \mathcal{P}_1, \mathtt{n}_1 \cdot \mathtt{n}_2 \rangle$, $G = \langle \mathcal{N}_2, \Sigma, \mathcal{P}_2, \mathtt{n}_2 \cdot \mathtt{n}_1 \rangle$ with $\mathcal{N}_1 = \mathcal{N}_2 = \{\mathtt{n}_1, \mathtt{n}_2, \mathtt{n}_3\}$ and

$$\begin{aligned} \mathcal{P}_1 = \mathcal{P}_2 = \{&\mathtt{n}_1 \to a_1\langle(\mathtt{n}_1 + \mathtt{n}_2)^\star\rangle, \mathtt{n}_2 \to a_1\langle(\mathtt{n}_2 + \mathtt{n}_3)^\star\rangle, \\ &\mathtt{n}_2 \to a_3\langle \mathtt{n}_2^\star \rangle, \mathtt{n}_3 \to a_3\langle(\mathtt{n}_1 + \mathtt{n}_2)^\star\rangle\}, \end{aligned}$$

and suppose we want to compute an RHG for $[\![S]\!] \cap [\![G]\!]$. We have

$$\begin{aligned} \partial_{\mathcal{N}_1^*}(\mathtt{n}_1 \cdot \mathtt{n}_2) \cup \bigcup_{\mathtt{n} \to a\langle s \rangle \in \mathcal{P}_1} \partial_{\mathcal{N}^*}(s) &= \{\mathtt{n}_1 \cdot \mathtt{n}_2, \mathtt{n}_2, \mathtt{n}_2^\star, (\mathtt{n}_1 + \mathtt{n}_2)^\star, (\mathtt{n}_2 + \mathtt{n}_3)^\star, 1\}, \\ s_1 = \mathtt{n}_1 \cdot \mathtt{n}_2,\ s_2 = \mathtt{n}_2,\ s_3 = \mathtt{n}_2^\star,\ s_4 &= (\mathtt{n}_1 + \mathtt{n}_2)^\star,\ s_5 = (\mathtt{n}_2 + \mathtt{n}_3)^\star,\ s_6 = 1 \\ \partial_{\mathcal{N}_2^*}(\mathtt{n}_2 \cdot \mathtt{n}_1) \cup \bigcup_{\mathtt{n} \to a\langle r \rangle \in \mathcal{P}_2} \partial_{\mathcal{N}^*}(r) &= \{\mathtt{n}_2 \cdot \mathtt{n}_1, \mathtt{n}_1, \mathtt{n}_2^\star, (\mathtt{n}_1 + \mathtt{n}_2)^\star, (\mathtt{n}_2 + \mathtt{n}_3)^\star, 1\}, \\ r_1 = \mathtt{n}_2 \cdot \mathtt{n}_1,\ r_2 = \mathtt{n}_1,\ r_3 = \mathtt{n}_2^\star,\ r_4 &= (\mathtt{n}_1 + \mathtt{n}_2)^\star,\ r_5 = (\mathtt{n}_2 + \mathtt{n}_3)^\star,\ r_6 = 1 \end{aligned}$$

and the following systems of equations:

$$\begin{cases} S(s_1) \doteq \mathtt{af}_2(S(s_1)) \doteq 0 + a_1\langle S(s_4)\rangle \cdot S(s_2), \\ S(s_2) \doteq \mathtt{af}_2(S(s_2)) \doteq 0 + (a_1\langle S(s_5)\rangle + a_3\langle S(s_3)\rangle) \cdot S(s_6), \\ S(s_3) \doteq \mathtt{af}_2(S(s_3)) \doteq 1 + (a_1\langle S(s_5)\rangle + a_3\langle S(s_3)\rangle) \cdot S(s_3), \\ S(s_4) \doteq \mathtt{af}_2(S(s_4)) \doteq 1 + (a_1\langle S(s_4)\rangle + a_1\langle S(s_5)\rangle + a_3\langle S(s_3)\rangle) \cdot S(s_4), \\ S(s_5) \doteq \mathtt{af}_2(S(s_5)) \doteq 1 + (a_1\langle S(s_5)\rangle + a_3\langle S(s_3)\rangle + a_3\langle S(s_4)\rangle) \cdot S(s_5), \\ S(s_6) \doteq \mathtt{af}_2(S(s_6)) \doteq 1 \end{cases} \tag{7}$$

and

$$\begin{cases} G(r_1) \doteq \mathtt{af}_2(G(r_1)) \doteq 0 + (a_1\langle G(r_5)\rangle + a_3\langle G(r_3)\rangle) \cdot G(r_2), \\ G(r_2) \doteq \mathtt{af}_2(G(r_2)) \doteq 0 + a_1\langle G(r_4)\rangle \cdot G(r_6), \\ G(r_3) \doteq \mathtt{af}_2(G(r_3)) \doteq 1 + (a_1\langle G(r_5)\rangle + a_3\langle G(r_3)\rangle) \cdot G(r_3), \\ G(r_4) \doteq \mathtt{af}_2(G(r_4)) \doteq 1 + (a_1\langle G(r_4)\rangle + a_1\langle G(r_5)\rangle + a_3\langle G(r_3)\rangle) \cdot G(r_4), \\ G(r_5) \doteq \mathtt{af}_2(G(r_5)) \doteq 1 + (a_1\langle G(r_5)\rangle + a_3\langle G(r_3)\rangle + a_3\langle G(r_4)\rangle) \cdot G(r_5), \\ G(r_6) \doteq \mathtt{af}_2(G(r_6)) \doteq 1. \end{cases} \tag{8}$$

We consider the set of symbols $\mathcal{Z} = \{z_{i,j} \mid 1 \leq i, j \leq 6\}$ to be used as placeholders for the languages of the set $\{[\![s_i]\!]_S \cap [\![r_j]\!]_G \mid 1 \leq i, j \leq 6\}$. By intersecting all pairs of equations of (7) and (8), we obtain a system of 36 equations that are valid for the assignment $\sigma : \mathcal{Z} \to \mathcal{P}(\mathcal{H}(\Sigma))$ defined by $\sigma(z_{i,j}) = [\![s_i]\!]_S \cap [\![r_j]\!]_G$:

$$\begin{cases} z_{1,1} \doteq_\sigma 0 + a_1\langle z_{4,5}\rangle \cdot z_{2,2} \\ z_{2,2} \doteq_\sigma 0 + a_1\langle z_{5,4}\rangle \cdot z_{6,6} \\ z_{3,3} \doteq_\sigma 1 + (a_1\langle z_{5,5}\rangle + a_3\langle z_{3,3}\rangle) \cdot z_{3,3} \\ z_{3,4} \doteq_\sigma 1 + (a_1\langle z_{5,4}\rangle + a_1\langle z_{5,5}\rangle + a_3\langle z_{3,3}\rangle) \cdot z_{3,4} \\ z_{4,3} \doteq_\sigma 1 + (a_1\langle z_{4,5}\rangle + a_1\langle z_{5,5}\rangle + a_3\langle z_{3,3}\rangle) \cdot z_{4,3} \\ z_{4,4} \doteq_\sigma 1 + (a_1\langle z_{4,4}\rangle + a_1\langle z_{4,5}\rangle + a_1\langle z_{5,4}\rangle + a_1\langle z_{5,5}\rangle + a_3\langle z_{3,3}\rangle) \cdot z_{4,4} \\ z_{4,5} \doteq_\sigma 1 + (a_1\langle z_{4,5}\rangle + a_1\langle z_{5,5}\rangle + a_3\langle z_{3,3}\rangle + a_3\langle z_{3,4}\rangle) \cdot z_{4,5} \\ z_{5,4} \doteq_\sigma 1 + (a_1\langle z_{5,4}\rangle + a_1\langle z_{5,5}\rangle + a_3\langle z_{3,3}\rangle + a_3\langle z_{4,3}\rangle) \cdot z_{5,4} \\ z_{5,5} \doteq_\sigma 1 + (a_1\langle z_{5,5}\rangle + a_3\langle z_{3,3}\rangle + a_3\langle z_{3,4}\rangle + a_3\langle z_{4,3}\rangle + a_3\langle z_{4,4}\rangle) \cdot z_{5,5} \\ z_{6,6} \doteq_\sigma 1 \\ \ldots \end{cases}$$

We have elided the equations that play no role in the specification of $z_{1,1}$. There are only 10 relevant equations and we can ignore the other 26.

