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Dynamic balancing of linkages by algebraic methods

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Doctoral Thesis

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Eidesstattliche Erklarung

Ich erkläre an Eides statt, dass ich die vorliegende Dissertation selbständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäss entnommenen Stellen als solche kenntlich gemacht habe.

Linz, April 2009.

Brian Moore

Abstract

A mechanism is statically balanced if, for any motion, it does not exert forces on the base. Moreover, if it does not exert torques on the base, the mechanism is said to be dynamically balanced. In 1969, Berkof and Lowen showed that in some cases, it is possible to balance mechanisms without adding additional components, simply by choosing the design parameters (i.e. length, mass, centre of mass, inertia) in an appropriate way. For the simplest linkages, some solutions were found but no complete characterization was given.

The aim of the thesis is to present a new systematic approach to obtain such complete classifications for 1 degree of freedom linkages. The method is based on the use of complex variables to model the kinematics of the mechanism. The static and dynamic balancing constraints are written as algebraic equations over complex variables and joint angular velocities. After elimination of the joint angular velocity variables, the problem is formulated as a problem of factorisation of Laurent polynomials. Using computer algebra, necessary and sufficient conditions can be derived.

Using this approach, a classification of all possible statically and dynamically balanced planar four-bar mechanisms is given. Sufficient and necessary conditions for static balancing of spherical linkages is also described and a formal proof of the non-existence of dynamically balanced spherical linkage is given. Finally, conditions for the static balancing of Bennett linkages are described.

Zusammenfassung

Ein Gelenkmechanismus heißt statisch ausbalanciert, wenn er durch seine Bewegungen keine Kräfte auf die Plattform ausübt. Wenn er darüber hinaus keine Drehmomente ausübt, nennt man ihn dynamisch ausbalanciert. Berkof und Lowen zeigten 1969, dass es in einigen Fällen möglich ist, Gelenkmechanismen ohne Zusatzelemente auszubalancieren, indem man die Design-Parameter (d.h. Länge, Masse, Schwerpunkt, Trägheitsmoment) geeignet wählt. Für den einfachsten Typ von Gelenkmechanismen fanden sie mehrere Lösungen, allerdings keine vollständige Charakterisierung aller Lösungen.

Ziel dieser Dissertation ist die Darstellung eines neuen systematischen Zugangs, der es erlaubt, eine solche vollständige Klassifizierung für Gelenkmechanismen mit einem Freiheitsgrad zu bestimmen. Die Methode basiert auf Modellierung der Kinematik durch komplexe Variablen. Die statischen und dynamischen Gleichgewichtsbedingungen werden als algebraische Gleichungen über komplexen und reellen Variablen geschrieben. Nach der Elimination der Variablen, die die Winkelgeschwindigkeiten represäntieren kann das Problem formuliert werden als Faktorisierungsproblem von Laurent Polynomen. Mit Hilfe der Computer-Algebra können notwendige und hinreichende Bedingungen hergeleitet werden.

Mit diesem Zugang wird die Klassifizierung aller statisch und dynamisch ausbalancierten planaren Viergelenksmechanismen bestimmt. Außerdem werden notwendige und hinreichende Bedingungen für statische ausbalancierte sphärische Viergelenksmechanismen hergeleitet, und es wird ein formaler Beweis für die Unmöglichkeit von dynamisch ausbalancierten sphärischen Viergelenksmechanismen gegeben. Zum Schluss werden die Bedingungen für statisch ausbalancierte Bennett-Mechanismen hergeleitet.

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Contents

Chapter 1

Introduction

1.1 Motivation and applications

When walking or running, we exert force and torque on the ground in order to move forward. This affects the environment in different ways. For example, a crowd walking on a bridge can create vibration leading to the so called *synchronous lateral excitation*, where the bridge starts moving laterally. This was the case of the Millenium bridge (Fig. 1.1) in London which had to be closed right after its opening in 2000 due to such effect.

Figure 1.1: Millenium bridge in London (Wikipedia)

The same principles apply to mechanisms. A mechanism consists of a set of bodies (also called links or bars) connected by joints. The motion is determined according to the force applied at the joints by actuators (i.e. motors) and the external forces applied to the mechanism (for example gravity). Since the mechanism is connected to a base, its motion exerts shaking force and shaking moment (i.e. force in rotation) at the connecting points between the mechanism and the base. These forces introduce vibrations and cause noise, wear and fatigue to the system which are most of the time not desired.

A mechanism is said to be *statically balanced* if the weight of the links does not produce any torque (or force) at the actuators under static conditions, for any configuration of the mechanism [36]. A mechanism is said to be dynamically balanced (or reactionless) if no reaction force (excluding gravity) or moment is transmitted to the base, at any time, for any arbitrary motion of the mechanism [82]. In this case, the mechanism will not transmit any vibration to its environment when it is operated.

Statically and dynamically balanced mechanisms are highly desirable for many engineering applications:

- Static balancing of robotic manipulators with large payloads is very important since zero actuator torques are required whenever the manipulator is at rest. Furthermore, only inertial forces and moments have to be sustained while the manipulator is moving [26]. A typical application is commercial flight simulators.

- Statically balanced mechanisms can be used to help patients in rehabilitation. For example, subjects that are unable to generate enough force to lift their leg will wear a special device as an exoskeleton on their leg to statically balance the leg. That is, this device will take away the weight of the leg in a passive way [1].

- The use of space manipulators to perform tasks induces force and moment at the base of the manipulator, (i.e. space shuttle, space station) therefore causing translation and rotation of the free-floating base (satellite, space station) [2, 29]. The same effect can be observed when an artificial satellite is moving its antenna and solar panel to achieve maximum performance. These effects must be compensated by using thrusters or flywheels. This is one of the reasons why such systems are constrained to move very slowly [85].

- Large telescopes such as the Large Binocular Telescope uses heavy mirrors which are being moved at high frequencies to compensate for atmospheric disturbances. To avoid exciting the structure of the telescope during the motion, it must be dynamically balanced [41, 36].

- For industrial high-speed applications, dynamic balancing improves the dynamic performance by reducing the disturbance due to dynamic force variation and enhances the control system [83]. In larger-scale precision devices and in systems with delicate equilibrium, eliminating or reducing the reactions on the base also significantly improves the

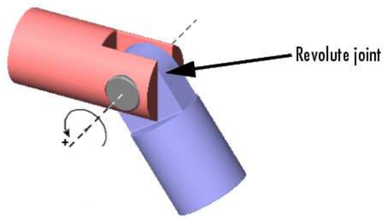


Figure 1.2: Revolute joint

general performance by reducing vibrations and thereby improving the accuracy [82].

1.2 4R Linkages

We now introduce the class of mechanisms which will be studied in this thesis. Mechanisms are classified (mainly) into two categories: serial and parallel mechanisms. A serial mechanism is constituted of a succession of rigid bodies, each of them being linked to its predecessor and its successor by rotational or translational joints [56]. A parallel mechanism or manipulator is a closed-loop kinematic chain whose end-effector is linked to the base by several independent kinematic chains [56]. Parallel mechanisms offer greater rigidity, higher speed and higher accuracy than serial mechanisms.

The simplest parallel mechanisms are so-called 4R mechanisms or 4R linkages. We will often refer to them simply as four-bar linkages. They are composed of four rigid bars and four revolute joints (see Fig. 1.2) forming a loop. In general, such linkages are not moveable, they are stiffed structures. Using Gruebler's criterion, they are actually triply stiff [39, 66]. However, for some special choice of the geometric design parameters, this linkage can move with one degree of freedom. Movable 4R linkages can be classified into the following three categories:

- Planar linkage
- Spherical linkage
- Bennett linkage

Different proofs of the existence and classification of 4R linkages are given in [4, 40, 19].

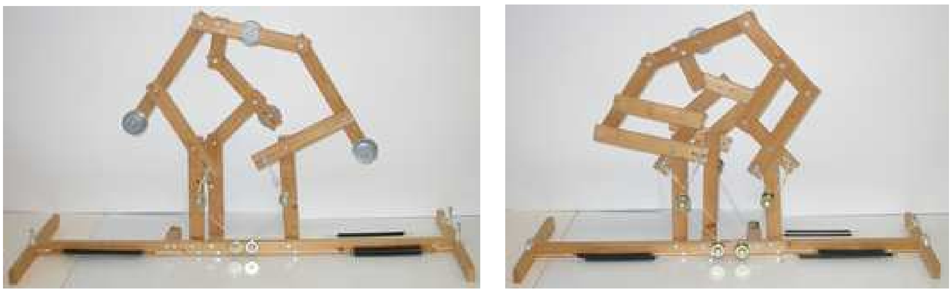


Figure 1.3: Planar 3-DOF mechanism balanced using counterweights and springs (left) and only springs (right) (Courtesy of Laval University)

1.3 Static balancing

Static balancing is usually achieved by using additional mechanical elements like elastic components [71, 26, 77, 42, 68] or counterweights, either directly mounted on the links of the mechanism [76, 48] or by using auxiliary system mounted on the system [77, 68]. Examples of statically balanced mechanisms built at Laval University are shown in Fig. 1.3 and 1.4. An excellent overview of the different approaches combined with drawings of different type of statically balanced mechanical systems is given in [11].

In some cases, it is possible to statically balanced linkages without using additional components. Studying four-bar linkages, Berkof and Lowen [14] developed in 1969 the *method of linearly independent vectors* which is based on an appropriate choice of the linkage lengths (i.e. the geometric design parameters) and mass distribution (i.e. the static design parameter) such that the centre of mass of the linkage remains fixed for any motion of the mechanism. They provided conditions for static balancing in terms of these design parameters. Based on this concept, several balancing techniques were developed in the literature (see for instance [8, 6]). In 1997, Gosselin [35] showed that the Berkof-Lowen method leads to sufficient conditions which are however not necessary. This was achieved by devising a statically balanced planar four-bar mechanism not fulfilling Berkof and Lowen balancing conditions. Although all possible cases were derived, no formal proof was given stating the completeness of this classification. In this work, we propose a method to prove that this classification is complete. The method is also extended to spatial linkages and a complete classification of statically balanced spherical and Bennett linkages is given.

1.4 Dynamic balancing

Several approaches can be used for dynamic balancing of mechanisms. A possible approach is to generate trajectories which minimize or eliminate

Figure 1.4: Spatial 6-DOF mechanism balanced using springs (Courtesy of Laval University)

reaction forces and torques [45, 25, 65, 3, 64, 52]. This approach is however quite restrictive and applicable only for special cases.

The classical method to obtain statically and dynamically balanced mechanisms consists of adding mass and inertia elements to the system such that the centre of mass remains fixed (i.e. it is statically balanced) and the angular momentum is zero for any motion. This can be achieved using several methods or "principles" which can be categorized as follows [73]:

- using counter-rotary counter-masses
- using separate counter-rotations
- using idler loops
- using a duplicated mechanism

The major drawback of these approaches is that a considerable amount of mass and inertia is added to the system, and as Kochev [46] stated: *"The price paid for shaking force and shaking moment balancing is discouraging"*. A comparison of these different methods is described in [73] to determine which method minimizes the additional mass and inertia requirement to balanced double-pendulum.

For the planar four-bar linkages, Berkof and Lowen [15] stated in 1971 that *"the shaking moment can generally not be totally eliminated without the addition of auxiliary linkages"*. That is, it is generally not possible to dynamically balance such mechanism without additional mechanical components. This study was based on a "generic" choice of the design parameters. However, in [67], special dynamically balanced linkages that do not require external counter-rotations were first revealed. An example of such mechanism is shown in Fig. 1.5. By combining dynamically balanced four-bar

Figure 1.5: Dynamically balanced planar 4R, Gabriel's mechanism (courtesy of Laval University)

linkages, mechanisms with more degrees of freedom have been synthesized, both planar [67] and spatial [38, 82]. However, no complete classification of dynamically balanced planar 4R linkages has been given. Several publications [45, 28] mentioned that it is very difficult to find or probably impossible to obtain a dynamically balanced 4R linkage without adding external components, i.e. by choosing the design parameters in an intelligent way, but no proof was given. In this thesis, we present a complete classification of dynamically balanced planar four-bar linkages and formally prove that spherical 4R linkages cannot be dynamically balanced.

1.5 Additional remarks

Parts of this thesis have been submitted or published in international journals [59, 37, 60]. The source code for the computations and examples described in this thesis can be found online [58].

Chapter 2

Preliminaries

This chapter contains the basic concepts, definitions and theorems that will be used thoughout this thesis.

2.1 Representation of trigonometric functions

This section describes several well-known and basic identities concerning trigonometric functions which will be heavily used in this document.

2.1.1 Complex numbers

Euler's formula states that, for any angle θ,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{2.1}$$

Clearly, we also have the following identities

$$\begin{aligned} \frac{e^{i\theta} + e^{-i\theta}}{2} &= \mathrm{Re}(e^{i\theta}) = \cos(\theta) \\ \frac{e^{i\theta} - e^{-i\theta}}{2i} &= \mathrm{Im}(e^{i\theta}) = \sin(\theta) \end{aligned} \tag{2.2}$$

where $\mathrm{Re}(e^{i\theta})$ represents the real part of $e^{i\theta}$ and $\mathrm{Im}(e^{i\theta})$ its imaginary part.

Considering this formula in the complex plane, we have

$$\mathbf{z} = \cos\theta + i\sin\theta = e^{i\theta} \tag{2.3}$$

where $\mathbf{z}$ is a unit complex number. Equations (2.2) can also be written in terms of the unit complex number $\mathbf{z}$ using (2.3):

$$\begin{aligned} \frac{\mathbf{z} + \mathbf{z}^{-1}}{2} &= \cos(\theta) \\ \frac{\mathbf{z} - \mathbf{z}^{-1}}{2i} &= \sin(\theta) \end{aligned} \tag{2.4}$$

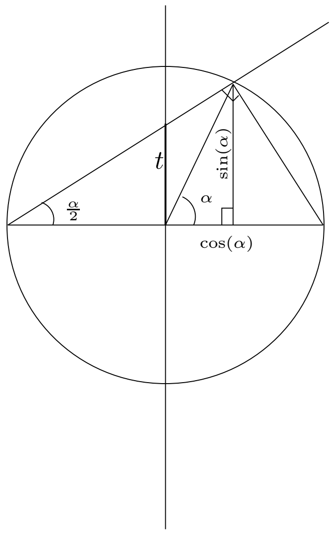


Figure 2.1: Parametrisation of the unit circle

Let $\mathbf{z}$ be a unit complex number as given in (2.3). The multiplication of an arbitrary complex number $\mathbf{x}$ by $\mathbf{z}$ corresponds to an anti-clockwise rotation of $\mathbf{x}$ by an angle θ in the complex plane. The multiplication of an arbitrary complex number $\mathbf{x}$ by $-\mathbf{z}$ corresponds to a clockwise rotation of $\mathbf{x}$ by an angle θ. Note also that $\mathbf{z}^{-1} = \overline{\mathbf{z}}$, since $\mathbf{z}$ is of unit length.

2.1.2 Tangent half-angle formulae

Tangent half-angle formulae are used to parametrize the unit circle. Given a point $p = (\cos(\alpha), \sin(\alpha))$ on the unit circle, draw a line passing through it and the point $(-1, 0)$. This line intersects the y axis at a point, with coordinate $y = t$ as shown in Fig. 2.1. The equation of this line is $y = t(1+x)$ and the intersection of the line with the unit circle can be expressed as

$$x^2 + y^2 - 1 = x^2 + (t(1+x))^2 - 1 = (t^2+1)\,x^2 + 2\,t^2\,x + (t^2-1) = 0 \quad (2.5)$$

Solving in terms of x we obtain $x = \frac{-t^2 \pm 1}{t^2+1}$ and therefore $x = -1$ or $x = \frac{1-t^2}{1+t^2}$. The x coordinate of the point p is $\cos(\alpha)$ and we obtain a parametrization for $\cos(\alpha)$. We can do the same for y and we obtain a parametrization of $\sin\alpha$ in terms of t. These parametrizations are given by

$$\cos(\alpha) = \frac{1-t^2}{1+t^2} \qquad \sin(\alpha) = \frac{2t}{1+t^2} \quad (2.6)$$

where t is a real parameter and corresponds to $\tan(\frac{\alpha}{2})$. Some values of t are shown in Fig. 2.2.

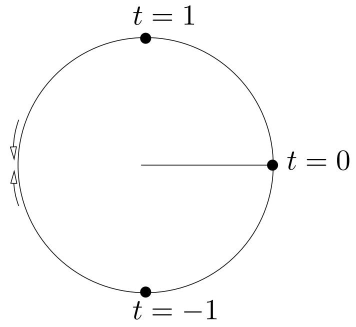


Figure 2.2: Some values of t in the complex plane

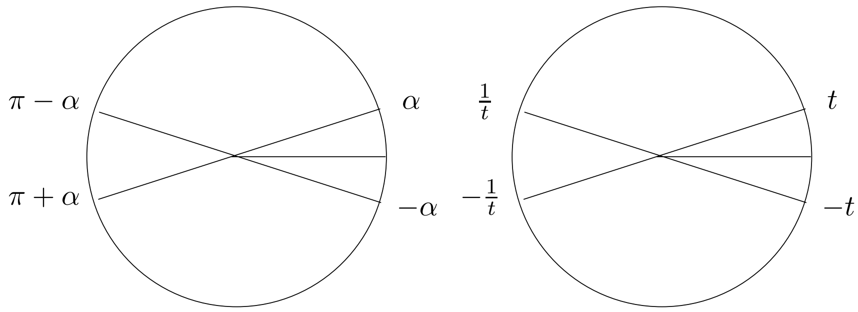


Figure 2.3: Tangent half-angle properties

Given t and α as in (2.6), we can look at different mappings (i.e. variable substitutions) of α in the interval $]0, \pi[$ and the induced mappings on the parameter t. These mapping are depicted in Fig. 2.3.

$$\begin{aligned} \alpha \to -\alpha \quad &\Longrightarrow \quad t \to -t \\ \alpha \to \pi - \alpha \quad &\Longrightarrow \quad t \to \tfrac{1}{t} \\ \alpha \to \pi + \alpha \quad &\Longrightarrow \quad t \to -\tfrac{1}{t} \end{aligned} \tag{2.7}$$

2.2 Basic algebra

In this section, we introduce basic algebra notions that are required in this thesis. For more details, we refer to classical algebra books [74, 44, 53, 49, 80, 23]. We start with some definitions and follow mainly the notation from [23].