Next, we define a new set $\mathcal{N}$ of nonterminals, one for every expression $a_i\langle z_{j,k}\rangle$ that occurs in the system of 10 relevant equations. In this case we define $\mathcal{N} := \{\mathtt{n}_1, \mathtt{n}_2, \mathtt{n}_3, \mathtt{n}_4, \mathtt{n}_5, \mathtt{n}_6, \mathtt{n}_7, \mathtt{n}_8\}$ where $\mathtt{n}_1$ replaces $a_1\langle z_{4,5}\rangle$, $\mathtt{n}_2$ replaces $a_1\langle z_{5,4}\rangle$, $\mathtt{n}_3$ replaces $a_1\langle z_{5,5}\rangle$, $\mathtt{n}_4$ replaces $a_3\langle z_{3,3}\rangle$, $\mathtt{n}_5$ replaces $a_3\langle z_{3,4}\rangle$, $\mathtt{n}_6$ replaces $a_3\langle z_{4,3}\rangle$, $\mathtt{n}_7$ replaces $a_1\langle z_{4,4}\rangle$, and $\mathtt{n}_8$ replaces $a_3\langle z_{4,4}\rangle$. We obtain the system of equations

$$\begin{cases} z_{1,1} \doteq_\theta 0 + \mathtt{n}_1 \cdot z_{2,2} \\ z_{2,2} \doteq_\theta 0 + \mathtt{n}_2 \cdot z_{6,6} \\ z_{3,3} \doteq_\theta 1 + (\mathtt{n}_3 + \mathtt{n}_4) \cdot z_{3,3} \\ z_{3,4} \doteq_\theta 1 + (\mathtt{n}_2 + \mathtt{n}_3 + \mathtt{n}_4) \cdot z_{3,4} \\ z_{4,3} \doteq_\theta 1 + (\mathtt{n}_1 + \mathtt{n}_3 + \mathtt{n}_4) \cdot z_{4,3} \\ z_{4,4} \doteq_\theta 1 + (\mathtt{n}_7 + \mathtt{n}_1 + \mathtt{n}_2 + \mathtt{n}_3 + \mathtt{n}_4) \cdot z_{4,4} \\ z_{4,5} \doteq_\theta 1 + (\mathtt{n}_1 + \mathtt{n}_3 + \mathtt{n}_4 + \mathtt{n}_5) \cdot z_{4,5} \\ z_{5,4} \doteq_\theta 1 + (\mathtt{n}_2 + \mathtt{n}_3 + \mathtt{n}_4 + \mathtt{n}_6) \cdot z_{5,4} \\ z_{5,5} \doteq_\theta 1 + (\mathtt{n}_3 + \mathtt{n}_4 + \mathtt{n}_5 + \mathtt{n}_6 + \mathtt{n}_8) \cdot z_{5,5} \\ z_{6,6} \doteq_\theta 1. \end{cases} \tag{9}$$

which is valid under assignment $\theta : \mathcal{Z} \cup \mathcal{N} \to \mathcal{P}(\mathcal{H}(\Sigma))$ defined by

$$\theta(\mathtt{n}_1) := a_1\langle\sigma(z_{4,5})\rangle, \theta(\mathtt{n}_2) := a_1\langle\sigma(z_{5,4})\rangle, \theta(\mathtt{n}_3) := a_1\langle\sigma(z_{5,5})\rangle,$$
$$\theta(\mathtt{n}_4) := a_3\langle\sigma(z_{3,3})\rangle, \theta(\mathtt{n}_5) := a_3\langle\sigma(z_{3,4})\rangle, \theta(\mathtt{n}_6) := a_3\langle\sigma(z_{4,3})\rangle,$$
$$\theta(\mathtt{n}_7) := a_1\langle\sigma(z_{4,4})\rangle, \theta(\mathtt{n}_8) := a_3\langle\sigma(z_{4,4})\rangle,$$

and $\theta(z_{i,j}) := \sigma(z_{i,j})$ if $z_{i,j} \in \mathcal{Z}$. System (9) can be put in the matrix form $\mathsf{U} \doteq_\theta \mathsf{E} + \mathsf{Q} \cdot \mathsf{U}$ where

$$\mathsf{U} = \begin{pmatrix} z_{1,1} \\ z_{2,2} \\ z_{3,3} \\ z_{3,4} \\ z_{4,3} \\ z_{4,4} \\ z_{4,5} \\ z_{5,4} \\ z_{5,5} \\ z_{6,6} \end{pmatrix}, \quad \mathsf{E} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

and Q is

$$\begin{pmatrix}
0 & \mathtt{n}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathtt{n}_2 \\
0 & 0 & \mathtt{n}_3 + \mathtt{n}_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \begin{matrix}\mathtt{n}_2 + \mathtt{n}_3 \\ +\mathtt{n}_4\end{matrix} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \begin{matrix}\mathtt{n}_1 + \mathtt{n}_3 \\ +\mathtt{n}_4\end{matrix} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \begin{matrix}\mathtt{n}_7 + \mathtt{n}_1 + \mathtt{n}_2 \\ +\mathtt{n}_3 + \mathtt{n}_4\end{matrix} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \begin{matrix}\mathtt{n}_1 + \mathtt{n}_3 \\ +\mathtt{n}_4 + \mathtt{n}_5\end{matrix} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \begin{matrix}\mathtt{n}_2 + \mathtt{n}_3 \\ +\mathtt{n}_4 + \mathtt{n}_6\end{matrix} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \begin{matrix}\mathtt{n}_3 + \mathtt{n}_4 + \mathtt{n}_5 \\ +\mathtt{n}_6 + \mathtt{n}_8\end{matrix} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

It follows that $\mathsf{U}_i \doteq_\sigma G'_i$ for $1 \leq i \leq 10$, where

$$\begin{aligned}
G'_i &:= \langle \mathcal{N}, \Sigma, \mathcal{P}, r'_i \rangle \quad \text{with } r'_i := (\mathsf{Q}^\star \cdot \mathsf{E})_i \qquad (1 \leq i \leq 10) \\
\mathcal{P} &:= \{\mathtt{n}_1 \to a_1\langle r'_7\rangle, \mathtt{n}_2 \to a_1\langle r'_8\rangle, \mathtt{n}_3 \to a_1\langle r'_9\rangle, \mathtt{n}_4 \to a_3\langle r'_3\rangle, \\
&\qquad \mathtt{n}_5 \to a_3\langle r'_4\rangle, \mathtt{n}_6 \to a_3\langle r'_5\rangle, \mathtt{n}_7 \to a_1\langle r'_6\rangle, \mathtt{n}_8 \to a_3\langle r'_6\rangle\}.
\end{aligned}$$

In particular, $[\![S]\!] \cap [\![G]\!] = [\![s_1]\!]_S \cap [\![r_1]\!]_G = [\![z_{1,1}]\!]_\sigma = [\![\mathsf{U}_1]\!]_\sigma = [\![G'_1]\!]$. □

5.3 Product Derivative of Regular Hedge Languages

The product derivative of word languages [18] has a straightforward generalization to hedge languages: If $H_1, H_2 \in \mathcal{P}(\mathcal{H}(\Sigma))$ then the *product derivative* of H_1 with respect to H_2 is the hedge language $\{h \mid \forall h_2 \in H_2.\, h_2 h \in H_1\}$. In other words, the product derivative of H_1 with respect to H_2 is the $\subseteq$-maximal solution of the relation $H_2.X \subseteq H_1$ in the algebra $\mathcal{A}_{\mathrm{HReg}}(\Sigma)$. We denote this hedge language by $H_2 \rhd H_1$.

In this section we prove that if H_1, H_2 are RHLs then $H_2 \rhd H_1$ is RHL too, and indicate a method to solve the following problem:

Given: $\Sigma = \{a_1, \ldots, a_m\}$, $S_0 \in \mathcal{G}(\mathcal{N}_1, \Sigma)$, and $G_0 \in \mathcal{G}(\mathcal{N}_2, \Sigma)$
Compute: RHG G' such that $[\![G']\!] = ([\![S_0]\!] \rhd [\![G_0]\!])$.

First, we define $\mathtt{lf}_a(M) := \bigcup_{G \in N}\{\langle a\langle G'\rangle, G''\rangle \mid \langle a\langle G'\rangle, G''\rangle \in \mathtt{lf}(G)\}$ for every $a \in \Sigma$ and $M \subseteq \partial_{\mathcal{H}(\Sigma)}(\{S_0, G_0\})$, and define inductively the ternary relation $M \lhd S \blacktriangleright N$ for $M, N \in \mathcal{P}(\partial_{\mathcal{H}(\Sigma)}(G_0))$ and $S \in \partial_{\mathcal{H}(\Sigma)}(S_0)$, by

$$\frac{[o(\mathtt{ini}(S)) = 1 \wedge \forall G \in M\,.o(\mathtt{ini}(G)) = 1 \wedge \forall G' \in N.\, o(\mathtt{ini}(G')) = 0]}{M \lhd S \blacktriangleright N}$$

$$\frac{M_1 \lhd S_1 \blacktriangleright \{G'_{1,i} \mid i \in I_1\} \quad M_2 \lhd S_2 \blacktriangleright \{G'_{2,i} \mid i \in I_2\} \quad [\alpha]}{\{G_1, \ldots, G_n\} \lhd S \blacktriangleright N}$$

where $M_k = \{G_{k,i} \mid i \in \{1, \ldots, n\}\}$ for $k \in \{1, 2\}$, and the side condition of the last inference rule is

$$\begin{aligned} \alpha \equiv\ & \{G_1, \ldots, G_n\} \cap N = \emptyset \wedge \langle a\langle S_1\rangle, S_2\rangle \in \mathtt{lf}(S) \\ & \wedge \forall i \in \{1, \ldots, n\}.\, \langle a\langle G_{1,i}\rangle, G_{2,i}\rangle \in \mathtt{lf}(G_i) \\ & \wedge \mathtt{lf}_a(N) = \{\langle a\langle G'_{1,i}\rangle, G'_{2,i}\rangle \mid 1 \le i \le m\} \wedge \{1, \ldots, m\} = I_1 \uplus I_2. \end{aligned}$$