Definition A **commutative ring** is a set R equipped with two binary operations $+ : R \times R \to R$ and $* : R \times R \to R$ (where $\times$ denotes the Cartesian product), called addition and multiplication. To qualify as a ring, the set and two operations, $(R, +, *)$, must satisfy the following requirements known as the ring axioms.

- $(R, +, *)$ is required to be an Abelian group under addition:
    1. Closure under addition. For all a, b in R, the result of the operation $a + b$ is also in R.
    2. Associativity of addition. For all a, b and c in R, the equation $(a + b) + c = a + (b + c)$ holds.
    3. Existence of additive identitiy. There exists an element 0 in R, such that for all elements a in R, the equation $0 + a = a + 0 = a$ holds.
    4. Existence of additive inverse. For each a in R, there exists an element b in R such that $a + b = b + a = 0$
    5. Commutativity of addition. For all a, b in R, the equation $a+b = b + a$ holds.
- $(R, +, *)$ is required to be a monoid under multiplication:
    1. Closure under multiplication. For all a, b in R, the result of the operation $a * b$ is also in R.
    2. Associativity of multiplication. For all a, b, and c in R, the equation $(a * b) * c = a * (b * c)$ holds.
    3. Existence of multiplicative identity. There exists an element 1 in R, such that for all elements a in R, the equation $1 * a = a$ holds.
    4. Commutativity of multiplication: For all a, b in R, the equation $a * b = b * a$ holds.
- The distributive laws:
    1. For all a, b and c in R, the equation $a * (b + c) = (a * b) + (a * c)$ holds.

Definition A nonzero element a in a commutative ring R is a **zero divisor** if there exists a nonzero element b in R such that $a * b = 0$.

Definition A **monomial** in the variables $z_1, z_2, ..., z_n$ is a product of the following form

$$z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \tag{2.8}$$

where the α_i are non negative integers, $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$.

Definition Let R be a commutative ring. A **polynomial** is a finite linear combinations of monomials with coefficients in R. The collection of all polynomials in the variables $z_1, ..., z_n$ with coefficients in R is denoted $R[z_1, ..., z_n]$. $R[z_1, ..., z_n]$ is a commutative ring.

Definition A **rational function** in the variables $z_1, ..., z_n$ with coefficients in the ring R is an expression of the form f/g, where f, g are polynomials and $g \neq 0$. The set of all rational functions is denoted by $R(z_1, ..., z_n)$.

Definition A **Laurent monomial** in the variables $z_1, ..., z_n$ is a product of the following form

$$z_1^{\beta_1} z_2^{\beta_2} \cdots z_n^{\beta_n} \tag{2.9}$$

where the $\beta_i \in \mathbb{Z}$, that is, they can also be negative integers.

Definition A **Laurent polynomial** over a ring R is a finite linear combination of Laurent monomials with coefficients in R. The collection of all Laurent polynomials in the variables $z_1, ..., z_n$ with coefficients in R is denoted by $R[z_1, z_1^{-1}, z_2, ..., z_n, z_n^{-1}]$ and is a commutative ring.

Definition The **support** of a Laurent polynomial g in n variables $z_1, ..., z_n$ is the set of all $\beta \in \mathbb{Z}^n$ such that the coefficient of the monomial z^β in g is different than 0.

Example Let g be the following Laurent polynomial in the variables z_1 and z_2 with nonzero coefficients a_i in a ring R

$$g = a_1 + a_2 z_1 + a_3 z_2 + a_4 z_1^{-1} z_2 + a_5 z_1^{-1} + a_6 z_2^{-1} + a_7 z_1 z_2^{-1} \tag{2.10}$$

The support of g is

$$\{(0,0), (1,0), (0,1), (-1,1), (-1,0), (0,-1), (1,-1)\} \tag{2.11}$$

2.3 Polytopes and polynomials

In this section, we describe how we can use the support of a polynomial (represented by a polytope) to study its reducibility/irreducibility.

A set $S \subset \mathbb{R}^n$ is a **convex set** if it contains the line segment connecting any two points in S. The **convex hull** of S is the smallest convex set containing it. If S contains finitely many elements, its convex hull is called **polytope**. In our case, we will consider the convex hull of a finite set of points with integer coordinates called lattice polytopes.

Definition Let g be a Laurent polynomial g in n variables $x_1, ..., x_n$. The **Newton polytope** of g is the convex hull of its support in $\mathbb{R}^n$. It is denoted by $\Pi(g)$ and is a convex set. If $n = 2$, it is called **Newton polygon**.

Definition The **corners** of a Newton polygon are the elements of the polygon that cannot be removed from the polygon without modifying its shape.

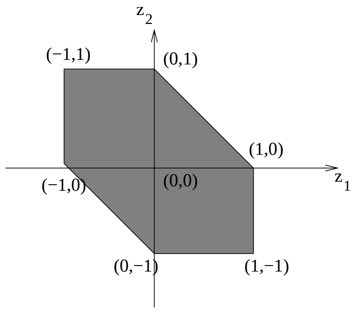


Figure 2.4: Newton polygon of g

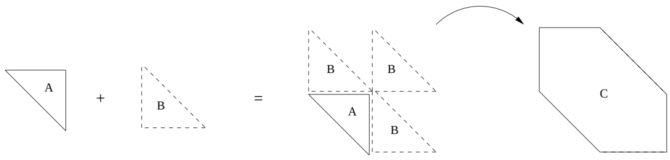


Figure 2.5: Minkowski sum.

Example Let g as defined in Example 2.2. The Newton polygon of g is given in Figure 2.4. The corners of g are all the points of its support described by (2.11) except $(0, 0)$.

Definition The **Minkowski sum** of two polytopes A and B in $\mathbb{R}^n$ is defined as:

$$A + B = \{a + b \mid a \in A \wedge b \in B\} \tag{2.12}$$

where $a + b$ corresponds to the normal vector sum in $\mathbb{R}^n$. Note that $A + B$ is also a convex polytope.

Example Let g_1 and g_2 be the following Laurent polynomials

$$\begin{aligned} g_1 &= a_1 z_1^{-1} + a_2 z_2^{-1} + a_3 \\ g_2 &= b_1 z_1 + b_2 z_2 + b_3 \end{aligned} \tag{2.13}$$

and $a_i, b_j \in \mathbb{R}^*$. Let $A = \Pi(g_1)$ and $B = \Pi(g_2)$ be the Newton polygons of g_1 and g_2. The Minkowski sum $A + B$ can be computed geometrically by moving B on the boundary of A and taking the convex hull as shown in Figure 2.5.

The following theorem states the relation between the product of Laurent polynomials and the Minkowsi sum of the corresponding Newton polytopes.

Theorem 1 *Let R a commutative ring without zero divisors and consider the ring of Laurent polynomials over R, denoted $R[z_1, z_1^{-1}, z_2, z_2^{-1}]$. Let g_1, g_2*

be two Laurent polynomials in $R[z_1, z_1^{-1}, z_2, z_2^{-1}]$*. Let* a_i*,* b_j *be nonzero element of* R*. Then*

$$\Pi(g_1 g_2) = \Pi(g_1) + \Pi(g_2) \tag{2.14}$$

We refer to Ostrowski [62, 63] for a proof.

Example Let g_1 and g_2 as defined in (2.13). Their product is given by

$$\begin{aligned} g_1 g_2 = & a_1 b_2 z_1^{-1} z_2 + a_1 b_3 z_1^{-1} + a_2 b_1 z_1 z_2^{-1} + a_2 b_3 z_2^{-1} \\ & + a_3 b_1 z_1 + a_3 b_2 z_2 + a_1 b_1 + a_2 b_2 + a_3 b_3 \end{aligned} \tag{2.15}$$

Comparing with Figure 2.5, it is clear that the Newton polygon of $g_1 g_2$ is equal to the Minkowski sum of their Newton polygons.

2.4 Factorisation of (parametric) polynomials

Consider a polynomial $g \in \mathbb{Z}[z]$, that is a polynomial in terms of the variable z with coefficients in $\mathbb{Z}$. Let $g = z^2 + 1$. g is an irreducible polynomial over $\mathbb{Z}$, meaning that there exists no $g_1, g_2 \in \mathbb{Z}[z]$ (not the identity) such that $g = g_1 g_2$. That is, factoring this polynomial requires the use of complex numbers which are not in the ground field (the field of the coefficients) $\mathbb{Z}$.

Consider the case when the coefficients of the polynomial g are unknown, that is, they are expressed in terms of some parameters. For example, consider $g \in \mathbb{Z}[x]$ with $g = z^2 + a$, where $a \in \mathbb{Z}$ but is unknown. When is this polynomial irreducible? When is it reducible? Can we find conditions on a such that it is irreducible/reducible? Clearly, if g is reducible over $\mathbb{Z}$, then there must be a factorisation of the form (up to scalar multiplication)

$$z^2 + a = (z + c_1)(z + c_2) = z^2 + (c_1 + c_2)z + c_1 c_2 \tag{2.16}$$

with $c_1, c_2 \in \mathbb{Z}$. Conditions can be easily found by comparing coefficients on both sides of (2.16) and we obtain

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 c_2 &= a \end{aligned} \tag{2.17}$$

Therefore, the polynomial is reducible if and only if

$$a = c_1 c_2 = c_1(-c_1) = -c_1^2 \tag{2.18}$$

That is, if and only if $a = -1, -4, -9, \ldots.$

In the previous example, we dealt with a univariate polynomial of degree 2 and we saw that the shape of the factorisation given by (2.16) is unique. But what if we consider multivariate polynomials? Can we also determine all possible factorisation shapes? We will now show that this can be achieved using the Newton polygons.

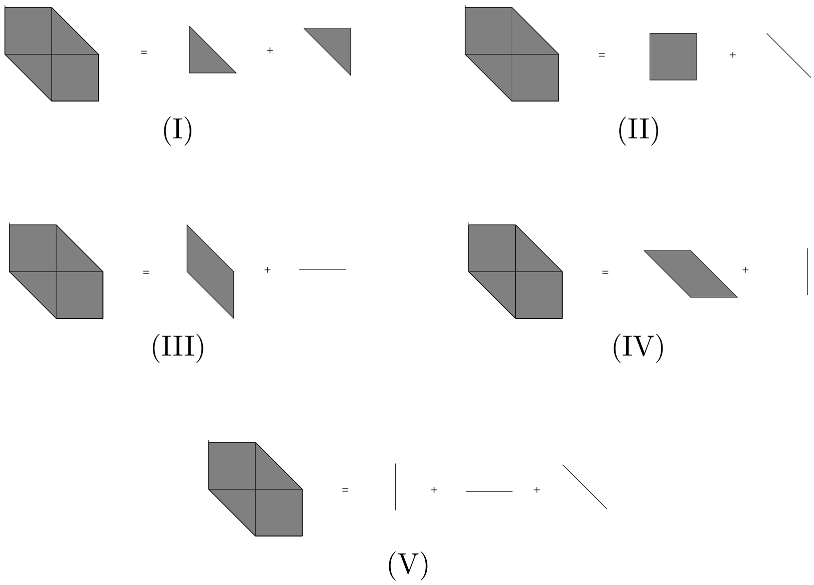


Table 2.1: Possible factorisations based on Newton polytopes and Minkowski sums (Example 2.4)

Let g be a polynomial with parametric coefficients. Using Theorem 1, one could look at all possible decomposition of $\Pi(g)$ into the Minkowski sum of Newton polygons and for each of these decomposition, find the values of the parameters such that this decomposition is valid. The complete decomposition of Newton polygons into the Minkowski sum of smaller problem is not an easy problem [27, 31, 30]. However, in our case, we will deal with polynomials with relatively simple Newton polygons, similar to the next example.

Example Let

$$g = a_1 + a_2 z_1 + a_3 z_2 + a_4 z_1^{-1} z_2 + a_5 z_1^{-1} + a_6 z_2^{-1} + a_7 z_1 z_2^{-1} \tag{2.19}$$

as defined in Example 2.2. To find all possible factorisations of g, it suffices to look at the decompositions of its Newton polygon into a Minkoski sum of Newton polygons. The possible decompositions are given in Table 2.1. These components, defined by Newton polygons, can be translated back to Laurent polynomials since every integral point (point with integer coordinates) in the Newton polygon corresponds to a monomial. Therefore, we have the following decompositions of g:

$$\begin{array}{rl} I: & (u_1 + u_2 z_1 + u_3 z_2)(v_1 + v_2 z_1^{-1} + v_3 z_2^{-1}) \\ II: & (u_1 + u_2 z_1 + u_3 z_1 z_2^{-1} + u_4 z_2^{-1})(v_1 + v_2 z_1^{-1} z_2) \\ III: & (u_1 + u_2 z_2 + u_3 z_1 + u_4 z_1 z_2^{-1})(v_1 + v_2 z_1^{-1}) \\ IV: & (u_1 + u_2 z_1^{-1} z_2 + u_3 z_2 + u_4 z_1)(v_1 + v_2 z_2^{-1}) \\ V: & (u_1 + u_2 z_2^{-1})(v_1 + v_2 z_1)(w_1 + w_2 z_1^{-1} z_2) \end{array}$$

with u_i, v_i and w_i being unknown coefficients. By coefficient comparison with the coefficients of g, one can obtain conditions for such decomposition to happen. However, not every decomposition yields a valid factorisation. A complete example is given in section 3.2.

2.5 Toric polynomial division

Several methods are known to deal with parametric polynomial systems including parametric Gröbner bases [78, 79, 57], triangular sets [75, 7], discriminant varieties [51, 61] and rational parametrisation [70]. For the problem studied here, we need to find an efficient way of dividing multivariate parametric polynomials in order to obtain a remainder which is expressed in the simplest possible form. To achieve this goal, we take optimal advantage of the Newton polygons of the equations, and so we use a well-known theorem of Ostrowski and a toric variant of polynomial division. This variant of the polynomial division algorithm is related to [69], which use similar ideas for factoring polynomials by taking advantage of the special shape of their Newton polytopes.

We introduce a polynomial division algorithm called toric polynomial division which will be used in sections 3, 4 and 5. It will be shown in section 3.3 that for the type of problem considered in this thesis, this algorithm leads to simpler equations.

Definition Assume that g is a Laurent polynomial such that its corner coefficients are not zero divisors. A finite subset Γ of $\mathbb{Z}^2$ is called a **remainder support set** with respect to g iff no polynomial multiple of g, except zero, has support contained in Γ.

Definition Let g be a Laurent polynomial such that its corner coefficients are invertible in R. Let f be an arbitrary Laurent polynomial. Then (q, r) is a **quotient remainder pair** for (f, g) iff the following conditions are fulfilled.

a) $f = qg + r$.
b) The support of r is contained in $\Pi(f)$.
c) The support of r is a remainder support set with respect to g.

Quotient remainder pairs are not unique. Here is a nondeterministic algorithm that computes quotient remainder pairs.

Algorithm 1 Toric Polynomial Division Algorithm

1: **Input:** f, g, such that g has invertible corner coefficients.
2: **while** $f \neq 0$ **do**
3: Select a linear functional $l : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto (ax + by)$ such that a/b is irrational.
4: Compute the point $P_f = (\alpha_1, \alpha_2) \in \text{Support}(f)$ that maximizes l.
5: Compute the point $P_g = (\beta_1, \beta_2) \in \text{Support}(g)$ that maximizes l.
6: **if** $\Pi(g) + P_f - P_g \subset \Pi(f)$ **then**
7: $m := \frac{\text{Coef}(f,P_f)}{\text{Coef}(g,P_g)} x^{\alpha_1 - \beta_1} y^{\alpha_2 - \beta_2}$;
8: $q := q + m$; $f := f - mg$;
9: **else**
10: $m := \text{Coef}(f, P_f) x^{\alpha_1} y^{\alpha_2}$;
11: $r := r + m$; $f := f - m$;
12: **end if**
13: **end while**
14: **Output:** q, r.

Theorem 2 *Algorithm 1 is correct.*

Proof The Newton polygon of f becomes smaller in each **while** loop, hence it is clear that Algorithm 1 terminates. Also, any monomial which is added to r is contained in $\Pi(f)$, hence it follows that r fulfills (b) in Definition 2.5.

No step in the algorithm changes the value of $f + qg + r$. Initially, this value is the given polynomial f, and in the end, this value is equal to $qg + r$. This shows that (a) in Definition 2.5 is fulfilled.

In order to prove (c) in Definition 2.5, we claim that the following is true throughout the execution of the algorithm: if h is any Laurent polynomial such that gh has support in $\Pi(f) \cup \text{Support}(r)$, then the coefficients of gh at the exponent vectors in $\text{Support}(r)$ are zero.

Initially, $\text{Support}(r)$ is empty and the claim is trivially true. If the claim is true before step 8, then it is also true after step 8, because this step does not change r and does not increase the Newton polygon of f.

Assume that for a certain Laurent polynomial h, the claim is true before step 11 and false after step 11. Then it follows that the coefficient of gh at P_f is not zero, because this is the only exponent vector which is new in r. The support of gh is also contained in the Newton polygon of f before step 11, hence P_f is the unique vector in $\text{Support}(gh)$ where l reaches a maximal value. Because P_g is the unique vector in $\text{Support}(g)$ where l reaches a maximal value, it follows that $(P_f - P_g) \in \text{Support}(h)$. Then $\Pi(g) + P_f - P_g \subset \Pi(gh)$ as a consequence of Theorem 1. But this implies that the **if** condition in step 6 is fulfilled for g and f before step 11, and therefore step 11 is not reached for such values of f and g.

Step	P_f	P_g	Computation
7: 8:	(0,2)	(0,1)	$M = \frac{c_{02}}{d_{01}} y$ $Q = \frac{c_{02}}{d_{01}} y$ $F = \left(c_{11} - \frac{c_{02}d_{10}}{d_{01}}\right) xy + \left(c_{01} - \frac{c_{02}d_{00}}{d_{01}}\right) y + c_{10}x + c_{00}$
10: 11:	(1,1)	(0,1)	$M = \left(c_{11} - \frac{c_{02}d_{10}}{d_{01}}\right) xy$ $R = \left(c_{11} - \frac{c_{02}d_{10}}{d_{01}}\right) xy$ $F = \left(c_{01} - \frac{c_{02}d_{00}}{d_{01}}\right) y + c_{10}x + c_{00}$
7: 8:	(0,1)	(0,1)	$M = \frac{1}{d_{01}} \left(c_{01} - \frac{c_{02}d_{00}}{d_{01}}\right) =: \frac{A}{d_{01}}$ $Q = \frac{c_{02}}{d_{01}} y + \frac{A}{d_{01}}$ $F = \left(c_{10} - \frac{Ad_{10}}{d_{01}}\right) x + \left(c_{00} - \frac{Ad_{00}}{d_{01}}\right)$
10: 11:	(1,0)	(0,1)	$M = \left(c_{10} - \frac{Ad_{10}}{d_{01}}\right) x$ $R = \left(c_{11} - \frac{c_{02}d_{10}}{d_{01}}\right) xy + \left(c_{10} - \frac{Ad_{10}}{d_{01}}\right) x$ $F = c_{00} - \frac{Ad_{00}}{d_{01}}$
10: 11:	(0,0)	(0,1)	$M = c_{00} - \frac{Ad_{00}}{d_{01}}$ $R = \left(c_{11} - \frac{c_{02}d_{10}}{d_{01}}\right) xy + \left(c_{10} - \frac{Ad_{10}}{d_{01}}\right) x + \left(c_{00} - \frac{Ad_{00}}{d_{01}}\right)$ $F = 0$

Table 2.2: Example: Toric polynomial division algorithm

It follows that the claim is true throughout the execution of Algorithm 1. In particular, it is true at the end, which shows that (c) in Definition 2.5 holds.

Example Let $f = c_{02}y^2 + c_{11}xy + c_{01}y + c_{10}x + c_{00}$ and $g = d_{01}y + d_{10}x + d_{00}$. The result of the polynomial division algorithm is shown in Table 2.2 for the linear functional $l : (x, y) \mapsto (\sqrt{2}x + 2y)$. Therefore, f is divisible by g if and only if $r = 0$, or in other words if all coefficients of r are zero:

$$c_{11} - \frac{c_{02}d_{10}}{d_{01}} = c_{10} - \frac{Ad_{10}}{d_{01}} = c_{00} - \frac{Ad_{00}}{d_{01}} = 0 \tag{2.20}$$

2.6 Common components of parametric polynomials

The problem considered in this section is to find all parameters such that two bivariate parametric polynomial have infinitely many common solutions. We introduce the quantifier $\underset{\infty}{\exists}$ for "*there exists infinitely many*" and $S^1 = \{\mathbf{z} \in \mathbb{C} \mid \ |\mathbf{z}| = 1\}$, that is the unit circle.

Theorem 3 *Let g be an irreducible Laurent polynomial. Let f be a Laurent polynomial (not necessarily irreducible). The following are equivalent:*

1. $\underset{\infty(\mathbf{z}_1,\mathbf{z}_2)\in(S^1)^2}{\exists} g(\mathbf{z}_1, \mathbf{z}_2) = 0 \Rightarrow f(\mathbf{z}_1, \mathbf{z}_2) = 0$

2. $\exists$ *Laurent polynomial* $k(\mathbf{z}_1, \mathbf{z}_2)$ *such that* $f = g \cdot k$

Proof 2) $\Longrightarrow$ 1): It is straightforward since if there exists k such that $f = g \cdot k$ and $g(\mathbf{z}_1, \mathbf{z}_2) = 0$, then $f(\mathbf{z}_1, \mathbf{z}_2) = 0$.

1) $\Longrightarrow$ 2): Assume indirectly that f is not a multiple of g in the ring of Laurent polynomials. Using Bernshtein's theorem [16], it follows that the number of common zeros in $\mathbb{C}^{*2}$ is at most equal to the normed mixed volume of $\Pi(f)$ and $\Pi(g)$. In particular, there are at most finitely many common zeroes. But since g has infinitely many zeros in $(S^1)^2$, this is a contradiction.

Theorem 3 reformulates the problem into a polynomial division problem that can be solved using the toric polynomial division algorithm described in Section 2.5.

2.7 Mechanics

In this section, we introduce the concept of rotation matrix, angular momentum and moment of inertia matrix. See [72] for a more detailed introduction to the subject.

Let $R_p(\theta)$ be the rotation matrix representing a rotation of angle θ about a unit vector $p = [u_x, u_y, u_z]^T$, i.e. $R_p(\theta)$ is given by

$$\begin{bmatrix} u_x^2 + (u_y^2 + u_z^2)\cos\theta & u_x u_y(1-\cos\theta) - u_z \sin\theta & u_x u_z(1-\cos\theta) + u_y\sin\theta \\ u_x u_y(1-\cos\theta) + u_z\sin\theta & u_y^2 + (u_x^2 + u_z^2)\cos\theta & u_y u_z(1-\cos\theta) - u_x\sin\theta \\ u_x u_z(1-\cos\theta) - u_y\sin\theta & u_y u_z(1-\cos\theta) + u_x\sin\theta & u_z^2 + (u_x^2+u_y^2)\cos\theta \end{bmatrix} \tag{2.21}$$

Therefore, the rotation matrix about the x, y and z axis are given by

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let us start with a single particle rotating about a fixed axis, say the z axis, with angular velocity ω. The particle has a certain mass m and its position is given by $\mathbf{r} = [r_x, r_y, r_z]$. The velocity $\mathbf{v}$ of the particle is given by:

$$\mathbf{v} = \omega \times \mathbf{r} \tag{2.22}$$

where $\omega = [0, 0, \omega_z]$. The angular momentum $\mathbf{l}$ of a particle is defined as the vector

$$\mathbf{l} = \mathbf{r} \times m\,\mathbf{v} \tag{2.23}$$

Now instead of a single particle, consider a rigid body. A rigid body can be imagined as the collection of n particles, $i = 1, \ldots, n$. The position of the particle i is given by

$$\mathbf{r}_i = [x_i, y_i, z_i] \tag{2.24}$$

Its velocity and angular momentum are

$$\mathbf{v}_i = \omega \ \times \mathbf{r}_i = [-\omega_z \, y_i, \omega_z \, x_i, 0] \tag{2.25}$$

$$\mathbf{l}_i = \mathbf{r}_i \ \times \ m_i \, \mathbf{v}_i = m_i \omega_z [-z_i x_i, -z_i y_i, x_i^2 + y_i^2] \tag{2.26}$$

Therefore the angular momentum of the body is given by

$$L = [L_x, L_y, L_z] \tag{2.27}$$

with

$$\begin{aligned} L_x &= -\sum m_i x_i z_i \omega_z \\ L_y &= -\sum m_i y_i z_i \omega_z \\ L_z &= \sum m_i (x_i^2 + y_i^2) \omega_z \end{aligned} \tag{2.28}$$

This is the angular momentum of a rigid body about the z axis. Now, rewrite L_x, L_y and L_z in the following form:

$$\begin{aligned} L_x =& - I_{xz} \omega_z \\ L_y =& - I_{yz} \omega_z \\ L_z =& I_{xz} \omega_z \end{aligned} \tag{2.29}$$

I_{xz} and I_{yz} are called the **products of inertia** of the body while I_{zz} is called the **moment of inertia**. For a rotation about a general axis, the angular momentum can be written as [72]

$$L = \begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = I\omega \tag{2.30}$$

with

$$\begin{aligned} I_{xx} &= \sum m_i \left(y_i^2 + z_i^2\right) \\ I_{yy} &= \sum m_i \left(x_i^2 + z_i^2\right) \\ I_{zz} &= \sum m_i \left(x_i^2 + y_i^2\right) \\ I_{xy} &= \sum m_i x_i y_i \\ I_{xz} &= \sum m_i x_i z_i \\ I_{yz} &= \sum m_i y_i z_i \end{aligned} \tag{2.31}$$

The matrix I is called the **moment of inertia tensor** or just **inertia tensor**. To compute the angular momentum as expressed in (2.30), I and ω must both be expressed in the same frame of reference. The following theorem states that the inertia tensor can be diagonalized, the statement and the proof can be found in [72].

Theorem 4 *For any rigid body and any point O there are three perpendicular axes through O called the principal axes such that the inertia tensor I is diagonal.*

Here are some properties of the moment of inertia matrix.

Theorem 5 *Let I as defined in* (2.30), (2.31)*. Then*

1. *I is symmetric*
2. *I is positive definite*
3. *Every eigenvalue of I is strictly smaller than the sum of the two other eigenvalues (assuming that we do not allow infinitely thin bars, in this case, we could have also equality).*
4. *Let E be the identity matrix. $F = trace(I)\, E - 2\, I$ is positive definite. In other words, all its eigenvalues are positive.*

Proof For 1. and 2., we refer to [72, 5] for a proof.

3) Using Theorem 4, there exists principal axes such that I can be diagonalized to another matrix I' with diagonal values $\lambda_1, \lambda_2, \lambda_3$. In other words, these elements are the eigenvalues of I' and of I as well since the eigenvalues remain unchanged by orthogonal transformations. Let $A = \sum m_i x_i^2$, $B = \sum m_i y_i^2$ and $C = \sum m_i z_i^2$. From (2.31), we can write:

$$\begin{aligned} \lambda_1 &= \sum m_i y_i^2 + \sum m_i z_i^2 = B + C \\ \lambda_2 &= \sum m_i x_i^2 + \sum m_i z_i^2 = A + C \\ \lambda_3 &= \sum m_i x_i^2 + \sum m_i y_i^2 = A + B \end{aligned} \tag{2.32}$$

Assuming we do not allow infinitely thin rods, $A > 0$, $B > 0$, $C > 0$ and hence

$$\begin{aligned} -\lambda_1 + \lambda_2 + \lambda_3 &= 2\, A \; > \; 0 \\ \lambda_1 - \lambda_2 + \lambda_3 &= 2\, B \; > \; 0 \\ \lambda_1 + \lambda_2 - \lambda_3 &= 2\, C \; > \; 0 \end{aligned} \tag{2.33}$$

In other words

$$\begin{aligned} \lambda_2 + \lambda_3 &> \lambda_1 \\ \lambda_1 + \lambda_3 &> \lambda_2 \\ \lambda_1 + \lambda_2 &> \lambda_3 \end{aligned} \tag{2.34}$$

This completes the proof.

4) From [50], the eigenvalues of $-2I$ are $-2\lambda_1$, $-2\lambda_2$ and $-2\lambda_3$. The trace of a symmetric matrix is equal to the sum of its eigenvalues

$$\mathrm{trace}(I) = \lambda_1 + \lambda_2 + \lambda_3 \tag{2.35}$$

Therefore, $\mathrm{trace}(I)\,E$ has three identical eigenvalues $\lambda_1 + \lambda_2 + \lambda_3$ and we obtain the eigenvalues of F by translation of the eigenvalues [50] of $-2\,I$ by $\lambda_1 + \lambda_2 + \lambda_3$. Therefore the eigenvalues of F are $\lambda_1 + \lambda_2 - \lambda_3$, $\lambda_1 - \lambda_2 + \lambda_3$ and $-\lambda_1 + \lambda_2 + \lambda_3$. Since all the eigenvalues of F are positive, F is positive definite.

Chapter 3

Planar linkage

In this section, we derive sufficient and necessary conditions for the static and dynamic balancing of planar 4R linkages. First, we derive a system of algebraic equations in the design parameters and the angles in configuration space, modelled by complex variables $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$, and eliminate the variable $\mathbf{z}_3$ immediately. In section 3 we eliminate the remaining configuration variables $\mathbf{z}_1, \mathbf{z}_2$. Then, we explain what we mean by "solving a system of equations and inequations", and perform this operation for our system of design parameters, after changing to parameters that make the equations linear. The complete characterisation of statically and dynamically balanced planar four-bar linkages is summarized in Table 3.4.

3.1 Model

3.1.1 Kinematic model

A planar four-bar linkage is shown in Fig. 3.1. It consists of four links: the base of length d which is fixed, and three moveable links of length l_1, l_2, l_3 respectively. We assume that all link lengths are strictly positive. Since the base is fixed, the mass properties of the base have no influence on the equations and will therefore be ignored. Each of the three moveable links has a mass m_i, a centre of mass whose position is defined by r_i and ψ_i and the axial moment of inertia I_i. The design of planar four-bar linkages consists in choosing these 16 design parameters (see Table 3.1).

The links are connected by revolute joints rotating about axes pointing in a direction orthogonal to the plane of motion. The joint angles are specified using the time variables $\theta_1(t), \theta_2(t)$ and $\theta_3(t)$ as shown in Fig. 3.1. The kinematics of planar linkages can be conveniently represented in the complex plane, using complex numbers to describe the linkage's configuration (Fig. 3.2) and the position of the centre of mass (Fig. 3.3). Referring to Figs. 3.2 and 3.3, let $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ be time dependent unit complex numbers and $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ unit complex numbers depending on the design parameters (actually only

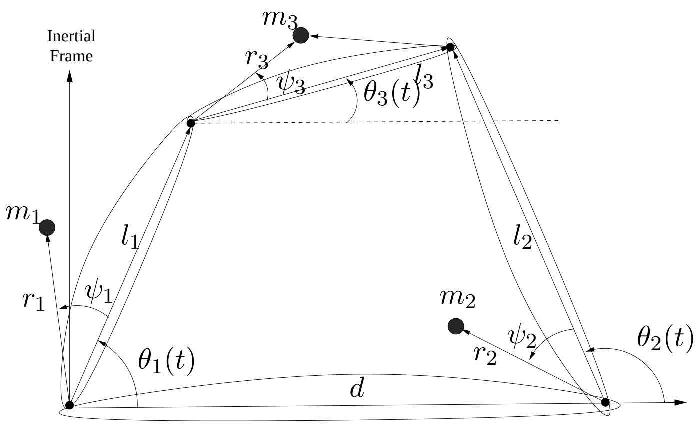


Figure 3.1: Four-bar linkage.

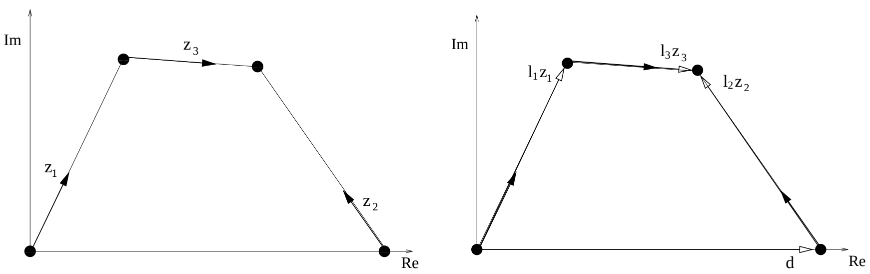


Figure 3.2: Complex representation for the kinematics.

on ψ_1, ψ_2, ψ_3). The orientation of $\mathbf{p}_i$ is specified relative to $\mathbf{z}_i$, i.e., it is attached to $\mathbf{z}_i$ and moves with it. If $\mathbf{p}_i$ coincides with $\mathbf{z}_i$, then $\mathbf{p}_i = 1$.

The dependency between the different joint angles is described by the following closure constraint:

$$\mathbf{z}_3 = G_1\mathbf{z}_1 + G_2\mathbf{z}_2 + G_3 \tag{3.1}$$

where $G_1, G_2, G_3 \in \mathbb{R}$ with $G_1 = \frac{-l_1}{l_3}, G_2 = \frac{l_2}{l_3}, G_3 = \frac{d}{l_3}$. The time derivative of $\mathbf{z}_j$ is given by:

$$\frac{d\mathbf{z}_j(t)}{dt} = \frac{de^{i\theta_j(t)}}{dt} = ie^{i\theta_j(t)}\dot{\theta}_j = i\mathbf{z}_j\dot{\theta}_j \tag{3.2}$$

Taking the time derivative of (3.1), we get a relationship between the joint angular velocities $\dot{\theta}_1, \dot{\theta}_2$ and $\dot{\theta}_3$, namely:

$$\mathbf{z}_3\dot{\theta}_3 = G_1\mathbf{z}_1\dot{\theta}_1 + G_2\mathbf{z}_2\dot{\theta}_2 \tag{3.3}$$

Since $\mathbf{z}_3$ is a unit complex number, $\mathbf{z}_3\overline{\mathbf{z}_3} = \mathbf{z}_3\mathbf{z}_3^{-1} = 1$ and therefore we

Type		Parameters
Geometric	Length	l_1, l_2, l_3, d
Static	Mass	m_1, m_2, m_3
	Centre of mass	$r_1, \psi_1, r_2, \psi_2, r_3, \psi_3$
Dynamic	Inertia	I_1, I_2, I_3

Table 3.1: Design parameters for the planar four-bar linkages.