($\{1, \ldots, m\} = I_1 \uplus I_2$ denotes the fact that $\{I_1, I_2\}$ is a partition of $\{1, \ldots, m\}$.) This relation is decidable because it is defined inductively on finite sets. Therefore, we can define and compute for every $a \in \Sigma$, $S \in \partial_{\mathcal{H}(\Sigma)}(S_0)$, and $M \subseteq \partial_{\mathcal{H}(\Sigma)}(G_0)$ with $\mathtt{lf}_a(M) = \{\langle a\langle G_i\rangle, G'_i\rangle \mid 1 \le i \le n\}$, the set

$$\begin{aligned} \mathtt{lq}_{a\langle S\rangle}(M) := \{\{G'_i \mid i \in I_1\} \mid & \{1, \ldots, m\} = I_1 \uplus I_2 \wedge \\ & \{G_i \mid i \in I_1\} \lhd S \blacktriangleright \{G_i \mid i \in I_2\}\}. \end{aligned}$$

Intuitively, $M \lhd G \blacktriangleright N$ holds iff there exists a hedge $h \in [\![G]\!]$ that belongs to the languages of all RHGs in M but to none of the languages of RHGs in N. This intuition is confirmed by the following lemma.

Lemma 7. *$M \lhd S \blacktriangleright N$ iff* $\left(\bigcap_{G \in M \cup \{S\}} [\![G]\!]\right) \setminus \bigcup_{G \in N} [\![G]\!] \neq \emptyset$.

Proof. First, we prove that $M \lhd S \blacktriangleright N$ implies $\left(\bigcap_{G \in M \cup \{S\}} [\![G]\!]\right) \setminus \bigcup_{G \in N} [\![G]\!] \neq \emptyset$. The proof is by induction on the size of the derivation of $M \lhd S \blacktriangleright N$. If the shortest derivation is

$$\frac{[o(\mathtt{ini}(S)) = 1 \wedge \forall G \in M\,.o(\mathtt{ini}(G)) = 1 \wedge \forall G' \in N.\, o(\mathtt{ini}(G')) = 0]}{M \lhd S \blacktriangleright N}$$

then $\epsilon \in \left(\bigcap_{G \in M \cup \{S\}} [\![G]\!]\right) \setminus \bigcup_{G \in N} [\![G]\!]$. Otherwise, $M = \{G_1, \ldots, G_n\}$ and the shortest derivation of $M \lhd S \blacktriangleright N$ is

$$\frac{\mathcal{D}_1 \quad \mathcal{D}_2 \quad [\alpha]}{\{G_1, \ldots, G_n\} \lhd S \blacktriangleright N}$$

with $\mathcal{D}_1$ a shortest derivation of $M_1 \lhd S_1 \blacktriangleright \{G'_{1,i} \mid i \in I_1\}$, $\mathcal{D}_2$ a shortest derivation of $M_2 \lhd S_2 \blacktriangleright \{G'_{2,i} \mid i \in I_2\}$ and the side condition $[\alpha]$ as shown above. By induction hypothesis for $\mathcal{D}_1$ and $\mathcal{D}_2$, there exist

$$h_k \in \left(\bigcap_{G \in \{G_{k,i} \mid i \in \{1,\ldots,n\}\} \cup \{S_k\}} [\![G]\!]\right) \setminus \bigcup_{G \in \{G'_{k,i} \mid i \in I_k\}} [\![G]\!]$$

for $k \in \{1,2\}$, and this implies $\langle a\langle h_1\rangle\, h_2\rangle \in \left(\bigcap_{G \in M \cup \{S\}} [\![G]\!]\right) \setminus \bigcup_{G \in N} [\![G]\!]$.

Next, we prove that $\left(\bigcap_{G \in M \cup \{S\}} [\![G]\!]\right) \setminus \bigcup_{G \in N} [\![G]\!] \neq \emptyset$ implies $M \lhd S \blacktriangleright N$. This proof is by induction on the size of the shortest hedge in $\left(\bigcap_{G \in M \cup \{S\}} [\![G]\!]\right) \setminus \bigcup_{G \in N} [\![G]\!]$. If $\epsilon \in \left(\bigcap_{G \in M \cup \{S\}} [\![G]\!]\right) \setminus \bigcup_{G \in N} [\![G]\!]$ then $o(\texttt{ini}(G)) = 1$ for all $G \in M \cup \{S\}$ and $o(\texttt{ini}(G)) = 0$ for all $G \in N$, therefore we can perform the derivation

$$\overline{M \lhd S \blacktriangleright N}$$

to conclude that $M \lhd S \blacktriangleright N$ holds. Otherwise, the shortest hedge in the set $\left(\bigcap_{G \in M \cup \{S\}} [\![G]\!]\right) \setminus \bigcup_{G \in N} [\![G]\!]$ is of the form $a\langle h_1\rangle\, h_2$. Suppose $M = \{G_1, \ldots, G_n\}$ and $\texttt{lf}_a(N) = \{\langle a\langle G'_{1,i}\rangle, G'_{2,i}\rangle \mid 1 \leq i \leq m\}$. From $a\langle h_1\rangle\, h_2 \in \left(\bigcap_{G \in M \cup \{S\}} [\![G]\!]\right) \setminus \bigcup_{G \in N} [\![G]\!]$ we learn that

- $\{G_1, \ldots, G_n\} \cap N = \emptyset$.
- $a\langle h_1\rangle\, h_2 \in [\![G_i]\!] \setminus \{\epsilon\} = \bigcup_{\langle a\langle G_1\rangle, G_2\rangle \in \texttt{lf}(G_i)} a\langle [\![G_1]\!]\rangle.[\![G_2]\!]$ for all $i \in \{1, \ldots, n\}$, thus for every $i \in \{1, \ldots, n\}$ there exists a pair $\langle a\langle G_{1,i}\rangle, G_{2,i}\rangle \in \texttt{lf}(G_i)$ with $h_1 \in [\![G_{1,i}]\!]$ and $h_2 \in [\![G_{2,i}]\!]$.
- $a\langle h_1\rangle\, h_2 \in [\![S]\!] \setminus \{\epsilon\} = \bigcup_{\langle a\langle S_1\rangle, S_2\rangle \in \texttt{lf}(S)} a\langle [\![S_1]\!]\rangle.[\![S_2]\!]$, thus there exists a pair $\langle a\langle S_1\rangle, S_2\rangle \in \texttt{lf}(S)$ with $h_1 \in [\![S_1]\!]$ and $h_2 \in [\![S_2]\!]$.
- $a\langle h_1\rangle\, h_2 \notin \bigcup_{G \in N} [\![G]\!]$, and since

$$\bigcup_{G \in N} [\![G]\!] \setminus \{\epsilon\} = \bigcup_{a \in \Sigma} \bigcup_{\langle a\langle G_1\rangle, G_2\rangle \in \texttt{lf}_a(N)} a\langle [\![G_1]\!]\rangle.[\![G_2]\!],$$
$$\texttt{lf}_a(N) = \{\langle a\langle G'_{1,i}\rangle, G'_{2,i}\rangle \mid 1 \leq i \leq m\},$$

we conclude that $a\langle h_1\rangle\, h_2 \notin \bigcup_{i=1}^{m} a\langle [\![G'_{1,i}]\!]\rangle.[\![G'_{2,i}]\!]$. We consider the sets

$$I_1 := \{i \in \{1, \ldots, m\} \mid h_1 \notin [\![G'_{1,i}]\!]\},$$
$$I_2 := \{1, \ldots, m\} \setminus I_1,$$

and note that $h_1 \notin \bigcup_{i \in I_1} [\![G'_{1,i}]\!]$ and $h_2 \notin \bigcup_{i \in I_2} [\![G'_{2,i}]\!]$.

Let $M_k := \{G_{k,i} \mid i \in \{1, \ldots, n\}\}$ for $k \in \{1,2\}$. Then

$$\|h_k\| < \|h\| \text{ and } h_k \in \left(\bigcap_{G \in M_k \cup \{S\}} [\![G]\!]\right) \setminus \bigcup_{G \in \{G'_{k,i} \mid i \in I_k\}} [\![G]\!] \quad \text{for } k \in \{1,2\},$$

and we can apply the induction hypothesis to infer the existence of derivations $\mathcal{D}_k$ for $M_k \lhd S_k \blacktriangleright \{G'_{k,i} \mid i \in I_k\}$ when $k \in \{1,2\}$. We can construct now the derivation

$$\frac{\mathcal{D}_1 \quad \mathcal{D}_2}{\{G_1,\ldots,G_n\} \lhd S \blacktriangleright N}$$

to conclude that $M \lhd S \blacktriangleright N$ holds. $\square$

Corollary 3. $\mathtt{lq}_{a\langle S\rangle}(M) = \{\partial_{a\langle h\rangle}(M) \mid a\langle h\rangle \in [\![a\langle S\rangle]\!]\}$.