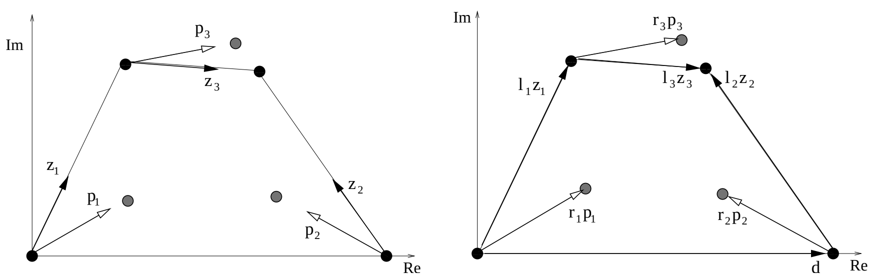


Figure 3.3: Complex representation for the centre of masses.

obtain the following geometric constraint (i.e. loop closure equation):

$$\begin{aligned} g &= (G_1\mathbf{z}_1 + G_2\mathbf{z}_2 + G_3)\left(G_1\mathbf{z}_1^{-1} + G_2\mathbf{z}_2^{-1} + G_3\right) - 1 \\ &= G_1G_2(\mathbf{z}_1^{-1}\mathbf{z}_2 + \mathbf{z}_1\mathbf{z}_2^{-1}) + G_1G_3(\mathbf{z}_1 + \mathbf{z}_1^{-1}) \\ &\qquad + G_2G_3(\mathbf{z}_2 + \mathbf{z}_2^{-1}) + (G_1^2 + G_2^2 + G_3^2 - 1) = 0 \end{aligned} \tag{3.4}$$

The time derivative of the geometric constraint (3.4) can be written as a linear combination of the joint angular velocities:

$$i(k_1\dot{\theta}_1 + k_2\dot{\theta}_2) = 0 \tag{3.5}$$

where

$$k_1 = G_1G_2(\mathbf{z}_1\mathbf{z}_2^{-1} - \mathbf{z}_1^{-1}\mathbf{z}_2) + G_1G_3(\mathbf{z}_1 - \mathbf{z}_1^{-1}) \tag{3.6}$$

$$k_2 = G_1G_2(\mathbf{z}_1^{-1}\mathbf{z}_2 - \mathbf{z}_1\mathbf{z}_2^{-1}) + G_2G_3(\mathbf{z}_2 - \mathbf{z}_2^{-1}) \tag{3.7}$$

It is noted that since k_1 and k_2 are purely imaginary, only one constraint equation, over the real set, is obtained.

3.1.2 Centre of mass

Let M be the total mass of the linkage ($M = m_1 + m_2 + m_3$). The position of the centre of mass of the linkage $\mathbf{rS}$ is:

$$\mathbf{rS} = \frac{1}{M}\left(\mathbf{rS}_1 + \mathbf{rS}_2 + \mathbf{rS}_3\right) \tag{3.8}$$

where $\mathbf{rS}_1$, $\mathbf{rS}_2$ and $\mathbf{rS}_3$ are the positions of the centre of mass of the three moving links expressed in the reference frame:

$$\begin{aligned} \mathbf{rS}_1 &= r_1\mathbf{p}_1\mathbf{z}_1 \\ \mathbf{rS}_2 &= (d + r_2\mathbf{p}_2\mathbf{z}_2) \\ \mathbf{rS}_3 &= (l_1\mathbf{z}_1 + r_3\mathbf{p}_3\mathbf{z}_3) \end{aligned} \tag{3.9}$$

Substituting (3.1) in (3.8), the variable $\mathbf{z}_3$ can be eliminated and the position of the centre of mass can be written in the following form:

$$\mathbf{rS} = \frac{1}{M}\left(\mathbf{F}_1\mathbf{z}_1 + \mathbf{F}_2\mathbf{z}_2 + \mathbf{F}_3\right) \tag{3.10}$$

where $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3 \in \mathbb{C}$:

$$\begin{aligned} \mathbf{F}_1 &= m_1r_1\mathbf{p}_1 + m_3l_1 + G_1m_3r_3\mathbf{p}_3 \\ \mathbf{F}_2 &= m_2r_2\mathbf{p}_2 + G_2m_3r_3\mathbf{p}_3 \\ \mathbf{F}_3 &= m_2d + G_3m_3r_3\mathbf{p}_3. \end{aligned} \tag{3.11}$$

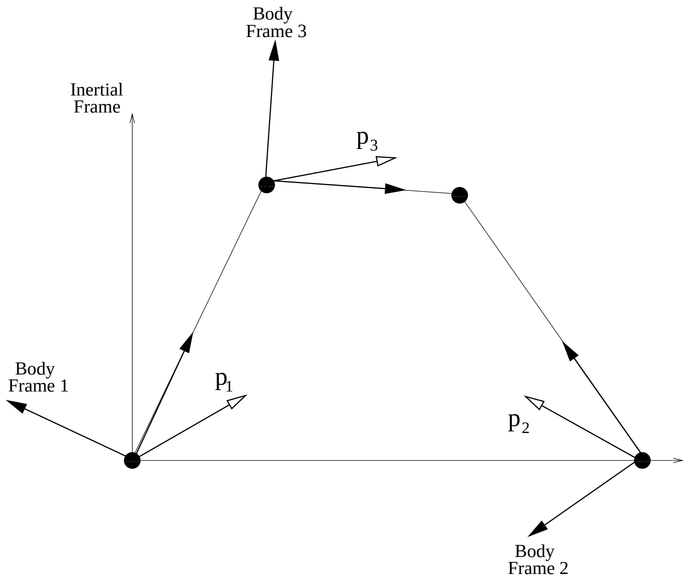


Figure 3.4: Unit vectors representation

3.1.3 Angular momentum

Since the linkage is planar, the contribution of the body i to the angular momentum is a scalar and can be given in the following form:

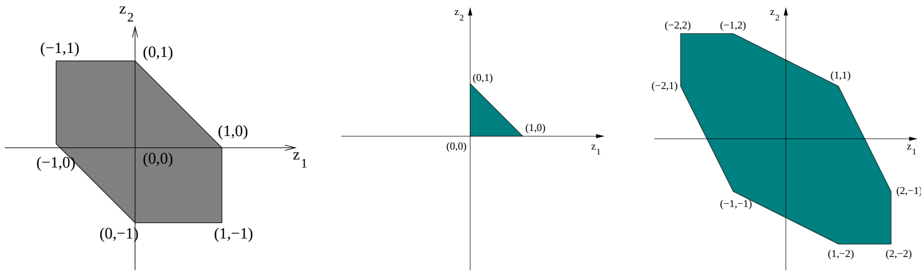
$$L_i = m_i \langle \mathbf{q}_i, -i\dot{\mathbf{q}}_i \rangle + I_i \dot{\theta}_i \tag{3.12}$$

where $\mathbf{q}_i$ and $\dot{\mathbf{q}}_i$ are respectively the position and the velocity of the centre of mass of body i with respect to a given inertial frame, I_i denotes the axial moment of inertia of body i with respect to its centre of mass and $J_i = I_i + m_i r_i^2$. $\langle *, * \rangle$ is the scalar product of planar vectors, i.e. $\langle u, v \rangle = \mathrm{Re}(u\overline{v}) = \frac{u\overline{v} + \overline{u}v}{2}$. The total angular momentum L of the system is given by the sum of the angular momentum of the links, (i.e. $L = L_1 + L_2 + L_3$). The angular momentum of the first body with respect to the inertial frame is:

$$\begin{aligned} L_1 &= \langle r_1 \mathbf{p}_1 \mathbf{z}_1, -im_1(ir_1 \mathbf{p}_1 \mathbf{z}_1 \dot{\theta}_1) \rangle + I_1 \dot{\theta}_1 \\ &= \langle r_1 \mathbf{p}_1 \mathbf{z}_1, m_1 r_1 \mathbf{p}_1 \mathbf{z}_1 \dot{\theta}_1 \rangle + I_1 \dot{\theta}_1 \\ &= J_1 \dot{\theta}_1 \end{aligned} \tag{3.13}$$

The contribution of the second body to the angular momentum is given by:

$$\begin{aligned} L_2 &= \left\langle (d + r_2 \mathbf{p}_2 \mathbf{z}_2), m_2 r_2 \mathbf{p}_2 \mathbf{z}_2 \dot{\theta}_2 \right\rangle + I_2 \dot{\theta}_2 = \\ &= \left[m_2 d r_2 \left(\frac{\mathbf{p}_2 \mathbf{z}_2 + \mathbf{p}_2^{-1} \mathbf{z}_2^{-1}}{2} \right) + J_2 \right] \dot{\theta}_2 \end{aligned} \tag{3.14}$$



$\Pi(g)$ $\Pi(f)$ $\Pi(k)$

Figure 3.5: Newton polygons of g, f and k.

For the third body, we get

$$L_3 = \left\langle (l_1\mathbf{z}_1 + r_3\mathbf{p}_3\mathbf{z}_3), m_3 \left(l_1\mathbf{z}_1\dot{\theta}_1 + r_3\mathbf{p}_3\mathbf{z}_3\dot{\theta}_3 \right) \right\rangle + I_3\dot{\theta}_3 \tag{3.15}$$

Substituting (3.1) and (3.3) into (3.15), we can eliminate $\mathbf{z}_3\dot{\theta}_3$ and $\dot{\theta}_3$ and obtain an expression in terms of $\mathbf{z}_1, \mathbf{z}_2, \dot{\theta}_1, \dot{\theta}_2$ only. The total angular momentum L of the linkage is then given by:

$$L = L_1 + L_2 + L_3 = k_3\dot{\theta}_1 + k_4\dot{\theta}_2 \tag{3.16}$$

where k_3 and k_4 are written as:

$$\begin{aligned} k_3 &= a_1\mathbf{z}_1 + a_2\mathbf{z}_1^{-1} + a_3\mathbf{z}_1\mathbf{z}_2^{-1} + a_4\mathbf{z}_1^{-1}\mathbf{z}_2 + a_5 \\ k_4 &= b_1\mathbf{z}_2 + b_2\mathbf{z}_2^{-1} + b_3\mathbf{z}_1\mathbf{z}_2^{-1} + b_4\mathbf{z}_1^{-1}\mathbf{z}_2 + b_5 \end{aligned} \tag{3.17}$$

where constants a_i, b_i can be obtained from (3.13), (3.14), (3.15), (3.16).

3.1.4 Problem formulation

In our settings, a mechanism is said to be statically balanced if the centre of mass of the mechanism remains stationary **for infinitely many configurations** (i.e. infinitely many choices of the joint angles). From (3.10), this condition can be formulated as:

$$f(\mathbf{z}_1, \mathbf{z}_2) = \mathbf{F}_1\mathbf{z}_1 + \mathbf{F}_2\mathbf{z}_2 - \mathbf{r}\mathbf{S}' = 0 \tag{3.18}$$

where $\mathbf{r}\mathbf{S}' = \mathbf{r}\mathbf{S}M - \mathbf{F}_3$ is a constant. In other words, the expression $\mathbf{F}_1\mathbf{z}_1 + \mathbf{F}_2\mathbf{z}_2$ must be constant. A mechanism is said to be dynamically balanced [82] if the centre of mass remains fixed (static balancing) and the total angular momentum is zero at all times, i.e.,

$$L = k_3(\mathbf{z}_1, \mathbf{z}_2)\dot{\theta}_1 + k_4(\mathbf{z}_1, \mathbf{z}_2)\dot{\theta}_2 = 0 \tag{3.19}$$

Therefore, (3.4), (3.5), (3.18), (3.19) have to be satisfied. Among these four equations, only two (3.5) and (3.19) depend (linearly) on the joint angular velocities and they can be rewritten in the following form:

$$\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.20}$$

If the rank of the matrix $A = \begin{bmatrix} k_1(\mathbf{z}_1,\mathbf{z}_2) & k_2(\mathbf{z}_1,\mathbf{z}_2) \\ k_3(\mathbf{z}_1,\mathbf{z}_2) & k_4(\mathbf{z}_1,\mathbf{z}_2) \end{bmatrix}$ is 2. Since the system is homogeneous, then the only solution is $\dot{\theta}_1 = \dot{\theta}_2 = 0$. In other words, the linkage is not moving. Therefore we must have:

$$k(\mathbf{z}_1,\mathbf{z}_2) := \det(A) = k_1(\mathbf{z}_1,\mathbf{z}_2)k_4(\mathbf{z}_1,\mathbf{z}_2) - k_2(\mathbf{z}_1,\mathbf{z}_2)k_3(\mathbf{z}_1,\mathbf{z}_2) = 0 \tag{3.21}$$

The Newton polygons of g, f and L are shown in Fig. 3.5. We therefore obtain a set of 3 algebraic equations (3.4),(3.18),(3.21) in terms of the unit complex variable $\mathbf{z}_1, \mathbf{z}_2$ and independent from the joint angular velocities. The problem can be formulated in the following way (see section 2.6):

PROBLEM FORMULATION

Let $(S^1)^2 = \{(\mathbf{z}_1,\mathbf{z}_2) \in \mathbb{C} \mid |\mathbf{z}_1| = 1 \text{ and } |\mathbf{z}_2| = 1\}$. Let g, f and k as defined in equations (3.4), (3.18) and (3.21) respectively.

Static balancing:

Find all possible geometric and static parameters such that:

$$\underset{\infty(\mathbf{z}_1,\mathbf{z}_2)\in(S^1)^2}{\exists} g(\mathbf{z}_1,\mathbf{z}_2) = 0 \Rightarrow f(\mathbf{z}_1,\mathbf{z}_2) = 0 \tag{3.22}$$

Dynamic balancing:

Find all possible geometric, static and dynamic parameters such that:

$$\underset{\infty(\mathbf{z}_1,\mathbf{z}_2)\in(S^1)^2}{\exists} g(\mathbf{z}_1,\mathbf{z}_2) = 0 \Rightarrow f(\mathbf{z}_1,\mathbf{z}_2) = k(\mathbf{z}_1,\mathbf{z}_2) = 0 \tag{3.23}$$

Using Theorem 3, we can reformulate the balancing problem as a factorisation problem of Laurent polynomials.

3.2 Kinematic modes

In order to use Theorem 3 to derive sufficient and necessary conditions for the static and dynamic balancing, the geometric constraint g defined by (3.4) must be irreducible. For some specific choices of the geometric parameters, the geometric constraint can be factored into several irreducible components. For example, if $l_1 = l_2 \neq l_3 = d$, the geometric constraint can be factored as:

$$g = g_A\, g_B = \frac{-l_1}{\mathbf{z}_1\mathbf{z}_2}\left(d\mathbf{z}_1\mathbf{z}_2 + l_1\mathbf{z}_1 - l_1\mathbf{z}_2 - d\right)(\mathbf{z}_1 - \mathbf{z}_2) \qquad (3.24)$$

The geometric constraint is therefore fulfilled if at least one of the components, either g_A or g_B is zero. Each of these components is called a kinematic mode.

In order to obtain a complete description of all reducible cases, we can classify all possible factorisations of g by looking at the decomposition of its Newton polygon ($\Pi(g)$) into the Minkowski sums of Newton polygons (see Section 2.4). All such decompositions are shown in Table 2.1.

There is no factorisation of the geometric constraint corresponding to the decomposition I in Table 2.1. The reason is that this decomposition corresponds to the factorisation of

$$g + 1 = (G_1\mathbf{z}_1 + G_2\mathbf{z}_2 + G_3)\left(G_1\mathbf{z}_1^{-1} + G_2\mathbf{z}_2^{-1} + G_3\right),$$

$g + 1$ and g have the same corner coefficients, which completely determine the factors. Moreover, g and $g + 1$ cannot have the same factorisation.

For the case II, it is possible to find valid geometric design parameters for this factorisation. We proceed as follow. Let

$$g = h_1 h_2 \qquad (3.25)$$

with

$$\begin{aligned} h_1 &= u_1 + u_2\mathbf{z}_1 + u_3\mathbf{z}_1\mathbf{z}_2^{-1} + u_4\mathbf{z}_2^{-1} \\ h_2 &= v_1 + v_2\mathbf{z}_1^{-1}\mathbf{z}_2 \end{aligned}$$

where $u_i, v_j \in \mathbb{R}$ are unknowns. Comparing the coefficients of the left and right hand side of equation (3.25), we obtain

$$\begin{aligned} \left[\mathbf{z}_1\mathbf{z}_2^{-1}\right] : G_1G_2 &= u_3v_1 \\ \left[\mathbf{z}_1^{-1}\mathbf{z}_2\right] : G_2G_1 &= u_1v_2 \\ [\mathbf{z}_1] : G_1G_3 &= u_2v_1 \\ \left[\mathbf{z}_1^{-1}\right] : G_3G_1 &= u_4v_2 \\ [\mathbf{z}_2] : G_2G_3 &= u_2v_2 \\ \left[\mathbf{z}_2^{-1}\right] : G_3G_2 &= u_4v_1 \\ [1] : G_1G_1 + G_2G_2 + G_3G_3 - 1 &= u_1v_1 + u_3v_2 \end{aligned}$$

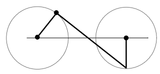
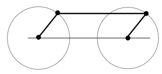
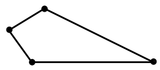
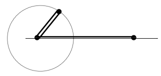
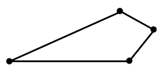
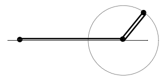
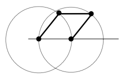
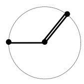
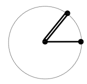
Case	Mode A	Mode B	Mode C
II Parallelogram $l_1 = l_2 \neq l_3 = d$	 $d\mathbf{z}_1\mathbf{z}_2 + l_1\mathbf{z}_1 - l_1\mathbf{z}_2 - d = 0$	 $\mathbf{z}_1 - \mathbf{z}_2 = 0$	
III Deltoid-1 $l_1 = l_3 \neq l_2 = d$	 $-d\mathbf{z}_1\mathbf{z}_2 - d\mathbf{z}_1 + l_1\mathbf{z}_2 + l_1\mathbf{z}_1^2 = 0$	 $\mathbf{z}_2 + 1 = 0$	
IV Deltoid-2 $l_1 = d \neq l_2 = l_3$	 $d\mathbf{z}_1\mathbf{z}_2 + l_2\mathbf{z}_1 - d\mathbf{z}_2 - l_2\mathbf{z}_2^2 = 0$	 $\mathbf{z}_1 - 1 = 0$	
V Rhomboid $l_1 = l_2 = l_3 = d$	 $\mathbf{z}_1 - \mathbf{z}_2 = 0$	 $\mathbf{z}_1 - 1 = 0$	 $\mathbf{z}_2 + 1 = 0$

Table 3.2: Kinematic modes for all physically possible reducible cases.

Solving this system of equations gives the following two constraints:

$$G_1^2 = G_2^2 \quad (3.26)$$
$$G_3^2 = 1 \quad (3.27)$$

In terms of the geometric parameters, this can be translated to

$$\left(\left(\frac{-l_1}{l_3}\right)^2 = \left(\frac{l_2}{l_3}\right)^2\right) \wedge \left(\left(\frac{d}{l_3}\right)^2 = 1\right). \quad (3.28)$$

and

$$l_1 = l_2 \quad (3.29)$$
$$l_3 = d. \quad (3.30)$$

Case II corresponds therefore to a parallelogram. That is, the length of the base is equal to the length of the coupler, while the lengths of the input and output crank are also equal. The same approach can be used for the other cases and the conditions on the geometric parameters are summarized in Table 3.2. This classification corresponds to the classification given in [32] and [17] (Chapter 11, p.426).

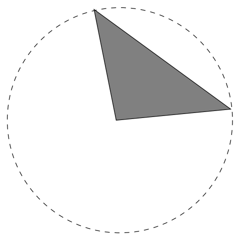


Figure 3.6: Degenerated case.

3.3 Static balancing

In this section, a complete characterisation of statically balanced planar four-bar linkages is given. We first start with the degenerated case for which one of the body lengths is 0. Then we investigate the case for which the geometric constraint is irreducible. Finally, we derive the conditions for all cases for which the geometric constraint is reducible: the parallelogram, the deltoid and the rhomboid.

3.3.1 Degenerated case

If one of the l_i (i=1,2,3) is zero and $d \neq 0$, clearly the linkage cannot move since it has no degree of freedom. Therefore, the only case to consider is the case when $d = 0$. If all other lengths are non-zero (i.e. $l_1 \neq 0$, $l_2 \neq 0$, $l_3 \neq 0$), we obtain a triangle rotating about the origin (see Figure 3.6). Clearly, this linkage is statically balanced if and only if the centre of mass of the linkage is at the origin. The same conditions are obtained if $d = 0$ and one of the l_i is also equal to 0. In this case, the linkage is a pendulum.

3.3.2 Irreducible case

Assume g is irreducible. Since the lengths of the bodies are strictly positive, G_1, G_2 and G_3 are different than zero. Therefore, the coefficients of all monomials of g are also non-zero and the Newton polygon of g cannot be smaller. However, we do not have such constraints on the coefficients of f, i.e.: $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$ could be equal to 0. Therefore, $\Pi(f)$ could be smaller. Using Theorem 3, the four-bar mechanism is statically balanced if and only if there exists a Laurent polynomials l such that:

$$f(\mathbf{z}_1, \mathbf{z}_2) = g(\mathbf{z}_1, \mathbf{z}_2)\, l(\mathbf{z}_1, \mathbf{z}_2) \tag{3.31}$$

Using Theorem 1 to study the Newton polygon representation of this product as a Minkowski sum, we obtain a relationship between the Newton polygons (see Fig. 3.7).

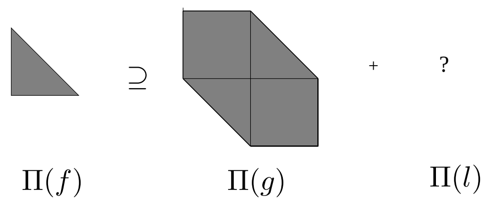


Figure 3.7: Newton polygons relation for the irreducible case

Clearly, such Newton polygons do not exist. The only solution is the trivial solution, $l = 0$. Therefore, f must be the zero polynomial. In other words $\mathbf{F}_1 = \mathbf{F}_2 = 0$ and we obtain the following condition for static balancing:

$$\begin{aligned} \mathbf{F}_1 &= m_1 r_1 \mathbf{p}_1 + m_2 l_1 + G_1 m_2 r_2 \mathbf{p}_2 = 0 \\ \mathbf{F}_2 &= m_3 r_3 \mathbf{p}_3 + G_2 m_2 r_2 \mathbf{p}_2 = 0 \end{aligned} \tag{3.32}$$

These conditions correspond to the conditions derived by Berkof and Lowen[14]. When g is irreducible, these conditions are necessary and sufficient.

3.3.3 Parallelogram

In the case of the parallelogram, we have two kinematic modes (see Table 3.2). For the mode A, the Newton polygon of the geometric constraint is a square. Using the same argument as in the irreducible case, the only solution for static balancing is $\mathbf{F}_1 = \mathbf{F}_2 = 0$. However, in mode B, the geometric constraint is $\mathbf{z}_1 - \mathbf{z}_2 = 0$, or in other words $\mathbf{z}_1 = \mathbf{z}_2$. Replacing $\mathbf{z}_1 = \mathbf{z}_2$ in the expression for the centre of mass position given by (3.18), we obtain

$$f(\mathbf{z}_1, \mathbf{z}_2) = \mathbf{F}_1\mathbf{z}_1 + \mathbf{F}_2\mathbf{z}_2 - \mathbf{r}\mathbf{S}' = (\mathbf{F}_1 + \mathbf{F}_2)\mathbf{z}_2 - \mathbf{r}\mathbf{S}' \tag{3.33}$$

This expression is constant if and only if $\mathbf{F}_1 + \mathbf{F}_2 = 0$. Note that the Berkof and Lowen solution ($\mathbf{F}_1 = \mathbf{F}_2 = 0$) is a subset of this solution.

3.3.4 Deltoid

In the deltoid case (say deltoid-1), sufficient and necessary conditions for static balancing in mode A are the Berkof and Lowen conditions $\mathbf{F}_1 = \mathbf{F}_2 = 0$. For mode B, we have $\mathbf{z}_2 = -1$ and hence

$$f = \mathbf{F}_1\mathbf{z}_1 + \mathbf{F}_2\mathbf{z}_2 - \mathbf{r}\mathbf{S}' = \mathbf{F}_1\mathbf{z}_1 - \mathbf{F}_2 - \mathbf{r}\mathbf{S}' \tag{3.34}$$

$f = 0$ for infinitely many values of $\mathbf{z}_1$ if and only if $\mathbf{F}_1 = 0$. By symmetry, we obtain that $\mathbf{F}_2 = 0$ for the deltoid-2 in mode B.

3.3.5 Rhomboid

For the rhomboid, using the same approach, we get all special cases of the deltoid and rhomboid. Solutions are summarized in Table 3.4.

3.4 Dynamic balancing

In this section, we derive sufficient and necessary conditions for the dynamic balancing for all cases.

3.4.1 Degenerate case

It is not possible to dynamically balance such degenerated linkages since all parts are rotating in the same direction.

3.4.2 Irreducible

The Newton polygon of $k(\mathbf{z}_1, \mathbf{z}_2)$, given by (3.21), is shown in Fig 3.5. We are looking for the conditions on the design parameters (Table 3.1) such that g is a component of k. In other words, we are looking for a Laurent polynomial l such that

$$k(\mathbf{z}_1, \mathbf{z}_2) = g(\mathbf{z}_1, \mathbf{z}_2)\, l(\mathbf{z}_1, \mathbf{z}_2) \tag{3.35}$$

To derive these conditions, we have to divide k by g and look for the conditions for which the remainder is zero. A first approach would be to consider both polynomials k and g as univariate polynomials in terms of, let say, $\mathbf{z}_1$. The polynomial division would yield a remainder polynomial in the variable $\mathbf{z}_2$. By setting all coefficients of this remainder polynomial to zero, we would obtain a set of equations in terms of the design parameters. However, using this approach, we get a system of 18 equations and the number of additions and multiplications as returned by the Maple function `codegen[cost]` are 487 and 2569. An alternative approach is to use the toric polynomial division algorithm described in Section 2.5. Using this approach, the remainder is much sparser than the remainder obtained by the usual polynomial division: the complexity gives only 80 additions and 200 multiplications. We obtain a set of conditions between the design parameters. Combining these constraints with the static balancing constraints described above, we obtain a set of equalities and inequalities (due to physical constraints) for the linkage to be dynamically balanced. Among these constraints, we have

$$l_2^2 J_3 + l_3^2 J_2 = 0 \tag{3.36}$$

recalling from section 3.1.3 that $J_i = m_i r_i^2 + I_i$. Therefore $I_2 = I_3 = r_2 = r_3 = 0$ which is physically not possible. Therefore, if g is irreducible, a planar four-bar linkage cannot be dynamically balanced.

3.4.3 Parallelogram

For the reducible cases, deriving the balancing conditions is made easier by the fact that all polynomials representing a kinematic mode (Table 3.2) are linear either in $\mathbf{z}_1$ or $\mathbf{z}_2$ or both. For example, for the parallelogram in mode A, the corresponding polynomial $d\mathbf{z}_1\mathbf{z}_2 + l_1\mathbf{z}_1 - l_1\mathbf{z}_2 - d$ is linear if considered as a polynomial in variable $\mathbf{z}_1$. One can solve for $\mathbf{z}_1$ in terms of $\mathbf{z}_2$ and substitute into the dynamic balancing equation (3.21) to obtain a univariate polynomial in $\mathbf{z}_2$, which should vanish for infinitely many values of $\mathbf{z}_2$. Therefore all coefficients, which are expressions in terms of the design parameters, of this univariate polynomial must vanish. This gives a set of equations in terms of the design parameters.

For the case of the parallelogram, we have $l_3 = d$, $l_1 = l_2 =: l$ with $l \neq d$. Consider first the mode A. In order to describe the solutions for the dynamic balancing, we introduce another set of parameters, namely

$$\mathbf{q}_i := m_i r_i \mathbf{p}_i, i = 1, 2, 3.$$

and $J_i := I_i + m_i r_i^2$ as defined in Section 3.1.3. Parameters $I_i, r_i, \mathbf{p}_i$ can then be eliminated easily and the balancing conditions become linear in $m_i, \mathbf{q}_i, J_i, i = 1, 2, 3$. For the dynamic balancing, we have the following constraints on the $\mathbf{q}_i$:

$$\begin{aligned} \mathbf{q}_1 &= \frac{l}{d}\mathbf{q}_3 - l m_3 \\ \mathbf{q}_2 &= -\frac{l}{d}\mathbf{q}_3 \end{aligned} \tag{3.37}$$

and two constraints relating the J_i:

$$\begin{aligned} J_1 &= \frac{d^2 + l^2}{d}\mathbf{q}_3 - J_3 - l^2 m_3 \\ J_2 &= \frac{d^2 - l^2}{d}\mathbf{q}_3 - J_3 \end{aligned} \tag{3.38}$$

A first consequence is that $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ must be real. The parameters fulfill also the inequality constraints

$$m_i > 0, J_i m_i - |\mathbf{q}_i|^2 > 0 \tag{3.39}$$

for $i = 1, 2, 3$. In particular, J_1 and J_2 must be positive. From (3.38), we get an upper bound for m_3, which must be larger than the lower bound from (3.39) ($i = 3$). This yields

$$(d\mathbf{q}_3 - J_3)(dJ_3 - l^2\mathbf{q}_3) > 0. \tag{3.40}$$

It follows that J_3 is contained in the open interval $(d\mathbf{q}_3, \frac{l^2}{d}\mathbf{q}_3)$. (Note that $\frac{d^2-l^2}{d}\mathbf{q}_3 > 0$ as a consequence of (3.38), that is why we know which of the

Body i	l_i	m_i	r_i	p_i	I_i
$i=1$	1	m_1	$\frac{1}{2}$	-1	$\frac{3m_1}{4}$
$i=2$	1	$\frac{m_1}{3}$	$\frac{1}{2}$	-1	$\frac{5m_1}{4}$
$i=3$	4	$\frac{2m_1}{3}$	1	1	$\frac{m_1}{2}$

Body i	l_i	m_i	r_i	p_i	I_i
$i=1$	1	m_1	$\frac{1}{2}$	-1	$\frac{m_1}{4}$
$i=2$	4	$\frac{m_1}{3}$	2	1	$\frac{m_1}{3}$
$i=3$	1	$\frac{m_1}{3}$	$\frac{1}{2}$	-1	$\frac{3m_1}{4}$

Table 3.3: Example of dynamically balanced linkages: On the left for the Parallelogram (Mode A), and on the right for the Deltoid-1 (Mode A).

two interval boundaries is bigger.) Then $\mathbf{q}_3 > 0$ and $d > l$ follows. From $J_2 > 0$ and (3.38), we get

$$\frac{l^2}{d}\mathbf{q}_3 < J_3 < \frac{d^2 - l^2}{d}\mathbf{q}_3, \tag{3.41}$$

from which $d \geq \sqrt{2}l$ follows.

Conversely, if $d \geq \sqrt{2}l$, then we can choose $\mathbf{q}_3 > 0$ arbitrarily and J_3 subject to (3.41), and m_3 between the upper and lower bound for m_3 derived above. Then (3.38) determines J_1 and J_2, which will then be positive and (3.37) determines $\mathbf{q}_1$ and $\mathbf{q}_2$, and finally m_1 and m_2 can be chosen so that inequality (3.39) is fulfilled.

In Table 3.3, an example of a dynamically balanced linkage in mode A is shown. For these design parameters, the geometric constraint (3.4) becomes:

$$g = \frac{-1}{16}\left(4\mathbf{z}_1\mathbf{z}_2 + \mathbf{z}_1 - \mathbf{z}_2 - 4\right)\left(\mathbf{z}_1 - \mathbf{z}_2\right) = 0 \tag{3.42}$$

and (3.21) can be written as

$$k = \frac{-3}{512}\frac{m_1\left(4\mathbf{z}_1\mathbf{z}_2 + \mathbf{z}_1 - \mathbf{z}_2 - 4\right)\left(\mathbf{z}_1 + \mathbf{z}_2\right)\left(\mathbf{z}_1^2 + 14\mathbf{z}_1\mathbf{z}_2 + \mathbf{z}_2^2\right)}{\mathbf{z}_1^2\mathbf{z}_2^2} \tag{3.43}$$

For this mode, the corresponding factor in the geometric constraint equation is $4\mathbf{z}_1\mathbf{z}_2 + \mathbf{z}_1 - \mathbf{z}_2 - 4$ and appears as a factor in k. Therefore, in this kinematic mode, k is always zero. Note that this linkage is also statically balanced since $\mathbf{F}_1 = \mathbf{F}_2 = 0$.

In mode B, the linkage cannot be dynamically balanced. This can be proven formally using polynomial division and is a direct consequence of the fact that all moving bodies of the linkage are rotating in the same direction.

3.4.4 Deltoid

For the Deltoid-1 case, $l_2 = d$ and $l_1 = l_3 =: l$ with $l \neq d$. Consider mode A. This is the second case where we get solutions that are physically realizable for the dynamic balancing. Again, we introduce the parameters $\mathbf{q}_i$ and J_i and eliminate $I_i, r_i, \mathbf{p}_i$ for $i = 1, 2, 3$. The balancing conditions are:

$$\mathbf{q}_1 = \mathbf{q}_3 - lm_3, \mathbf{q}_2 = -\frac{d}{l}\mathbf{q}_3 \tag{3.44}$$

$$J_1 = J_3 - l^2 m_3, J_2 = \frac{l^2 - d^2}{l}\mathbf{q}_3 - J_3. \tag{3.45}$$

It follows that $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ must be real. The parameters fulfill again the inequality constraints

$$m_i > 0, J_i m_i - |\mathbf{q}_i|^2 > 0 \tag{3.46}$$

for $i = 1, 2, 3$; again it follows that J_1 and J_2 must be positive. From equation (3.45), we get an upper bound for m_3, which must be larger than the lower bound given by (3.46) ($i = 3$). This yields

$$(J_3 - lq_3)(J_3 + l\mathbf{q}_3) > 0. \tag{3.47}$$

This is equivalent to the statement $J_3 > |l\mathbf{q}_3|$. From (3.45), we obtain an upper bound for J_3, namely $\left|\frac{d^2-l^2}{l}\mathbf{q}_3\right|$. The lower bound must be larger than the upper bound, hence $\mathbf{q}_3 < 0$ and $d \geq \sqrt{2}l$.

Conversely, if $d \geq \sqrt{2}l$, then we can choose $\mathbf{q}_3 < 0$ arbitrarily and J_3 between $-(l\mathbf{q}_3)$ and $\frac{d^2-l^2}{l}\mathbf{q}_3$. Then we choose m_3 between the upper and lower bound for m_3 derived above. Next, (3.45) determines J_1 and J_2, which will then be positive. Then (3.44) determines $\mathbf{q}_1$ and $\mathbf{q}_2$, and finally m_1 and m_2 can be chosen so that inequality (3.46) is fulfilled.

An example of such a dynamically balanced linkage is shown in Table 3.3 (right). For these parameters, we have

$$g = -4\,\frac{(\mathbf{z}_2 + 1)\left(\mathbf{z}_2 - 4\,\mathbf{z}_1\,\mathbf{z}_2 + {\mathbf{z}_1}^2 - 4\,\mathbf{z}_1\right)}{\mathbf{z}_1\,\mathbf{z}_2} \tag{3.48}$$

$$k = 4\,\frac{\left(\mathbf{z}_2 - 4\,\mathbf{z}_1\,\mathbf{z}_2 + {\mathbf{z}_1}^2 - 4\,\mathbf{z}_1\right)\left(4\,\mathbf{z}_2\,{\mathbf{z}_1}^2 + 4\,{\mathbf{z}_1}^2 - \mathbf{z}_1\,\mathbf{z}_2 + {\mathbf{z}_2}^2\mathbf{z}_1 - 4\,{\mathbf{z}_2}^3 - 4\,{\mathbf{z}_2}^2\right)}{{\mathbf{z}_2}^2{\mathbf{z}_1}^2} \tag{3.49}$$

Therefore, for mode A (i.e.: $\mathbf{z}_2 - 4\,\mathbf{z}_1\,\mathbf{z}_2 + {\mathbf{z}_1}^2 - 4\,\mathbf{z}_1 = 0$), we obtain that $k = 0$. Moreover, $\mathbf{F}_1 = \mathbf{F}_2 = 0$ and the linkage is dynamically balanced.