Next, we consider the relation $S \rhd M \rightsquigarrow N$ for $M, N \in \mathcal{P}(\partial_{\mathcal{H}(\Sigma)}(G_0))$ and $S \in \partial_{\mathcal{H}(\Sigma)}(S_0)$, defined inductively by the inference rules:

$$\frac{[o(\mathtt{ini}(S)) = 1]}{S \rhd M \rightsquigarrow M} \quad \frac{S'_1 \rhd M' \rightsquigarrow N \quad [\langle a\langle S_1\rangle, S'_1\rangle \in \mathtt{lf}(S) \wedge M' \in \mathtt{lq}_{a\langle S_1\rangle}(M)]}{S \rhd M \rightsquigarrow N}.$$

Intuitively, the relation $S \rhd M \rightsquigarrow N$ holds iff there exists $h \in [\![S]\!]$ such that $N = \partial_h(M)$. This ternary relation is decidable because it is defined inductively on finite sets. Finally, we show that the following result holds.

Lemma 8. *If $S \in \partial_{\mathcal{H}(\Sigma)}(S_0)$ and $M \subseteq \partial_{\mathcal{H}(\Sigma)}(G_0)$, then*

$$\left([\![S]\!] \rhd \bigcup_{G\in M} [\![G]\!]\right) = \bigcap_{S\rhd M\rightsquigarrow N} \left(\bigcup_{G\in N} [\![G]\!]\right).$$

Proof. For every $n \in \mathbb{N}$ we define the ternary relation

$$S \rhd M \rightsquigarrow_n N :\Leftrightarrow \text{there is a derivation of length } n \text{ of } S \rhd M \rightsquigarrow N$$

and note that

$$\left([\![S]\!] \rhd \bigcup_{G\in M} [\![G]\!]\right) = \bigcap_{n\in\mathbb{N}} \bigcap_{\substack{h\in[\![S]\!] \\ |h|=n}} h^{-1}\left(\bigcup_{G\in M} [\![G]\!]\right),$$

$$\bigcap_{S\rhd M\rightsquigarrow N} \left(\bigcup_{G\in N} [\![G]\!]\right) = \bigcap_{n\in\mathbb{N}} \bigcap_{S\rhd M\rightsquigarrow_n N} \left(\bigcup_{G\in N} [\![G]\!]\right).$$

Thus it is sufficient to prove that

$$\left\{h^{-1}\left(\bigcup_{G\in M} [\![G]\!]\right) \,\middle|\, h \in [\![S]\!] \wedge |h| = n\right\} = \left\{\bigcup_{G\in N} [\![G]\!] \,\middle|\, S \rhd M \rightsquigarrow_n N\right\} \tag{10}$$

holds for every $S \in \partial_{\mathcal{H}(\Sigma)}(S_0)$, $M \subseteq \partial_{\mathcal{H}(\Sigma)}(G_0)$, and $n \in \mathbb{N}$. We prove this fact by induction on n. If $n = 0$ then

$$\left\{h^{-1}\left(\bigcup_{G\in M} [\![G]\!]\right) \,\middle|\, h \in [\![S]\!] \wedge |h| = 0\right\} = \left\{\bigcup_{G\in M} [\![G]\!] \,\middle|\, \epsilon \in [\![S]\!]\right\} \quad \text{and}$$

$$\left\{\bigcup_{G\in N} [\![G]\!] \,\middle|\, S \rhd M \rightsquigarrow_0 N\right\} = \left\{\bigcup_{G\in M} [\![G]\!] \,\middle|\, o(\mathtt{ini}(S)) = 1\right\}.$$

But $o(\texttt{ini}(S)) = 1$ iff $\epsilon \in [\![S]\!]$, therefore (10) holds for $n = 0$. If $n > 0$ then

$$\left\{h^{-1}\left(\bigcup_{G\in M}[\![G]\!]\right)\,\middle|\, h \in [\![S]\!] \wedge |h| = n\right\} =$$

$$\bigcup_{\langle a\langle S_1\rangle, S_1'\rangle\in\texttt{lf}(S)} \left\{(a\langle h_1\rangle\, h_1')^{-1}\bigcup_{G\in M}[\![\texttt{af}_2(G)]\!]\,\middle|\, h_1 \in [\![S_1]\!] \wedge h_1' \in [\![S_1']\!] \wedge |h_1'| = n-1\right\}$$

$$= \bigcup_{\langle a\langle S_1\rangle, S_1'\rangle\in\texttt{lf}(S)} \Big\{{h_1'}^{-1}\big(a\langle h_1\rangle^{-1} \bigcup_{\langle a\langle G_1\rangle, G_1'\rangle\in\texttt{lf}_a(M)} a\langle[\![G_1]\!]\rangle.[\![G_1']\!]\big)\,\Big|\, h_1 \in [\![S_1]\!] \wedge h_1' \in [\![S_1']\!] \wedge |h_1'| = n-1\Big\}$$

$$= \bigcup_{\langle a\langle S_1\rangle, S_1'\rangle\in\texttt{lf}(S)}\ \bigcup_{M'\in\texttt{lq}_{a\langle S_1\rangle}(M)} \left\{{h_1'}^{-1}\bigcup_{G_1'\in M'}[\![G_1']\!]\,\middle|\, h_1' \in [\![S_1']\!] \wedge |h_1'| = n-1\right\}.$$

By induction hypothesis, we have

$$\left\{{h_1'}^{-1}\bigcup_{G_1'\in M'}[\![G_1']\!]\,\middle|\, h_1' \in [\![S_1']\!] \wedge |h_1'| = n-1\right\} = \left\{\bigcup_{G\in N}[\![G]\!]\,\middle|\, S_1' \rhd M' \rightsquigarrow_{n-1} N\right\}$$

Since the only way to infer $S \rhd M \rightsquigarrow_n N$ is

$$\frac{S_1' \rhd M' \rightsquigarrow_{n-1} N \quad [\langle a\langle S_1\rangle, S_1'\rangle \in \texttt{lf}(S) \wedge M' \in \texttt{lq}_{a\langle S_1\rangle}(M)]}{S \rhd M \rightsquigarrow_n N},$$

we conclude that

$$\bigcup_{\langle a\langle S_1\rangle, S_1'\rangle\in\texttt{lf}(S)}\ \bigcup_{M'\in\texttt{lq}_{a\langle S_1\rangle}(M)} \left\{\bigcup_{G\in N}[\![G]\!]\,\middle|\, S_1' \rhd M' \rightsquigarrow_{n-1} N\right\} = \left\{\bigcup_{G\in N}[\![G]\!]\,\middle|\, S \rhd M \rightsquigarrow_n N\right\}.$$

This shows that (10) holds for $n > 0$ too. □

The Algorithm. Lemma 8 enables the computation of an RHG for $[\![S_0]\!] \rhd [\![G_0]\!]$ as follows. Suppose $G_0 = \langle \mathcal{N}_2, \Sigma, \mathcal{P}_2, r_1\rangle \in \mathcal{G}(\mathcal{N}_2, \Sigma)$. Then

1. We compute the finite set $\mathfrak{N} := \{N \mid S_0 \rhd \{G_0\} \rightsquigarrow N\}$.
2. Since $[\![S_0]\!] \rhd [\![G_0]\!] = \bigcap_{N\in\mathfrak{N}}\bigcup_{G\in N}[\![G]\!]$ and every $N \in \mathfrak{N}$ is a subset of $\{G_0(r) \mid r \in \partial_{\mathcal{N}_2^*}(r_1)\}$, we can apply distributivity of intersection over union to express $[\![S_0]\!] \rhd [\![G_0]\!]$ as union of intersections of languages produced by partial derivatives of G_0. In this way we obtain $([\![S_0]\!] \rhd [\![G_0]\!]) = \bigcup_{M\in\mathfrak{M}}\bigcap_{G\in M}[\![G]\!] = \bigcup_{M\in\mathfrak{M}_0}\bigcap_{G\in M}[\![G]\!]$ where $\mathfrak{M} \subseteq \mathcal{P}(\{G_0(r) \mid r \in \partial_{\mathcal{N}_2^*}(r_1)\})$ and $\mathfrak{M}_0$ is the set of $\subseteq$-minimal elements of $\mathfrak{M}$.

3. We use the differential calculus described in Sect. 4 to compute $\{r_1, \ldots, r_p\} := \partial_{\mathcal{N}_2^*}(r_1) \cup \bigcup_{\mathtt{n} \to a\langle r\rangle \in \mathcal{P}_2} \partial_{\mathcal{N}_2^*}(r)$, $\mathsf{A} \in \mathcal{M}_{p,p}(\mathcal{T}_{\mathrm{HReg}}(\Sigma, \mathcal{X}))$ with $\mathcal{X} := \{x_1, \ldots, x_p\}$, and $\mathsf{C} \in \mathcal{M}_{p,1}(\mathcal{T}_{\mathrm{Reg}}(\emptyset))$ such that the matriceal equality

$$\begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} \doteq_\sigma \mathsf{C} + \mathsf{A} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} \tag{11}$$

is an instance of an LS valid under the assignment $\sigma : \mathcal{X} \to \mathcal{P}(\mathcal{H}(\Sigma))$ defined by $\sigma(x_i) := [\![r_i]\!]_{G_0}$ for all $1 \leq i \leq p$.

4. Let $\{J_1, \ldots, J_{2^p}\}$ be an enumeration of the nonempty subsets of $\{1, \ldots, p\}$, $\mathcal{Z} := \{z_l \mid 1 \leq l < 2^p\}$ be a set of fresh symbols, and $\theta : \mathcal{Z} \to \mathcal{P}(\mathcal{H}(\Sigma))$ be the assignment defined by $\theta(z_l) := \bigcap_{j \in J_l} \sigma(x_j) = \bigcap_{j \in J_l} [\![r_j]\!]_{G_0}$ for all $1 \leq l < 2^p$.
We compute $\mathfrak{J} := \{j \mid (1 \leq j \leq 2^p) \wedge \{G_0(r_i) \mid i \in J_j\} \in \mathfrak{M}_0\}$. Note that

$$([\![S_0]\!] \rhd [\![G_0]\!]) = \bigcup_{j \in \mathfrak{J}} \bigcap_{i \in J_j} [\![G_0(r_i)]\!].$$

5. For every $j \in \{1, \ldots, 2^p - 1\}$ with $J_j = \{j_1, \ldots, j_k\}$ we intersect the equalities

$$\begin{cases} x_{j_1} \doteq_\sigma \mathsf{C}_{j_1} + \sum_{l=1}^{p} \mathsf{A}_{j_1,l} \cdot x_l \\ \vdots \\ x_{j_k} \doteq_\sigma \mathsf{C}_{j_k} + \sum_{l=1}^{p} \mathsf{A}_{j_k,l} \cdot x_l \end{cases}$$

in order to obtain the equality

$$z_j \doteq_\theta \min\{\mathsf{C}_{j_i} \mid 1 \leq i \leq k\} + \sum_{m=1}^{2^p-1} \mathsf{B}_{j,m} \cdot z_m \tag{12}$$

where $\sum_{l=1}^{2^p-1} \mathsf{B}_{j,l} \cdot z_l$ is produced from $\bigcap_{m=1}^{k} (\sum_{l=1}^{p} \mathsf{A}_{j_m,l} \cdot x_l)$ by distributing "$\cap$" and "$\cdot$" over "$+$", and accounting only for the summands of the form $\bigcap_{m=1}^{k} a\langle x'_m\rangle \cdot x''_m$, which are to be replaced by $a\langle z_q\rangle \cdot z_v$ with q and v defined uniquely by $\{x_i \mid i \in J_q\} = \{x'_1, \ldots, x'_k\}$ and $\{x_i \mid i \in J_v\} = \{x''_1, \ldots, x''_k\}$.
Note that the elements of B are sums of regular expressions from $\{a\langle z_j\rangle \mid a \in \Sigma, 1 \leq j < 2^p\}$. Therefore, the equalities (12) render an LS with solution $([\![z]\!]_1, \ldots, [\![z]\!]_{2^p-1})$. By solving this LS we can produce RHGs for all nonempty intersections of languages from $\{[\![G_0(r_i)]\!] \mid 1 \leq i \leq p\}$, and there are at most $2^p - 1$ such intersections. Since we need to compute RHGs only for $\bigcap_{i \in J_j} [\![G_0(r_i)]\!]$ when $j \in \mathfrak{J}$, we can often remove irrelevant equations from the LS and obtain an LS of much smaller size. By solving this LS we compute at once RHGs for all the needed intersections. The computed RHGs have the same set of productions and same set of nonterminals, that is, they make up a set of the form $\{\langle \mathcal{N}, \Sigma, \mathcal{P}, r'_i\rangle \mid 1 \leq i \leq m\}$. Therefore $([\![S_0]\!] \rhd [\![G_0]\!]) = \bigcup_{i=1}^{m} [\![\langle \mathcal{N}, \Sigma, \mathcal{P}, r'_i\rangle]\!] = [\![G']\!]$ where $G' := \langle \mathcal{N}, \Sigma, \mathcal{P}, r'_1 + \ldots + r'_m\rangle$.

Example 8. Let's consider the RHGs

$$S = \langle \mathcal{N}_1, \Sigma, \mathcal{P}_1, \mathtt{m}_a \cdot \mathtt{m}_c^\star + \mathtt{m}_b \cdot \mathtt{m}_c^\star \rangle, \qquad G = \langle \mathcal{N}_2, \Sigma, \mathcal{P}_2, \mathtt{n}_a \cdot \mathtt{n}_3^\star + \mathtt{n}_b \cdot \mathtt{n}_4^\star \rangle$$

over the alphabet $\Sigma = \{a, b, c\}$ with $\mathcal{N}_1 = \{\mathtt{m}_a, \mathtt{m}_b, \mathtt{m}_c\}$, $\mathcal{N}_2 = \{\mathtt{n}_a, \mathtt{n}_b, \mathtt{n}_3, \mathtt{n}_4\}$,

$$\begin{aligned}
\mathcal{P}_1 &= \{\mathtt{m}_a \to a\langle 1 + \mathtt{m}_c\rangle, \mathtt{m}_b \to b\langle 1 + \mathtt{m}_c\rangle, \mathtt{m}_c \to c\langle 1 + \mathtt{m}_c\rangle\},\\
\mathcal{P}_2 &= \{\mathtt{n}_a \to a\langle 1 + \mathtt{n}_3\rangle, \mathtt{n}_b \to b\langle \mathtt{n}_4^\star\rangle, \mathtt{n}_3 \to c\langle \mathtt{n}_4^\star\rangle, \mathtt{n}_4 \to c\langle 1 + \mathtt{n}_3\rangle\},
\end{aligned}$$

and suppose we want to compute an RHG for $[\![S]\!] \rhd [\![G]\!]$. In this case we have

$$\begin{aligned}
\partial_{\mathcal{N}_1^*}(\mathtt{ini}(S)) \cup \bigcup_{\mathtt{m} \to a\langle s\rangle \in \mathcal{P}_1} \partial_{\mathcal{N}_1^*}(s) &= \{s_1, s_2, s_3, s_4\},\\
\partial_{\mathcal{N}_2^*}(\mathtt{ini}(G)) \cup \bigcup_{\mathtt{n} \to a\langle r\rangle \in \mathcal{P}_2} \partial_{\mathcal{N}_2^*}(r) &= \{r_1, r_2, r_3, r_4, r_5\},
\end{aligned}$$

where $s_1 = \mathtt{m}_a \cdot \mathtt{m}_c^\star + \mathtt{m}_b \cdot \mathtt{m}_c^\star$, $s_2 = \mathtt{m}_c^\star$, $s_3 = 1$, $s_4 = 1 + \mathtt{m}_c$, $r_1 = \mathtt{n}_a \cdot \mathtt{n}_3^\star + \mathtt{n}_b \cdot \mathtt{n}_4^\star$, $r_2 = \mathtt{n}_3^\star$, $r_3 = \mathtt{n}_4^\star$, $r_4 = 1 + \mathtt{n}_3$, $r_5 = 1$.