For mode B, dynamic balancing is not possible for a similar reason to the parallelogram case.

3.4.5 Rhomboid

In the case of the rhomboid, all lengths are equal (i.e. $l_1 = l_2 = l_3 = d$). The three kinematic modes correspond respectively to the parallelogram mode B, the deltoid-1 mode B and the deltoid-2 mode B. Therefore, the balancing constraints can be easily derived from these cases and are summarized in Table 3.4.

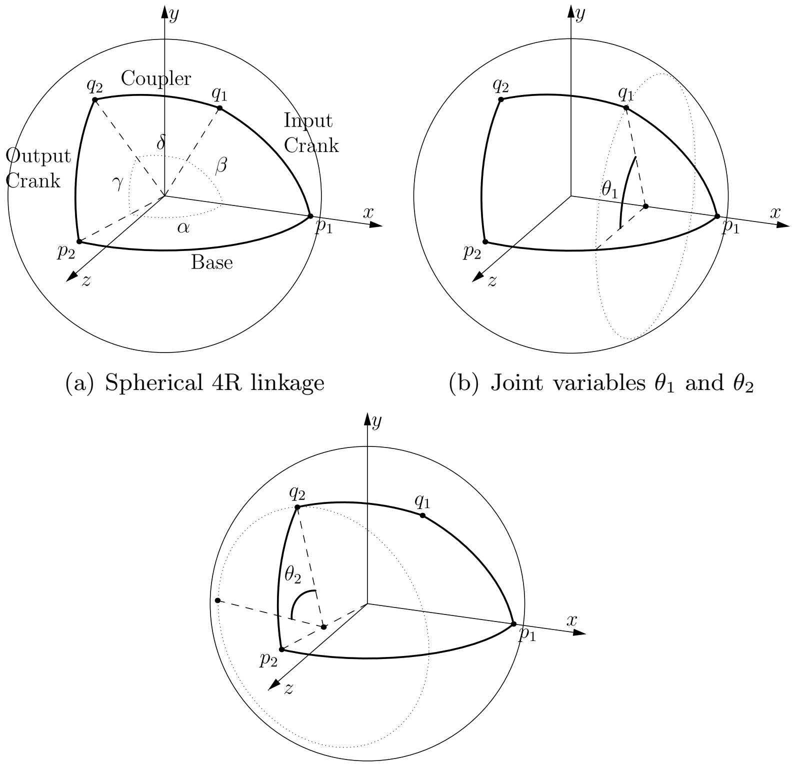
Case	Kinematic mode	Static balancing	Dynamic balancing
Irreducible		$\mathbf{F}_1 = \mathbf{F}_2 = 0$	no
Parallelogram	A	$\mathbf{F}_1 = \mathbf{F}_2 = 0$	possible iff $d \geq \sqrt{2}\, l_2$
	B	$\mathbf{F}_1 + \mathbf{F}_2 = 0$	no
Deltoid-1	A	$\mathbf{F}_1 = \mathbf{F}_2 = 0$	possible iff $d \geq \sqrt{2}\, l_3$
	B	$\mathbf{F}_1 = 0$	no
Deltoid-2	A	$\mathbf{F}_1 = \mathbf{F}_2 = 0$	possible iff $d \geq \sqrt{2}\, l_3$
	B	$\mathbf{F}_2 = 0$	no
Rhomboid	A	$\mathbf{F}_1 = 0$	no
	B	$\mathbf{F}_2 = 0$	no
	C	$\mathbf{F}_1 + \mathbf{F}_2 = 0$	no

Table 3.4: Balancing constraints for planar four-bar mechanisms.

Chapter 4

Spherical linkage

The aim of this chapter is to completely characterise statically and dynamically balanced spherical 4R linkages. The chapter is organised as follows. In section 4.1, we derive a system of algebraic equations in terms of the design parameters and the joint angles in the configuration space - modelled by complex variables $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ - and eliminate $\mathbf{z}_3$ immediately. That is, the kinematic constraints as well as the static and dynamic balancing constraints are modelled as algebraic equations in terms of the variables $\mathbf{z}_1$ and $\mathbf{z}_2$ and can be considered as parametric complex curves over $\mathbb{C}^2$. In section 4.2, based on [33, 34], we investigate for which lengths of the bars, the algebraic equations representing the kinematic constraints decompose into irreducible components called kinematic modes. It is important to look at these cases since they lead to a different set of balancing conditions. In section 4.3, we derive sufficient and necessary conditions on the design parameters for static balancing, for each kinematic mode. In section 4.4, we formally prove that it is not possible to dynamically balance such linkages, using a case by case analysis of all kinematic modes. Finally, section 4.5 describes the case where one of the links has length 0 or π (antipodal), this case being treated separately.



(a) Spherical 4R linkage

(b) Joint variables θ_1 and θ_2

(c)

Figure 4.1: Spherical 4R linkage

4.1 Model

4.1.1 Kinematics

A spherical 4R linkage is shown in Fig. 4.1(a). It consists of four bars: the base ($\overline{p_1p_2}$) which is fixed, the input crank ($\overline{p_1q_1}$) and the output crank ($\overline{p_2q_2}$) which are connected to the base, and the coupler ($\overline{q_1q_2}$). The joints are revolute joints and their axes of rotation intersect in a point. This point is considered as the origin of a sphere of radius 1, on which the joints are moving. Although the joints do not have to move on the sphere, this model is equivalent.

The lengths of the bars are measured by the angles between two successive axes of rotation. The base has length α, the input crank β, the output crank γ and the coupler δ. These parameters are called the geometric design parameters and denoted by Γ_G.

Two joints, p_1 and p_2 are fixed on the base and the two other joints, q_1 and q_2, are moving on two circles inscribed in a plane perpendicular to their axes of rotation. Therefore, the orientation of the input crank and output crank can be represented by two angles: θ_1 and θ_2 (see Fig. 4.1(b)

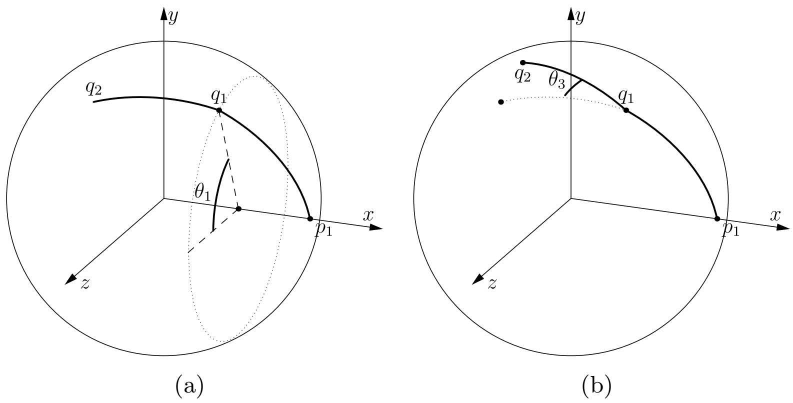


Figure 4.2: Computation of q_2'

and 4.1(c)). θ_3 represents the orientation of the coupler with respect to the input crank as shown in Fig. 4.2(b). The positions of the joints are given by:

$$p_1 = [1, 0, 0]^T \tag{4.1}$$

$$p_2 = R_y(\alpha)p_1 \tag{4.2}$$

$$q_1 = R_x(\theta_1)R_y(\beta)p_1 \tag{4.3}$$

$$q_2 = R_y(\alpha)R_x(\theta_2)R_y(\gamma)p_1 \tag{4.4}$$

$$q_2' = R_{q_1}(\theta_3)R_x(\theta_1)R_y(\delta)R_y(\beta)p_1 \tag{4.5}$$

The computation of q_2' requires some explanation. First consider the input crank and the coupler as one long body on the equator of the sphere (in the x-z plane). Then we successively rotate the point representing the end of this long body about the x axis by θ_1 (Fig. 4.2(a)) and by θ_3 about the axis q_1 (Fig. 4.2(b)). q_2 represents the position of the joint computed from the input crank side and q_2' the position computed from the output crank side, and should therefore be equal, i.e.

$$v(\theta_1, \theta_2, \theta_3) := q_2 - q_2' = [v_x, v_y, v_z]^T = 0 \tag{4.6}$$

This equation can be used to explicitely express θ_3 in terms of θ_1 and θ_2. Since v_x does not depend on $\sin(\theta_3)$ but only (and linearly) on $\cos(\theta_3)$, it can be solved in terms of $\cos(\theta_3)$. Considering

$$v_y \cos(\theta_1) + v_z \sin(\theta_1) = 0 \tag{4.7}$$

we obtain an expression depending on $\sin(\theta_3)$, but independent of $\cos(\theta_3)$. Therefore we can solve for $\sin(\theta_3)$. We obtain the following relations:

$$\begin{aligned} \cos\theta_3 &= \frac{-\cos(\alpha)\cos(\gamma) + \sin(\alpha)\cos(\theta_2)\sin(\gamma) + \cos(\delta)\cos(\beta)}{\sin(\beta)\sin(\delta)} \\ \sin\theta_3 &= \frac{\cos(\theta_1)\sin(\theta_2)\sin(\gamma) - \sin(\theta_1)\sin(\alpha)\cos(\gamma) - \sin(\theta_1)\cos(\alpha)\cos(\theta_2)\sin(\gamma)}{\sin(\delta)} \end{aligned} \tag{4.8}$$

These equations do not hold if $\sin\beta = 0$ or $\sin\delta = 0$. In this case, one of the bars is antipodal. This case will be treated separately in section 4.5. The geometric constraint (i.e. closure equation) can be written without θ_3 as:

$$g(\theta_1, \theta_2) := \mid q_2 - q_1 \mid - \cos\delta = 0 \tag{4.9}$$

since the distance between q_1 and q_2 is fixed and corresponds to the length of the coupler. Up to now, the equations have been written in terms of trigonometric expressions, both for the variables $(\theta_1, \theta_2, \theta_3)$ and the parameters $(\alpha, \beta, \gamma, \delta)$. A set of algebraic equations is desirable since algebraic tools could be used to solve the problem. To make the equations algebraic, we first replace the variables θ_1, θ_2 and θ_3 by the unit complex variables $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ using the following identities:

$$\cos(\theta_j) = \frac{\mathbf{z}_j + \overline{\mathbf{z}_j}}{2} \qquad \sin(\theta_j) = \frac{\mathbf{z}_j - \overline{\mathbf{z}_j}}{2i} \tag{4.10}$$

Similarly, we replace the parameters α, β, γ and δ by a, b, c and d respectively using the tangent half-angles substitutions, as shown in (2.6). Using these substitutions and the fact that $\overline{\mathbf{z}_i} = \mathbf{z}_i^{-1}$, the geometric constraint (4.9) can be written as a Laurent polynomial in terms of the complex variables $\mathbf{z}_1$ and $\mathbf{z}_2$:

$$g(\mathbf{z}_1, \mathbf{z}_2) = \langle q_2 - q_1, \overline{q_2 - q_1} \rangle - \frac{1 - d^2}{1 + d^2} = 0 \tag{4.11}$$

After simplification:

$$\begin{aligned} g(\mathbf{z}_1, \mathbf{z}_2) = &- a^2 bc \left(\mathbf{z}_1\mathbf{z}_2 + \mathbf{z}_1^{-1}\mathbf{z}_2^{-1}\right) + ac(b-1)(b+1)\left(\mathbf{z}_2 + \mathbf{z}_2^{-1}\right) \\ &+ ab(c-1)(c+1)\left(\mathbf{z}_1 + \mathbf{z}_1^{-1}\right) + bc\left(\mathbf{z}_1\mathbf{z}_2^{-1} + \mathbf{z}_1^{-1}\mathbf{z}_2\right) \\ &+ (-a^2 - b^2 - c^2 + d^2 - a^2b^2c^2 + a^2b^2d^2 + a^2c^2d^2 + b^2c^2d^2)/(1 + d^2) \end{aligned} \tag{4.12}$$

The time derivative of (4.12) can be written as a linear combination of the joint angular velocities $\dot{\theta_1}$ and $\dot{\theta_2}$:

$$k_1(\mathbf{z}_1, \mathbf{z}_2)\dot{\theta_1} + k_2(\mathbf{z}_1, \mathbf{z}_2)\dot{\theta_2} = 0 \tag{4.13}$$

where k_1 and k_2

$$\begin{aligned} k_1 &= - a^2 bc \left(\mathbf{z}_1\mathbf{z}_2 - \mathbf{z}_1^{-1}\mathbf{z}_2^{-1}\right) + ab(c-1)(c+1)\left(\mathbf{z}_1 - \mathbf{z}_1^{-1}\right) + bc\left(\mathbf{z}_1\mathbf{z}_2^{-1} - \mathbf{z}_1^{-1}\mathbf{z}_2\right) \\ k_2 &= - a^2 bc \left(\mathbf{z}_1\mathbf{z}_2 - \mathbf{z}_1^{-1}\mathbf{z}_2^{-1}\right) + ac(b-1)(b+1)\left(\mathbf{z}_2 - \mathbf{z}_2^{-1}\right) + bc\left(-\mathbf{z}_1\mathbf{z}_2^{-1} + \mathbf{z}_1^{-1}\mathbf{z}_2\right) \end{aligned} \tag{4.14}$$

For computing the angular momentum, we will also need an explicit expression of $\mathbf{z}_3$ and its time derivative in terms of $\mathbf{z}_1, \mathbf{z}_2, \theta_1$ and θ_2. Using Euler's formula:

$$\mathbf{z}_3 = \cos\theta_3 + i\sin\theta_3 \tag{4.15}$$

where $\sin\theta_3$ and $\cos\theta_3$ are given by (4.8), $\mathbf{z}_3$ can be written explicitely in terms of $\mathbf{z}_1$ and $\mathbf{z}_2$. Note, that (4.15) can be simplified by

$$\mathbf{z}_3 = \cos\theta_3 + i\sin\theta_3 + \frac{1}{2(1 + a^2)b(1 + c^2)d}\, g \tag{4.16}$$

which is equivalent since the last component is multiplied by g which must be zero and several terms are simplified. Taking the time derivative, we obtain a relation between the joint angular velocities of the following form:

$$\mathbf{z}_3\dot{\theta}_3 = k_5(\mathbf{z}_1, \mathbf{z}_2)\dot{\theta}_1 + k_6(\mathbf{z}_1, \mathbf{z}_2)\dot{\theta}_2 \tag{4.17}$$

Replacing (4.16) in (4.17), we can explicitely write $\dot{\theta}_3$ as a linear expression in terms of $\dot{\theta}_1$ and $\dot{\theta}_2$ with coefficients being rational functions in terms of the variables $\mathbf{z}_1$ and $\mathbf{z}_2$

$$\dot{\theta}_3 = k_7(\mathbf{z}_1, \mathbf{z}_2)\dot{\theta}_1 + k_8(\mathbf{z}_1, \mathbf{z}_2)\dot{\theta}_2 \tag{4.18}$$

4.1.2 Centre of mass

Four design parameters are necessary for specifying the mass and centre of mass position of each bar. Since the base is fixed and does not influence the dynamics of the linkage, 12 static design parameters are needed. The set of static parameters is denoted by Γ_S. For the input crank, the position of the centre of mass, denoted by $\mathbf{rS}_1$, can be written as:

$$\mathbf{rS}_1 = r_1 R_x(\theta_1 + \psi_1) R_y(\xi_1) p_1 \tag{4.19}$$

where r_1, ψ_1, ξ_1 specify the position of the centre of mass of input crank and p_1 as defined in (4.1). Similarly, the position of the centre of mass of the output crank $\mathbf{rS}_2$ is given by:

$$\mathbf{rS}_2 = r_2 R_y(\alpha) R_x(\theta_2 + \psi_2) R_y(\xi_2) p_1 \tag{4.20}$$

and for the coupler ($\mathbf{rS}_3$)

$$\mathbf{rS}_3 = c_1 q_1 + c_2 q_2 + c_3(q_1 \times q_2) \tag{4.21}$$

where the real parameters c_1, c_2 and c_3 represents the position of the centre of mass of the coupler. Note that q_1, q_2 and $q_1 \times q_2$ are in general not orthogonal but linearly independent, except if $\sin\delta = 0$. But this case has been excluded previously. Thus, the centre of mass of the linkage $\mathbf{rS}$ is:

$$\mathbf{rS} = \frac{1}{M}(m_1\mathbf{rS}_1 + m_2\mathbf{rS}_2 + m_3\mathbf{rS}_3) = [\mathbf{rS}_x, \mathbf{rS}_y, \mathbf{rS}_z]^T \tag{4.22}$$

where $M = m_1 + m_2 + m_3$ is the total mass of the linkage.