First, we spell out the ternary relation $M \lhd S(s_i) \blacktriangleright N$ for this particular example. It is defined inductively for $M, N \subseteq \{G(r_j) \mid 1 \le j \le 5\}$ and $1 \le i \le 4$ by the inference rules described by the following schemata:

$$\frac{[i \in \{2,3,4\}]}{\{G(r_2), G(r_3), G(r_4), G(r_5)\} \lhd S(s_i) \blacktriangleright \{G(r_1)\}}$$

$$\frac{\{G(r_4)\} \lhd S(s_4) \blacktriangleright \emptyset \quad \{G(r_2)\} \lhd S(s_2) \blacktriangleright \emptyset \quad [N \subseteq \{G(r_i) \mid 2 \le i \le 5\}]}{\{G(r_1)\} \lhd S(s_1) \blacktriangleright N}$$

$$\frac{\emptyset \lhd S(s_4) \blacktriangleright \emptyset \quad \emptyset \lhd S(s_2) \blacktriangleright \emptyset \quad [N \subseteq \{G(r_i) \mid 2 \le i \le 5\}]}{\emptyset \lhd S(s_1) \blacktriangleright N}$$

$$\frac{\emptyset \lhd S(s_4) \blacktriangleright \{G(r_4)\} \quad \emptyset \lhd S(s_2) \blacktriangleright \emptyset \quad [G(r_1) \in N]}{\emptyset \lhd S(s_1) \blacktriangleright N}$$

$$\frac{\emptyset \lhd S(s_4) \blacktriangleright \emptyset \quad \emptyset \lhd S(s_2) \blacktriangleright \{G(r_2)\} \quad [G(r_1) \in N]}{\emptyset \lhd S(s_1) \blacktriangleright N}$$

$$\frac{\{G(r_3)\} \lhd S(s_4) \blacktriangleright \emptyset \quad \{G(r_3)\} \lhd S(s_2) \blacktriangleright \emptyset \quad [N \subseteq \{G(r_i) \mid 2 \le i \le 5\}]}{\{G(r_1)\} \lhd S(s_1) \blacktriangleright N}$$

$$\frac{\emptyset \lhd S(s_4) \blacktriangleright \emptyset \quad \emptyset \lhd S(s_2) \blacktriangleright \emptyset \quad [N \subseteq \{G(r_i) \mid 2 \le i \le 5\}]}{\emptyset \lhd S(s_1) \blacktriangleright N}$$

$$\frac{\emptyset \lhd S(s_4) \blacktriangleright \{G(r_3)\} \quad \emptyset \lhd S(s_2) \blacktriangleright \emptyset \quad [G(r_1) \in N]}{\emptyset \lhd S(s_1) \blacktriangleright N}$$

$$\frac{\emptyset \lhd S(s_4) \blacktriangleright \emptyset \quad \emptyset \lhd S(s_2) \blacktriangleright \{G(r_3)\} \quad [G(r_1) \in N]}{\emptyset \lhd S(s_1) \blacktriangleright N}$$

$$\frac{\{G(r_3), G(r_4)\} \lhd S(s_4) \blacktriangleright \emptyset \ \{G(r_2), G(r_3), G(r_5)\} \lhd S(s_{i+1}) \blacktriangleright \emptyset \ \begin{bmatrix} i \in \{1,2\} \wedge \\ N \subseteq \{G(r_1), G(r_5)\} \end{bmatrix}}{\{G(r_2), G(r_3), G(r_4)\} \lhd S(s_{2i}) \blacktriangleright N}$$

$$\frac{\{G(r_3)\} \lhd S(s_4) \blacktriangleright N_1 \quad \{G(r_2)\} \lhd S(s_{i+1}) \blacktriangleright N_2 \quad [i \in \{1,2\} \wedge G(r_2) \notin N]}{\{G(r_2)\} \lhd S(s_{2i}) \blacktriangleright N}$$

$$\frac{\{G(r_4)\} \lhd S(s_4) \blacktriangleright N_1 \quad \{G(r_3)\} \lhd S(s_{i+1}) \blacktriangleright N_2 \quad [i \in \{1,2\} \wedge G(r_3) \notin N]}{\{G(r_3)\} \lhd S(s_{2i}) \blacktriangleright N}$$

$$\frac{\{G(r_3)\} \lhd S(s_4) \blacktriangleright N_1 \quad \{G(r_5)\} \lhd S(s_{i+1}) \blacktriangleright N_2 \quad [i \in \{1,2\} \wedge G(r_4) \notin N]}{\{G(r_4)\} \lhd S(s_{2i}) \blacktriangleright N}$$

$$\frac{\{G(r_3), G(r_4)\} \lhd S(s_4) \blacktriangleright N_1 \quad \{G(r_2), G(r_3)\} \lhd S(s_{i+1}) \blacktriangleright N_2 \quad \begin{bmatrix} i \in \{1,2\} \wedge \\ G(r_2), G(r_3) \notin N \end{bmatrix}}{\{G(r_2), G(r_3)\} \lhd S(s_{2i}) \blacktriangleright N}$$

$$\frac{\{G(r_3)\} \lhd S(s_4) \blacktriangleright N_1 \quad \{G(r_2), G(r_5)\} \lhd S(s_{i+1}) \blacktriangleright N_2 \quad \begin{bmatrix} i \in \{1,2\} \wedge \\ G(r_2), G(r_4) \notin N \end{bmatrix}}{\{G(r_2), G(r_4)\} \lhd S(s_{2i}) \blacktriangleright N}$$

$$\frac{\{G(r_3), G(r_4)\} \lhd S(s_4) \blacktriangleright N_1 \quad \{G(r_3), G(r_5)\} \lhd S(s_{i+1}) \blacktriangleright N_2 \quad \begin{bmatrix} i \in \{1,2\} \wedge \\ G(r_3), G(r_4) \notin N \end{bmatrix}}{\{G(r_3), G(r_4)\} \lhd S(s_{2i}) \blacktriangleright N}$$

where (N_1, N_2) in the last 6 schemata is a pair of sets of RHGs from:

- $\{(\emptyset, \emptyset)\}$ if $N \subseteq \{G(r_1), G(r_5)\}$,
- $\{(\emptyset, \{G(r_2)\}), (\{G(r_3)\}, \emptyset)\}$ if $G(r_2) \in N$ and $G(r_3), G(r_4) \notin N$,
- $\{(\emptyset, \{G(r_3)\}), (\{G(r_4)\}, \emptyset)\}$ if $G(r_3) \in N$ and $G(r_2), G(r_4) \notin N$,
- $\{(\emptyset, \{G(r_5)\}), (\{G(r_3)\}, \emptyset)\}$ if $G(r_4) \in N$ and $G(r_2), G(r_3) \notin N$,
- $\{(\emptyset, \{G(r_3), G(r_5)\}), (\{G(r_4)\}, \{G(r_5)\}), (\{G(r_3)\}, \{G(r_3)\}), (\{G(r_3), G(r_4)\}, \emptyset)\}$ if $G(r_2) \notin N$ and $G(r_3), G(r_4) \in N$,
- $\{(\emptyset, \{G(r_2), G(r_5)\}), (\{G(r_3)\}, \{G(r_5)\}), (\{G(r_3)\}, \{G(r_2)\}), (\{G(r_3)\}, \emptyset)\}$ if $G(r_3) \notin N$ and $G(r_2), G(r_4) \in N$,
- $\{(\emptyset, \{G(r_2), G(r_3)\}), (\{G(r_3)\}, \{G(r_3)\}), (\{G(r_4)\}, \{G(r_2)\}), (\{G(r_3), G(r_4)\}, \emptyset)\}$ if $G(r_4) \notin N$ and $G(r_2), G(r_3) \in N$,
- $\{(\emptyset, \{G(r_2), G(r_3), G(r_5)\}), (\{G(r_3)\}, \{G(r_3), G(r_5)\}), (\{G(r_4)\}, \{G(r_2), G(r_5)\}), (\{G(r_3)\}, \{G(r_2), G(r_3)\}), (\{G(r_3), G(r_4)\}, \{G(r_2)\}), (\{G(r_3)\}, \{G(r_3)\}), (\{G(r_3), G(r_4)\}, \{G(r_5)\}), (\{G(r_3), G(r_4)\}, \emptyset)\}$ if $G(r_2), G(r_3), G(r_4) \in N$.

It follows that

- $\mathtt{lf}_a(\{G\}) = \{\langle a\langle G(r_4)\rangle, G(r_2)\rangle\}$ and $\mathtt{lq}_{a\langle S(s_4)\rangle}(\{G\}) = \{\{G(r_2)\}\}$ because we can infer $\{G(r_4)\} \lhd S(s_4) \blacktriangleright \emptyset$ but not $\emptyset \lhd S(s_4) \blacktriangleright \{G(r_4)\}$.
- $\mathtt{lf}_b(\{G\}) = \{\langle b\langle G(r_3)\rangle, G(r_3)\rangle\}$ and $\mathtt{lq}_{b\langle S(s_4)\rangle}(\{G\}) = \{\{G(r_3)\}\}$ because we can infer $\{G(r_3)\} \lhd S(s_4) \blacktriangleright \emptyset$ but not $\emptyset \lhd S(s_4) \blacktriangleright \{G(r_3)\}$.
- $\mathtt{lf}_c(\{G(r_2)\}) = \{\langle c\langle G(r_3)\rangle, G(r_2)\rangle\}$ and $\mathtt{lq}_{c\langle S(s_4)\rangle}(\{G(r_2)\}) = \{\{G(r_2)\}\}$ because we can infer $\{G(r_3)\} \lhd S(s_4) \blacktriangleright \emptyset$ but not $\emptyset \lhd S(s_4) \blacktriangleright \{G(r_3)\}$.
- $\mathtt{lf}_c(\{G(r_3)\}) = \{\langle c\langle G(r_4)\rangle, G(r_3)\rangle\}$ and $\mathtt{lq}_{c\langle S(s_4)\rangle}(\{G(r_3)\}) = \{\{G(r_3)\}\}$ because we can infer $\{G(r_4)\} \lhd S(s_4) \blacktriangleright \emptyset$ but not $\emptyset \lhd S(s_4) \blacktriangleright \{G(r_4)\}$.