4.1.3 Angular momentum

The angular velocity of the input crank, output crank and coupler in terms of the fixed coordinate frame can be written as:

$$\begin{aligned} \omega_1 &= p_1\dot{\theta}_1 \\ \omega_2 &= p_2\dot{\theta}_2 \\ \omega_3 &= q_1\dot{\theta}_3 + p_1\dot{\theta}_1 \end{aligned} \tag{4.23}$$

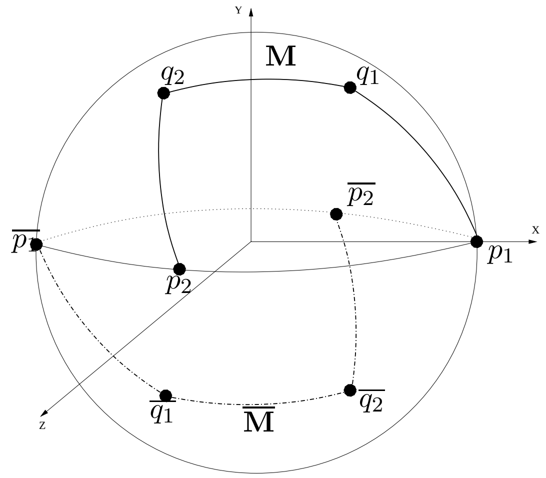


Figure 4.3: Mirror linkage of the spherical 4R linkage

where $\dot{\theta}_3$ is given by (4.18). Let L_j be the angular momentum of bar j in terms of the fixed coordinate frame. From [55], it can be written in the following form:

$$L_j = I_j\omega_j + \mathbf{rS}_j \times m_j\dot{\mathbf{rS}}_j \qquad (4.24)$$

where ω_j is the angular velocity and I_j is the inertia matrix of the bar j in the reference frame. The total angular momentum of the spherical linkage is given by:

$$L = \sum_{j=1}^{3} L_j = \sum_{j=1}^{3} I_j\omega_j + \sum_{j=1}^{3} \mathbf{rS}_j \times m_j\dot{\mathbf{rS}}_j \qquad (4.25)$$

Note that (4.25) depends on all design parameters including the static parameters.

Consider the linkage $\overline{\mathbf{M}}$ obtained by the reflection of the spherical linkage at the origin (see Fig. 4.3). Denote $\widetilde{\mathbf{M}}$ the combined linkage formed by the original spherical linkage $\mathbf{M}$ and its mirror image $\overline{\mathbf{M}}$. Each bar of $\widetilde{\mathbf{M}}$, for example the input crank, is made of the input crank of $\mathbf{M}$ glued with its mirror image. Let $\widetilde{\mathbf{rS}_j}$ be the centre of mass position of bar j in $\widetilde{\mathbf{M}}$. Then

$$\widetilde{\mathbf{rS}_j} = 0 \qquad j = 1, 2, 3 \qquad (4.26)$$

and the angular momentum of $\widetilde{\mathbf{M}}$, denoted $\widetilde{L}$, is:

$$\widetilde{L} = \sum_{j=1}^{3} \widetilde{L_j} = \sum_{j=1}^{3} \widetilde{I}_j\omega_j \qquad (4.27)$$

Moreover, it is clear that since L is equal to the angular momentum of $\overline{\mathbf{M}}$, denoted $\overline{L}$, then

$$\widetilde{L} = L + \overline{L} = 2L \qquad (4.28)$$

Clearly, $\widetilde{L} = 0$ if and only if $L = 0$. Notice that the angular momentum $\widetilde{L}$ is independent of the static design parameters.

Let I_1 be the inertia matrix of the input crank, I_2 the inertia matrix of the output crank and I_3 the inertia matrix of the coupler, in their respective centre of mass frame, i.e.

$$I_j = \begin{bmatrix} I_{jxx} & I_{jxy} & I_{jxz} \\ I_{jxy} & I_{jyy} & I_{jyz} \\ I_{jxz} & I_{jyz} & I_{jzz} \end{bmatrix} \tag{4.29}$$

The contributions of the input and the output crank to the angular momentum written in the fixed coordinate frame are given by:

$$\begin{aligned} \widetilde{L_1} &= R_x(\theta_1)\, I_1\, R_x(\theta_1)^T \omega_1 \\ &= R_x(\theta_1)\, I_1\, R_x(\theta_1)^T p_1 \dot{\theta}_1 \end{aligned} \tag{4.30}$$

$$\begin{aligned} \widetilde{L_2} &= R_y(\alpha)\, R_x(\theta_2)\, I_2\, R_x(\theta_2)^T R_y(\alpha)^T\, \omega_2 \\ &= R_y(\alpha)\, R_x(\theta_2)\, I_2\, R_x(\theta_2)^T R_y(\alpha)^T\, p_2\, \dot{\theta}_2 \\ &= R_y(\alpha)\, R_x(\theta_2)\, I_2\, R_x(\theta_2)^T R_y(\alpha)^T\, R_y(\alpha)\, p_1\, \dot{\theta}_2 \\ &= R_y(\alpha)\, R_x(\theta_2)\, I_2\, R_x(\theta_2)^T p_1\, \dot{\theta}_2 \end{aligned} \tag{4.31}$$

with ω_1, ω_2 are given by (4.23). Using this representation, only 3 inertial parameters of the input crank and output crank have an influence on the angular momentum: I_{jxx}, I_{jxy} and I_{jxz} $(j = 1, 2)$. For the coupler, the angular momentum of the coupler in the fixed coordinate frame is given by

$$\widetilde{L_3} = R_{q_1}(\theta_3)\, R_x(\theta_1)\, I_3\, R_x(\theta_1)^T\, R_{q_1}(\theta_3)^T\, \omega_3 \tag{4.32}$$

Using (4.16), L_3 can be written independently of $\mathbf{z}_3$. The total angular momentum can be written in the following form:

$$\begin{aligned} \widetilde{L} &= \widetilde{L_1} + \widetilde{L_2} + \widetilde{L_3} \\ &= [L_x, L_y, L_z]^T \\ &= \begin{bmatrix} k_{3x}(\mathbf{z}_1, \mathbf{z}_2) \\ k_{3y}(\mathbf{z}_1, \mathbf{z}_2) \\ k_{3z}(\mathbf{z}_1, \mathbf{z}_2) \end{bmatrix} \dot{\theta}_1 + \begin{bmatrix} k_{4x}(\mathbf{z}_1, \mathbf{z}_2) \\ k_{4y}(\mathbf{z}_1, \mathbf{z}_2) \\ k_{4z}(\mathbf{z}_1, \mathbf{z}_2) \end{bmatrix} \dot{\theta}_2 \end{aligned} \tag{4.33}$$

where $k_{3x}, k_{3y}, k_{3z}, k_{4x}, k_{4y}, k_{4z}$ are Laurent polynomials in terms of $\mathbf{z}_1$ and $\mathbf{z}_2$ and depending on the geometric parameters Γ_G and the dynamic parameters Γ_D (see Table 4.1).

4.1.4 Problem formulation

In our settings, the linkage is statically balanced if the centre of mass of the mechanism remains stationary for infinitely many configurations (i.e.

Γ_G	$\{a, b, c, d\}$
Γ_S	$\{m_1, r_1, \psi_1, \xi_1, m_2, r_2, \psi_2, \xi_2, m_3, c_1, c_2, c_3\}$
Γ_D	$\{I_{1xx}, I_{1xy}, I_{1xz}, I_{2xx}, I_{2xy}, I_{2xz}, I_{3xx}, I_{3xy}, I_{3xz}, I_{3yy}, I_{3yz}, I_{3zz}\}$

Table 4.1: List of design parameters: Geometric parameters (Γ_G), static parameters (Γ_S) and dynamic parameters (Γ_D)

infinitely many choices of the joint angles). Therefore we want

$$F_x := \mathbf{rS}_x - d_1 = 0 \qquad F_y := \mathbf{rS}_y - d_2 = 0 \qquad F_z := \mathbf{rS}_z - d_3 = 0 \quad (4.34)$$

where d_1, d_2, d_3 are constants. A mechanism is said to be dynamically balanced [82] if is it statically balanced and the total angular momentum is zero at all times, i.e.

$$\widetilde{L_x} = \widetilde{L_y} = \widetilde{L_z} = 0 \quad (4.35)$$

Therefore, (4.11), (4.13), (4.34), (4.35) have to be satisfied. Among these four set of equations, only (4.13) and (4.35) depend (linearly) on the joint angular velocities and they can be rewritten (in the x coordinate, for example) in the following form:

$$\begin{bmatrix} k_1 & k_2 \\ k_{3x} & k_{4x} \end{bmatrix} \begin{bmatrix} \dot{\theta_1} \\ \dot{\theta_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (4.36)$$

If the rank of the matrix $A_x = \begin{bmatrix} k_1 & k_2 \\ k_{3x} & k_{4x} \end{bmatrix}$ is 2, since the system is homogeneous, then the only solution is $\dot{\theta_1} = \dot{\theta_2} = 0$. In other words, the linkage is not moving. Therefore, $\det(A_x)$ must be 0. The same holds for the y and z coordinates and we obtain the following set of equations:

$$\begin{aligned} k_x := \det(A_x) &= k_1 k_{4x} - k_2 k_{3x} = 0 \\ k_y := \det(A_y) &= k_1 k_{4y} - k_2 k_{3y} = 0 \\ k_z := \det(A_z) &= k_1 k_{4z} - k_2 k_{3z} = 0 \end{aligned} \quad (4.37)$$

Using this notation, the static and dynamic balancing problem can be formulated as follow:

PROBLEM FORMULATION

Let $(S^1)^2 = \{(\mathbf{z}_1, \mathbf{z}_2) \in \mathbb{C} \mid |\mathbf{z}_1| = 1 \text{ and } |\mathbf{z}_2| = 1\}$. Let g, f_x, f_y, f_z, k_x, k_y, k_z as defined by equations (4.11), (4.34) and (4.37).

Force balancing:

Find all possible geometric and static parameters such that:

$$\underset{\infty(\mathbf{z}_1,\mathbf{z}_2)\in(S^1)^2}{\exists}\{g(\mathbf{z}_1,\mathbf{z}_2) = 0 \Rightarrow (f_x(\mathbf{z}_1,\mathbf{z}_2) = 0 \wedge f_y(\mathbf{z}_1,\mathbf{z}_2) = 0 \wedge f_z(\mathbf{z}_1,\mathbf{z}_2) = 0)\} \tag{4.38}$$

Moment balancing:

For a statically balanced linkage, find all possible geometric and dynamic parameters such that:

$$\underset{\infty(\mathbf{z}_1,\mathbf{z}_2)\in(S^1)^2}{\exists}\{g(\mathbf{z}_1,\mathbf{z}_2) = 0 \Rightarrow (k_x(\mathbf{z}_1,\mathbf{z}_2) = 0 \wedge k_y(\mathbf{z}_1,\mathbf{z}_2) = 0 \wedge k_z(\mathbf{z}_1,\mathbf{z}_2) = 0)\} \tag{4.39}$$

Using Theorem 3, we can reformulate the balancing problem as a factorisation problem of Laurent polynomials. Theorem 3 required g to be an irreducible polynomial. We will see in the next section, that for some choice of the geometric parameters, the geometric constraint g is not irreducible.

Name	Design parameters	$\alpha+\beta+\gamma+\delta=2\pi$?	Gibson-Selig
B1	$\alpha=\delta\neq\beta=\gamma$	no	II2 Parallelogram
B2	$\alpha=\beta\neq\gamma=\delta$	no	II2 Kite
B3	$\alpha=\gamma\neq\beta=\delta$	no	II2 Kite
B4	$\alpha=\beta=\gamma=\delta$	no	II3
B5	$\alpha+\delta=\gamma+\beta=\pi$	yes	III2
B6	$\alpha+\beta=\gamma+\delta=\pi$	yes	III2
B7	$\alpha+\gamma=\delta+\beta=\pi$	yes	III2
B8	$\beta=\alpha\neq\delta=\gamma$	yes	III3
B9	$\gamma=\alpha\neq\delta=\beta$	yes	III3
B10	$\delta=\alpha\neq\gamma=\beta$	yes	III3
B11	$\alpha=\beta=\gamma=\delta=\frac{\pi}{2}$	yes	III4

Table 4.2: Classification of reducible cases for non-antipodal linkages (Gibson-Selig)

4.2 Kinematic modes

For a generic choice of the geometric parameters $\alpha, \beta, \gamma, \delta$, $g(\mathbf{z}_1, \mathbf{z}_2)$ is an irreducible Laurent polynomial. However, we must find all geometric parameters such that $g(\mathbf{z}_1, \mathbf{z}_2)$ is reducible, and investigate the balancing conditions for all these cases. If g is reducible, it decomposes into the product of k Laurent polynomials (i.e. $g = g_1...g_k$) and each of these components corresponds to a physical configuration of the linkage and is called kinematic mode.

In [33, 34], Gibson and Selig gave a complete classification of all cases when the geometric constraint is reducible. In their first paper [33], they embed the configuration space in a complex projective space by homogenization of the equations and complexification of the variables. The set of all solutions of this algebraic system forms the linkage variety. Then, they find all conditions on the design parameters for the curve to have finite singularity, that is singularities which are not at infinity. In their second paper [34], they investigate the nature of these singularities and they show how their disposition determines the irreducible components into which the linkage variety decomposes. The complete list of reducible cases is given in Table 4.2.

However, for some cases of linkages mentioned in Table 4.2, it is possible to prove that they can be mapped to an equivalent linkage corresponding to another reducible case. This will simplify the classification and reduce the number of reducible cases to investigate. Assume for the moment, that none of the lengths are $\frac{\pi}{2}$, i.e.: the parameters a, b, c, d are different than ± 1. The following mappings leave the geometric constraint g unchanged, i.e. they are different representations of the same linkage:

$$\begin{array}{ll} \text{R1}: & \left[a \to -a,\ \mathbf{z}_1 \to -\frac{1}{\mathbf{z}_1}, \mathbf{z}_2 \to -\frac{1}{\mathbf{z}_2}\right] \\ \text{R2}: & [b \to -b,\ \mathbf{z}_1 \to -\mathbf{z}_1] \\ \text{R3}: & [c \to -c,\ \mathbf{z}_2 \to -\mathbf{z}_2] \\ \text{R4}: & [d \to -d] \end{array} \tag{4.40}$$

The parameters which are not explicitely mentioned in the mappings remain unchanged under the transformation. For example, in R1, c and d are not changed. The mappings given by R1-R4 can be used to map a linkage with any number of negative parameters a, b, c or d to an equivalent linkage with strictly positive parameters. The following mappings also leave the geometric constraint unchanged:

$$\begin{array}{ll} \text{S1}: & \left[a \to -\frac{1}{a},\ b \to \frac{1}{b},\ \mathbf{z}_1 \to -\mathbf{z}_1,\ \mathbf{z}_2 \to \frac{1}{\mathbf{z}_2}\right] \\ \text{S2}: & \left[a \to -\frac{1}{a},\ c \to \frac{1}{c},\ \mathbf{z}_1 \to \frac{1}{\mathbf{z}_1},\ \mathbf{z}_2 \to -\mathbf{z}_2\right] \\ \text{S3}: & \left[a \to -\frac{1}{a},\ d \to \frac{1}{d},\ \mathbf{z}_1 \to \frac{1}{\mathbf{z}_1}\right] \\ \text{S4}: & \left[a \to -\frac{1}{a},\ d \to \frac{1}{d},\ \mathbf{z}_2 \to \frac{1}{\mathbf{z}_2}\right] \\ \text{S5}: & \left[b \to \frac{1}{b},\ c \to \frac{1}{c},\ \mathbf{z}_1 \to -\frac{1}{\mathbf{z}_1},\ \mathbf{z}_2 \to -\frac{1}{\mathbf{z}_2}\right] \\ \text{S6}: & \left[b \to -\frac{1}{b},\ d \to \frac{1}{d}\right] \\ \text{S7}: & \left[c \to -\frac{1}{c},\ d \to \frac{1}{d}\right] \end{array} \tag{4.41}$$

Using rules S1-S7 and R1-R4 and given any two parameters from the set $\{a, b, c, d\}$ in the interval $[1, \infty)$, they can be mapped to the interval $(0, 1]$. In other words, it is always possible to find an equivalent linkage with at most one parameter outside of the interval $(0, 1]$, that is, with a length between 0 and $\frac{\pi}{2}$. Moreover, we can choose which length will be outside of the interval $(0, 1]$ as shown in the following example.

Example Let $a = \frac{1}{3}, b = \frac{2}{5}, c = 4, d = \frac{1}{2}$. The numerator of the geometric constraint is:

$$2739\mathbf{z}_1\mathbf{z}_1 + 252\mathbf{z}_1 + 450\mathbf{z}_2 + 450\mathbf{z}_2\mathbf{z}_1^2 - 360\mathbf{z}_2^2 - 360\mathbf{z}_1^2 + 40\mathbf{z}_1^2\mathbf{z}_2^2 + 40 + 252\mathbf{z}_1\mathbf{z}_2^2 \tag{4.42}$$

Only the value of c is outside the interval $(0, 1]$. Assume we want to find the equivalent linkage such that only the value of d is outside the interval $(0, 1]$. Applying rule S2, and then S3 yields the following mapping:

$$\text{S3} \circ \text{S2} : [\mathbf{z}_1, \mathbf{z}_2, a, b, c, d] \to \left[\mathbf{z}_1, -\mathbf{z}_2, a, b, \frac{1}{c}, \frac{1}{d}\right] \tag{4.43}$$

Applying this mapping on the geometric constraint, and replacing the values of the parameters, the numerator of the geometric constraint is the same as (4.42). The denominator is different up to a factor, but does not affect the solutions of the equations.