Next, we spell out the inference steps of $S \rhd M \rightsquigarrow N$ that can lead to a derivation of $S \rhd \{G\} \rightsquigarrow N$:

$$\dfrac{}{S(s_2) \rhd \{G(r_2)\} \rightsquigarrow \{G(r_2)\}} \qquad \dfrac{}{S(s_2) \rhd \{G(r_3)\} \rightsquigarrow \{G(r_3)\}}$$

$$\dfrac{S(s_2) \rhd \{G(r_2)\} \rightsquigarrow N}{S \rhd \{G\} \rightsquigarrow N} \qquad \dfrac{S(s_2) \rhd \{G(r_3)\} \rightsquigarrow N}{S \rhd \{G\} \rightsquigarrow N}$$

$$\dfrac{S(s_2) \rhd \{G(r_2)\} \rightsquigarrow N}{S(s_2) \rhd \{G(r_2)\} \rightsquigarrow N} \qquad \dfrac{S(s_2) \rhd \{G(r_3)\} \rightsquigarrow N}{S(s_2) \rhd \{G(r_3)\} \rightsquigarrow N}$$

It follows that $\mathfrak{N} := \{N \mid S \rhd \{G\} \rightsquigarrow N\} = \{\{G(r_2)\}, \{G(r_3)\}\}$, therefore

$$(\llbracket S \rrbracket \rhd \llbracket G \rrbracket) = \llbracket G(r_2) \rrbracket \cap \llbracket G(r_3) \rrbracket.$$

We have

$$\begin{cases} G(r_1) \doteq a\langle G(r_4)\rangle \cdot G(r_2) + b\langle G(r_3)\rangle \cdot G(r_3) \\ G(r_2) \doteq 1 + c\langle G(r_3)\rangle \cdot G(r_2) \\ G(r_3) \doteq 1 + c\langle G(r_4)\rangle \cdot G(r_3) \\ G(r_4) \doteq 1 + c\langle G(r_3)\rangle \cdot G(r_5) \\ G(r_5) \doteq 1 \end{cases} \tag{13}$$

from which we infer that the following LS

$$\begin{cases} z_{2,3} \doteq_\sigma 1 + c\langle z_{3,4}\rangle \cdot z_{2,3} \\ z_{3,4} \doteq_\sigma 1 + c\langle z_{3,4}\rangle \cdot z_{3,5} \\ z_{3,5} \doteq_\sigma 1 \end{cases}$$

is valid under the assignment $\sigma : \{z_{2,3}, z_{3,4}, z_{3,5}\} \to \mathcal{P}(\mathcal{H}(\Sigma))$ defined by

$$\begin{aligned} \sigma(z_{2,3}) &:= \llbracket G(r_2) \rrbracket \cap \llbracket G(r_3) \rrbracket, \\ \sigma(z_{3,4}) &:= \llbracket G(r_3) \rrbracket \cap \llbracket G(r_4) \rrbracket, \\ \sigma(z_{3,5}) &:= \llbracket G(r_3) \rrbracket \cap \llbracket G(r_5) \rrbracket. \end{aligned}$$

By introducing the nonterminal $\mathtt{q}$ as placeholder for $c\langle z_{3,4}\rangle$ and defining

$$\begin{pmatrix} r'_1 \\ r'_2 \\ r'_3 \end{pmatrix} := \begin{pmatrix} \mathtt{q} & 0 & 0 \\ 0 & 0 & \mathtt{q} \\ 0 & 0 & 0 \end{pmatrix}^\star \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \mathtt{q}^\star & 0 & 0 \\ 0 & 1 & \mathtt{q} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \mathtt{q}^\star \\ \mathtt{q} + 1 \\ 1 \end{pmatrix}$$

we learn that $(\llbracket S \rrbracket \rhd \llbracket G \rrbracket) = \llbracket G' \rrbracket$ where $G' := \langle \{\mathtt{q}\}, \Sigma, \{\mathtt{q} \to c\langle \mathtt{q} + 1\rangle\}, \mathtt{q}^\star\rangle$.

Note that we can use (13) to produce 31 language equations for the nonempty intersections of RHLs from $\{\llbracket G(r_i) \rrbracket \mid 1 \leq i \leq 5\}$, but we need only 3 equations to produce an LS whose solution contains RHGs for the intersections that show up in the characterization of the product derivative $\llbracket S \rrbracket \rhd \llbracket G \rrbracket$. □

6 Factorizations of Regular Hedge Languages

The following notions are straightforward generalizations of the notions with the same name in the theory of regular word languages [7].

A product of hedge languages $F_1.\dots.F_n$ is a *subfactorization* of a hedge language E if and only if $F_1.\dots.F_n \subseteq E$. The languages $F_1,\dots,F_n$ are called the *terms* of the subfactorization. A term F_i is *maximal* if it can not be increased without violating the hedge language inclusion. A *factorization* of E is a subfactorization in which every term is maximal. A subfactorization $F'_1.\dots.F'_n$ of E *dominates* another factorization $F_1.\dots.F_n$ of E if $F_i \subseteq F'_i$ for all $1 \leq i \leq n$. A *factor* of E is any hedge language which is a term in some factorization of E. A *left* (resp. *right*) *factor* of E is one which can be the leftmost (resp. rightmost) term in some factorization of E.

The following are well known properties of regular languages, and their proofs carry over without change to regular hedge languages:

1. Any subfactorization of E is dominated by some factorization of E in which all terms initially $\subseteq$-maximal remain unchanged [7, Ch. 6, Thm. 1].
2. Any left factor is a left term in some 2-term factorization. Any factor is the central term in some 3-term factorization. Any right factor is the right term in some 2-term factorization [7, Ch. 6, Thm. 2].
3. The condition that $L.R$ be a factorization of E defines a 1-1 correspondence between the left and right factors [7, Ch. 6, Thm. 3].

It is well known that regular languages have finitely many factors, they are all regular, and can be computed effectively [7]. We show that this result holds for regular hedge languages too.

From now on we assume that E is a regular hedge language generated by the regular hedge grammar $G = \langle \mathcal{N}, \Sigma, \mathcal{P}, r_1 \rangle \in \mathcal{G}(\mathcal{N}, \Sigma)$. We also consider the RHG $\mathtt{S} = \langle \{\mathtt{n}\}, \Sigma, \{\mathtt{n} \to a\langle \mathtt{n}^\star \rangle \mid a \in \Sigma\}, \mathtt{n}^\star \rangle$ that generates the RHL $\mathcal{H}(\Sigma)$.

6.1 Right Factors

In this section we show that E has finitely many right factors, that all right factors are RHLs, and give a method to compute RHGs for them.

We define the sets

$$\begin{aligned}
\mathfrak{N} &:= \{N \mid \mathtt{S} \rhd \{G\} \rightsquigarrow N\}, \\
\mathfrak{G} &:= \left\{ G\left(\sum_{G' \in N} \mathtt{ini}(G') \right) \middle| N \in \mathfrak{N} \right\}, \\
\mathfrak{L} &:= \{ [\![G']\!] \mid G' \in \mathfrak{G} \},
\end{aligned}$$

and note that that $\mathfrak{N}$ is finite and decidable, therefore $\mathfrak{G}$ is a finite and computable set too. Suppose $\mathfrak{G} = \{G_1, \dots, G_p\}$. We will show that the set of right factors of $[\![G]\!]$ is $\left\{ \bigcap_{i \in \mathcal{I}} [\![G_i]\!] \middle| \mathcal{I} \subseteq \{1,\dots,p\} \right\}$. To prove this fact, we recall from the proof of Lemma 8 that the following equality holds for all $S \in \partial_{\mathcal{H}(\Sigma)}(\mathtt{S})$, $M \subseteq \partial_{\mathcal{H}(\Sigma)}(G)$, and $n \in \mathbb{N}$:

$$\left\{h^{-1}\left(\bigcup_{G' \in M} [\![G']\!]\right) \,\middle|\, h \in [\![S]\!] \wedge |h| = n\right\} = \left\{\bigcup_{G' \in N} [\![G']\!] \,\middle|\, S \rhd M \rightsquigarrow_n N\right\}. \quad (14)$$

Since $[\![\mathtt{S}]\!] = \mathcal{H}(\Sigma)$, a first consequence of (14) is that

$$h^{-1}[\![G]\!] \in \left\{\bigcup_{G' \in N} [\![G']\!] \,\middle|\, \mathtt{S} \rhd \{G\} \rightsquigarrow_{|h|} N\right\} \subseteq \mathfrak{L}$$

for all $h \in \mathcal{H}(\Sigma)$. It follows that, if R is a right factor of E then there exists $L \subseteq \mathcal{H}(\Sigma)$ such that

$$R = (L \rhd E) = \bigcap_{h \in L} h^{-1}[\![G]\!] = \bigcap_{i \in \mathcal{I}} [\![G_i]\!] \quad \text{for some } \mathcal{I} \subseteq \{1, \ldots, p\}.$$

This shows that every right factor is a finite intersection of RHLs, thus it is an RHL too. Moreover, if we know $\mathcal{I}$ then we can compute an RHG for R with the algorithm for intersection presented in Sect. 5.2. Since $\mathcal{I}$ ranges over the subsets of $\{1, \ldots, p\}$, we learn that the set of right factors of E has at most 2^p elements, thus it is a finite set.

Another consequence of (14) is that for every $\mathcal{I} \subseteq \{1, \ldots, p\}$ there exist $\{n_i \mid i \in \mathcal{I}\} \subseteq \mathbb{N}$ and $\{h_i \mid i \in \mathcal{I} \wedge |h_i| = n_i\} \subseteq \mathcal{H}(\Sigma)$ such that

$$\mathtt{S} \rhd \{G\} \rightsquigarrow_{n_i} N_i \quad \text{and} \quad [\![G_i]\!] = \bigcap_{G' \in N_i} [\![G']\!] = h_i^{-1}[\![G]\!] \quad \text{for all } i \in \mathcal{I}.$$

Therefore $(\{h_i \mid i \in \mathcal{I}\} \rhd E) = \bigcap_{i \in \mathcal{I}} h_i^{-1}[\![G]\!] = \bigcap_{i \in \mathcal{I}} [\![G_i]\!]$ is a right factor of E.

Thus, the set of right factors of E is indeed $\left\{\bigcap_{i \in \mathcal{I}} [\![G_i]\!] \,\middle|\, \mathcal{I} \subseteq \{1, \ldots, p\}\right\}$, and we can use the algorithm for intersection presented in Sect. 5.2 to compute an RHG for each of them.

We conclude with the following algorithm for the computation of a set of regular hedge grammars $\mathtt{RFG}(G)$ for the right factors of $[\![G]\!]$:

1. Compute the finite set $\mathfrak{N} = \{N \mid \mathtt{S} \rhd \{G\} \rightsquigarrow N\}$ where the ternary relation $S \rhd M \rightsquigarrow N$ is defined inductively as shown in Sect. 5.3 on the finite set $\partial_{\mathcal{H}(\Sigma)}(\mathtt{S}) \times \mathcal{P}(\partial_{\mathcal{H}(\Sigma)}(G)) \times \mathcal{P}(\partial_{\mathcal{H}(\Sigma)}(G))$. Suppose $n = \|\mathtt{ini}(G)\| + 1$. Note that $\partial_{\mathcal{H}(\Sigma)}(\mathtt{S}) = \{\mathtt{S}\}$ and $\partial_h(G) \subseteq \{G(r) \mid r \in \partial_{\mathcal{N}^*}(\mathtt{ini}(G))\}$ for all $h \in \mathcal{H}(\Sigma)$. Thus $\partial_{\mathcal{H}(\Sigma)}(\mathtt{S})$ has one element and $\partial_{\mathcal{H}(\Sigma)}(G)$ has at most 2^n elements, and we learn that:
   - the ternary relation $S \rhd M \rightsquigarrow N$ is defined on a set with at most 2^{2n} elements,
   - $\mathfrak{N} \subseteq \mathcal{P}(\{G(r) \mid r \in \partial_{\mathcal{N}^*}(\mathtt{ini}(G))\})$, which has at most 2^n elements.
2. The set of right factors of $[\![G]\!]$ is $\left\{\bigcap_{N \in \mathfrak{M}} \bigcup_{G' \in N} [\![G']\!] \,\middle|\, \mathfrak{M} \subseteq \mathfrak{N}\right\}$. Because of distributivity of intersection over union, we can express every right factor of E as a finite union of intersections of RHLs from the set $\{[\![G(r)]\!] \mid r \in \partial_{\mathcal{N}^*}(\mathtt{ini}(G))\}$. The computation of regular expressions for this union of intersections of partial hedge derivatives can be done as indicated in Sect. 5.3.

6.2 Left Factors

Note that L is a left factor of E iff L^{s} is a right factor of E^{s}. Therefore, we can compute a finite set $\mathtt{LFG}(G)$ of RHGs for all left factors of the language generated by G as follows:

1. We compute a set $\mathtt{RFG}(G)$ of RHGs for all right factors of $[\![G]\!]$, as indicated in the previous section.
2. We define $\mathtt{LFG}(G) := \{S^{\mathsf{s}} \mid S \in \mathtt{RFG}(G^{\mathsf{s}})\}$.

6.3 Factor Matrix

We have already seen that $\mathtt{LFG}(G)$ and $\mathtt{RFG}(G)$ are finite sets and that there is a 1-1 correspondence between them. First, we compute $\{S_1, \ldots, S_p\} := \mathtt{LFG}(G)$, and then we compute with the algorithm presented in Sect. 5.3 the set

$$\mathtt{RFG}(G) := \{G_i \mid i \in \{1, \ldots, p\} \wedge (G_i \text{ is RHG for the RHL } [\![S_i]\!] \rhd [\![G]\!])\} .$$

Note that $[\![S_1]\!].[\![G_1]\!]$, …, $[\![S_p]\!].[\![G_p]\!]$ are all 2-term factorizations of $[\![G]\!]$. Like in the case of regular word languages, we define the *factor matrix* of the RHL E as $\boldsymbol{E} \in \mathcal{M}_{p,p}(\mathcal{P}(\mathcal{H}(\Sigma)))$ with

$$\boldsymbol{E}_{i,j} := \text{the } \subseteq\text{-maximal solution of the subfactorization } [\![S_i]\!].X.[\![G_j]\!] \text{ of } [\![G]\!]$$

for every $i, j \in \{1, \ldots, p\}$.

Theorem 1. *Each $\boldsymbol{E}_{i,j}$ is a factor of E, and each factor of E is one of $\boldsymbol{E}_{i,j}$. There exist unique indices $l, r \in \{1, \ldots, p\}$ such that $E = [\![S_r]\!] = [\![G_l]\!] = \boldsymbol{E}_{l,r}$ and $[\![S_i]\!] = \boldsymbol{E}_{l,i}$, $[\![G_i]\!] = \boldsymbol{E}_{i,r}$ for each $i \in \{1, \ldots, p\}$. Hence the factors of the RHL E naturally form a square matrix among the entries of which is E.*

Proof. Identical to the proof of [7, Ch. 6, Thm. 4] for regular word languages. □

Note that $\boldsymbol{E}_{i,j} = ([\![S_i]\!] \rhd [\![S_j]\!])$ for all $i, j \in \{1, \ldots, p\}$, and we can use the product derivative algorithm from Sect. 5.3 to compute a $p \times p$ matrix $\boldsymbol{G}$ of RHGs such that $[\![\boldsymbol{G}]\!]_{i,j} = ([\![S_i]\!] \rhd [\![S_j]\!])$ for every $i, j \in \{1, \ldots, p\}$.

The factors of an RHL have the following remarkable properties.

Theorem 2. *Let E be an RHL and $\boldsymbol{E} \in \mathcal{M}_{p,p}(\mathcal{P}(\mathcal{H}(\Sigma)))$ be its factor matrix. Then we have:*

1. *$\epsilon \in \boldsymbol{E}_{i,i}$ for all $i \in \{1, \ldots, p\}$.*
2. *$\boldsymbol{E}_{i,j}.\boldsymbol{E}_{j,k} \subseteq \boldsymbol{E}_{i,k}$ for all $i, j, k \in \{1, \ldots, p\}$.*
3. *If $F_1, F_2 \in \mathcal{P}(\mathcal{H}(\Sigma))$ then $F_1.F_2 \subseteq \boldsymbol{E}_{i,j}$ if and only if there exists $k \in \{1, \ldots, p\}$ such that $F_1 \subseteq \boldsymbol{E}_{i,k}$ and $F_2 \subseteq \boldsymbol{E}_{k,j}$.*
4. *If $F_1, \ldots, F_j \in \mathcal{P}(\mathcal{H}(\Sigma))$ then $F_1. \ldots .F_j \subseteq E$ if and only if there exist indices $k_1, \ldots, k_{j-1} \in \{1, \ldots, p\}$ for which $F_1 \subseteq \boldsymbol{E}_{l,k_1}$, $F_j \subseteq \boldsymbol{E}_{k_{j-1},r}$ and $F_i \subseteq \boldsymbol{E}_{k_{i-1},k_i}$ for all $1 < i < j$.*

Proof. Identical to the proof of [7, Ch. 6, Thm. 6] for regular word languages. □

7 Conclusion

Linear systems of hedge language equations are an alternative representation of regular hedge languages that is suitable for the computation of intersection, quotient, product derivative, and factor matrix of RHLs. We have identified algorithms for the translation of RHGs into LSs (Lemma 3 and Corollary 1), for the translation of LSs into RHGs (Sect. 3), and for the computation of RHGs for the quotient (Sect. 5.1), intersection (Sect. 5.2), product derivative (Sect. 5.3) and elements of the factor matrix (Sect. 6) of RHLs represented by RHGs.

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