Name	Design parameters	Name
B1	$\alpha = \delta \neq \beta = \gamma$	Parallelogram
B2	$\alpha = \beta \neq \gamma = \delta$	Kite
B3	$\alpha = \gamma \neq \beta = \delta$	Kite
B4	$\alpha = \beta = \gamma = \delta$	Rhomboid
$B5 - 1$	$\alpha = \delta = \pi/2, \gamma + \beta = \pi$	Equatorial parallelogram
$B5 - 2$	$\alpha + \delta = \pi, \gamma = \beta = \pi/2$	Equatorial parallelogram
$B6 - 1$	$\alpha = \beta = \pi/2, \gamma + \delta = \pi$	Equatorial Kite
$B6 - 2$	$\alpha + \beta = \pi, \gamma = \delta = \pi/2$	Equatorial Kite
$B7 - 1$	$\alpha = \gamma = \pi/2, \delta + \beta = \pi$	Equatorial Kite
$B7 - 2$	$\alpha + \gamma = \pi, \delta = \beta = \pi/2$	Equatorial Kite
B11	$\alpha = \beta = \gamma = \delta = \frac{\pi}{2}$	Equatorial Rhomboid

Table 4.3: Classification of reducible cases for non-antipodal linkages (simplified version)

Using these results, we can reduce the number of non-antipodal reducible cases. For the case B5, we have $\alpha + \delta = \gamma + \beta = \pi$, and therefore we can write $\alpha = \pi - \delta$ and $\beta = \pi - \gamma$. Using (2.7), we obtain that $a = \frac{1}{d}$ and $b = \frac{1}{c}$. Using rule S7, we can find an equivalent new linkage using the mapping:

$$[a, b, c, d, \mathbf{z}_1, \mathbf{z}_2] \to [a, b, \frac{1}{c}, \frac{1}{d}, \mathbf{z}_1, \mathbf{z}_2] =: [a', b', c', d', \mathbf{z}_1', \mathbf{z}_2'] \tag{4.44}$$

Note, that for this linkage

$$\begin{aligned} a' &= a = \frac{1}{d} = d' \\ b' &= b = \frac{1}{c} = c' \end{aligned} \tag{4.45}$$

Therefore we are in the parallelogram case B1 since $a' = d'$ and $b' = c'$. Similarly, we can show that case B6 corresponds to case B2, B7 to B3, and cases B8, B9, B10 to B4. However, it is still required to check case B11 and cases B5, B6, B7 when exactly two parameters are equal to 1. A simplified list of all cases is given in Table 4.3 and the corresponding decompositions of the geometric constraint into irreducible components (i.e. the kinematic modes) are given in Table 4.4. In conclusion, there exists 6 different reducible linkages to study: the parallelogram, the kite, the rhomboid and their equatorial versions.

Case	Mode	Equation
B1	A	$(a\mathbf{z}_1\mathbf{z}_2 + b\mathbf{z}_1 - b\mathbf{z}_2 - a)$
	B	$(ab\mathbf{z}_1\mathbf{z}_2 - \mathbf{z}_1 + \mathbf{z}_2 - ab)$
B2	A	$\left(a^2c\mathbf{z}_1\mathbf{z}_2^2 + a(c^2-1)\mathbf{z}_1\mathbf{z}_2 - c\mathbf{z}_1 + a(1-c^2)\mathbf{z}_2 + c\mathbf{z}_2^2 - a^2c\right)$
	B	$(-1 + \mathbf{z}_1)$
B3	A	$\left(a^2d\mathbf{z}_1^2\mathbf{z}_2 - b\mathbf{z}_1^2 + a(1-b^2)\mathbf{z}_1\mathbf{z}_2 + a(1-b^2)\mathbf{z}_1 - b\mathbf{z}_2 + a^2b\right)$
	B	$(\mathbf{z}_2 + 1)$
B4	A	$\left(a^2\mathbf{z}_1\mathbf{z}_2 - \mathbf{z}_1 + \mathbf{z}_2 - a^2\right)$
	B	$(\mathbf{z}_1 - 1)$
	C	$(\mathbf{z}_2 + 1)$
$\mathrm{B5}-1$	A	$(-\mathbf{z}_1\mathbf{z}_2 + b\mathbf{z}_1 + b\mathbf{z}_2 - 1)$
	B	$(b\mathbf{z}_1\mathbf{z}_2 + \mathbf{z}_1 + \mathbf{z}_2 + b)$
$\mathrm{B5}-2$	A	$(a\mathbf{z}_1\mathbf{z}_2 - \mathbf{z}_1 - \mathbf{z}_2 + a)$
	B	$(a\mathbf{z}_1\mathbf{z}_2 + \mathbf{z}_1 + \mathbf{z}_2 + a)$
$\mathrm{B6}-1$	A	$(c^2\mathbf{z}_1\mathbf{z}_2 + c^2\mathbf{z}_2 - c\mathbf{z}_1 + c\mathbf{z}_1\mathbf{z}_2^2 - c\mathbf{z}_2^2 - \mathbf{z}_2\mathbf{z}_1 - \mathbf{z}_2 + c)$
	B	$(\mathbf{z}_1 + 1)$
$\mathrm{B6}-2$	A	$(-\mathbf{z}_1 + a^2\mathbf{z}_1\mathbf{z}_2^2 - \mathbf{z}_2^2 + a^2)$
	B	$(\mathbf{z}_1 + 1)$
$\mathrm{B7}-1$	A	$(-\mathbf{z}_1b^2 + \mathbf{z}_2\mathbf{z}_1b^2 - b\mathbf{z}_2\mathbf{z}_1^2 + b\mathbf{z}_2 - b\mathbf{z}_1^2 + b - \mathbf{z}_2\mathbf{z}_1 + \mathbf{z}_1)$
	B	$(\mathbf{z}_2 - 1)$
$\mathrm{B7}-2$	A	$(a^2\mathbf{z}_2\mathbf{z}_1^2 - \mathbf{z}_2 + \mathbf{z}_1^2 - a^2)$
	B	$(\mathbf{z}_2 - 1)$
B11	A	$(\mathbf{z}_1 - 1)$
	B	$(\mathbf{z}_1 + 1)$
	C	$(\mathbf{z}_2 - 1)$
	D	$(\mathbf{z}_2 + 1)$

Table 4.4: Factorisation of the geometric constraint in the non-antipodal cases (simplified version)

4.3 Static balancing

For the irreducible case and for all kinematic modes of the reducible cases (Table 4.4), we use the toric polynomial division for finding sufficient and necessary conditions on the design parameters for the centre of mass to be fixed in x, y and z coordinates.

We show the calculations for the case when the geometric constraint g given by (4.12) is irreducible. g can be written as

$$g = \sum_{j=-1}^{1} \sum_{k=-1}^{1} g_{jk} \mathbf{z}_1^j \mathbf{z}_2^k \tag{4.46}$$

where the coefficients g_{ij} are rational functions in terms of the geometric parameters Γ_G. Similarly, from (4.34), the x component of the centre of mass position can be written as

$$\mathbf{rS}_x = \sum_{j=-1}^{1} \sum_{k=-1}^{1} f_{jk} \mathbf{z}_1^j \mathbf{z}_2^k \tag{4.47}$$

where the coefficients f_{ij} are rational functions in terms of the geometric parameters and the static parameters Γ_S. According to Theorem 3, we must find a polynomial k such that

$$\mathbf{rS}_x = gk + d_1 \tag{4.48}$$

That is, since $g(\mathbf{z}_1, \mathbf{z}_2) = 0$ for valid unit complex values of $\mathbf{z}_1, \mathbf{z}_2$, $\mathbf{rS}_x$ should be a constant. Using the polynomial division/remainder algorithm, we can divide $\mathbf{rS}_x$ by g. Note that g and $\mathbf{rS}_x$ have the same support (same set of monomials) and therefore, k can only be a constant. For example, the coefficient of the monomial $\mathbf{z}_1\mathbf{z}_2$ in g and $\mathbf{rS}_x$ are g_{11} and f_{11} repectively. Let Q be the quotient and R be the remainder. We can write:

$$\mathbf{rS}_x = Qg + R = \left(\frac{f_{11}}{g_{11}}\right) g + \left(\mathbf{rS}_x - \frac{f_{11}}{g_{11}} g\right) \tag{4.49}$$

where $g_{11} = -2a^2bc(1+d^2)$ and is different than zero in the non-antipodal case. The remainder $R = \mathbf{rS}_x - \frac{f_{11}}{g_{11}} g$ should be a constant and therefore all coefficients of the monomials of R must be zero. In other words, the coefficients of $\mathbf{z}_1^{-1}\mathbf{z}_2, \mathbf{z}_1^{-1}, \mathbf{z}_1^{-1}\mathbf{z}_2^{-1}, \mathbf{z}_2^{-1}, \mathbf{z}_2, \mathbf{z}_1\mathbf{z}_2^{-1}, \mathbf{z}_1$ must all be zeros. This gives a set of 7 constraints in terms of the design parameters. Repeating the same procedure for the y and z coordinate of the centre of mass yields another 14 constraints. Among these constraint, one is of the form

$$2im_3bcc_3 = 0 \tag{4.50}$$

Cases	Conditions to be satisfied
Purely rotational motion with $\mathbf{z}_1 = \pm 1$	$\{c_3 = 0\} \cup \mathbf{F}_2$
Purely rotational motion with $\mathbf{z}_2 = \pm 1$	$\{c_3 = 0\} \cup \mathbf{F}_1$
Other kinematic modes	$\{c_3 = 0\} \cup \mathbf{F}_1 \cup \mathbf{F}_2$

Table 4.5: Necessary and sufficient conditions for static balancing.

Since m_3, b, c are non-zero values, c_3 (see (4.21)) must be 0. The remaining constraints are easy to solve and the following two sets of conditions, denoted $\mathbf{F}_1$ and $\mathbf{F}_2$, must be fulfilled:

$$\mathbf{F}_1 = \left\{\mathbf{p}_1 = \pm 1, m_1 r_1 \sin \xi_1 \mathbf{p}_1 (b^2 + 1) + 2bc_1 m_3 = 0\right\} \tag{4.51}$$

$$\mathbf{F}_2 = \left\{\mathbf{p}_2 = \pm 1, m_2 r_2 \sin \xi_2 \mathbf{p}_2 (c^2 + 1) + 2cc_2 m_3 = 0\right\} \tag{4.52}$$

where $\mathbf{p}_1$ and $\mathbf{p}_2$ are complex valued parameters corresponding to ψ_1 and ψ_2 using equations (4.10), i.e.

$$\cos(\psi_j) = \frac{\mathbf{p}_j + \overline{\mathbf{p}_j}}{2} \qquad \sin(\psi_j) = \frac{\mathbf{p}_j - \overline{\mathbf{p}_j}}{2i} \tag{4.53}$$

Using the same approach, the balancing conditions for kinematic modes of all reducible cases (Table 4.4) can be derived. They can be classified in 3 categories:

1. Purely rotational motion with $\mathbf{z}_1 = \pm 1$:
   B3(B), B4(C), B7(B), B8(C), B9(C), B10(C), B11(C), B11(D).
2. Purely rotational motion with $\mathbf{z}_2 = \pm 1$:
   B2(B), B4(B), B6(B), B8(B), B9(B), B10(B), B11(A), B11(B).
3. Other

For each of these category, the necessary and sufficient conditions for static balancing are shown in Table 4.5.

4.4 Dynamic balancing

In this section, we prove that it is not possible to dynamically balance spherical 4R linkages, using case by case distinction. For the irreducible case and all the reducible cases summarized in Table 4.4, we have to show that (4.39) cannot be fulfilled. Although, the same approach as in Section 4.3 is used to derive the constraints between the design parameters, we need a method to prove that there is no solutions. The following method[1] is applied for all kinematic modes g'.

1. **Elimination of the variables**: Using polynomial division, divide k_x by g'. The remainder of the polynomial division must be 0 for g' to be a component of k_x. From this condition on the remainder, we obtain a set of equations in terms of the geometric design parameters Γ_g and the dynamic parameters Γ_d. That is, these equations are independent of the joint variables $\mathbf{z}_1$ and $\mathbf{z}_2$ which have been eliminated. Using the same procedure for the y and z component (replace k_x by k_y and k_z), two additional sets of constraints are obtained. All these m constraints, denoted by $f_1(\Gamma_g, \Gamma_d), ..., f_m(\Gamma_g, \Gamma_d)$ should be fulfilled for the linkage to be dynamically balanced in kinematic mode g'.

2. **Parameters reduction**: The constraints obtained in step (1) are linear equations in terms of the dynamic parameters Γ_d and non-linear in terms of the geometric parameters Γ_g. Therefore, we can rewrite the equations in the following matrix form:

$$Ax = \begin{bmatrix} \frac{\delta f_1}{\delta I_{1xx}} & \frac{\delta f_1}{\delta I_{1xy}} & \cdots & \frac{\delta f_1}{\delta I_{3zz}} \\ \frac{\delta f_2}{\delta I_{1xx}} & \frac{\delta f_2}{\delta I_{1xy}} & \cdots & \frac{\delta f_2}{\delta I_{3zz}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\delta f_m}{\delta I_{1xx}} & \frac{\delta f_m}{\delta I_{1xy}} & \cdots & \frac{\delta f_m}{\delta I_{3zz}} \end{bmatrix} \begin{bmatrix} I_{1xx} \\ I_{1xy} \\ \cdots \\ I_{3zz} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix} \tag{4.54}$$

where x is a vector with the 12 dynamic parameters (element of Γ_d), A is a $m \times 12$ matrix. In all cases, the number of constraints (m) is larger than the number of dynamic parameters (12). If A is full rank, $x = 0$ and all dynamic parameters must be equal to 0, which is physically impossible. In this case, we have proven that there are no solutions (go to step 4). Otherwise, the system can be solved for at least $rank(A)$ parameters (the dependent parameters), where the other parameters are called free parameters. The choice of the dependent

[1]The method has been implemented in Maple, except step 3 where Mathematica has been used.

and free parameters is very important for the efficiency of the following steps.

3. **Positivity constraints**: From step 2, we obtain an infinite number of solutions. That is, by choosing any values for the free parameters and the geometric parameters, we obtain values for the dependent parameters. However, many of these solutions are not valid since the design parameters are subject to several inequalities. For example, the mass of each bar is positive and the inertia matrix should fulfilled the properties described in Theorem 5. In some cases, with a smart choice of the free parameters, it is possible to show that such constraints cannot be fulfilled. Alternatively, one can use quantifier elimination tools like cylindrical algebraic decomposition tools [20], such as QEPCAD[18], Redlog[24] or Mathematica[81] to prove that no solution exists, or to find an instance of a solution.

4. **Deficient rank checking**: In step 2, the rank of the matrix A is computed symbolically for generic parameters. However, for some special choice of the parameters, the rank might drop. This happens when the determinant of all $\text{rank}(A) \times \text{rank}(A)$ submatrices A' are 0. In practice however, one does not have to look at all such submatrices, but only at a few to get a list of candidates. Then the rank of the matrix A can be checked.

Example Consider the case B1-A where $\alpha = \delta \neq \beta = \gamma$ and $\alpha+\beta+\gamma+\delta \neq 2\pi$. In other words, $d = a$ and $c = b$ which can be replaced in g, k_x, k_y and k_z from (3.4) and (3.19).

1. Applying the toric polynomial division algorithm to divide k_x, k_y and k_z by g, a set of 28 equations ($m = 28$) is obtained. They are linear equations in the 12 dynamic parameters and non-linear in terms of the geometric parameters a, b.

2. A is of dimension 28×12 and depends on the geometric parameters a, b. The rank of the matrix for generic parameters is 9. We can solve this linear system in terms of the following 9 parameters:

$$\{I_{1xx}, I_{1xy}, I_{1xz}, I_{2xx}, I_{2xy}, I_{2xz}, I_{3xy}, I_{3xz}, I_{3yz}\} \tag{4.55}$$

   The solution will express these parameters in terms of a, b and of the free dynamic parameters I_{3xx}, I_{3yy}, I_{3zz}. In particular, we obtain

$$I_{1xy} = I_{2xy} = I_{3xy} = I_{3yz} = 0 \tag{4.56}$$

3. Since these free parameters are principal moments of inertia, they should all be positive. Therefore, we are looking for positive values of a, b, I_{3xx}, I_{3yy}, I_{3zz} such that

- $I_{1xx} > 0$
- $I_{2xx} > 0$
- $B = \text{Tr}(I_3)E_3 - 2I_3$ is positive definite

Using quantifier elimination tools, it is possible to prove that there are no such solutions. However, for some values of the parameters, we "almost" get a solution and this can be used to validate the method. For example, let $a = \frac{1}{5}$, $b = \frac{25}{128}$, $I_{3xx} = 1$, $I_{3yy} = \frac{7218}{7219}$, $I_{3zz} = \frac{1}{32}$. From these values of the free parameters, we can compute the values of the dependent parameters using the equations obtained in Step (2). In particular, we obtain

$$I_{1xx} = 0.900, I_{2xx} = 0.158, I_3 = \begin{bmatrix} 1 & 0 & 1.429 \\ 0 & 1.000 & 0 \\ 1.429 & 0 & 0.031 \end{bmatrix} \tag{4.57}$$

which fulfill the first 2 conditions (i.e. $I_{1xx} > 0, I_{2xx} > 0$) but not the third condition, since $\det(B) = -0.25$ and therefore, I_3 is not a valid inertia matrix. Replacing these values in the original equations of the angular momentum, one can prove that the solutions are valid, and that the angular momentum of the linkage vanishes.

4. Taking the determinant of a 9×9 submatrix A' of A of rank 9, we obtain an expression of the following form:

$$\begin{aligned} \text{A'} =& ab(a+b)(a-b)(a+1)(a-1)(a+i)(a-i)(b-i)(b+i) \\ & \left[i(-2b^2 + ab^3 - ab) + (ab^2 - a + 2a^2 b)\right] \\ & \left[i(2b^2 - ab^3 + ab) + (ab^2 - a + 2a^2 b)\right] \end{aligned} \tag{4.58}$$

Many of these factors cannot be zero by the assumptions on the parameters (i.e. a and b are positive real numbers and $a \neq b$). However, for $a = 1$ corresponding to $\alpha = \pi/2$, we obtain that the determinant vanishes. Replacing $a = 1$ in A, we get that the rank is still 9, so the rank does not drop. For the complex component $i(-2b^2 + ab^3 - ab) + (ab^2 - a + 2a^2 b)$ both the real and the imaginary component should be 0. This happens only if $a = b = 0$ or $a = -i, b = i$. This is not possible. Therefore, there are no parameters for which the rank of the matrix A drops.

Using this method on the irreducible case as well as on all reducible cases, it can be shown that there exist no dynamically balanced spherical 4R linkage.

4.5 Antipodal cases

In this section, we consider the antipodal cases, that is when one of the bars has length 0 or π. We do not use the equations developed in Section 4.1 since it was assumed that none of the bar's was antipodal.

Consider the case when one of the bars is antipodal. If $\alpha = 0 \mod \pi$, then the linkage can be seen as a triangle moving on a sphere (take $\alpha = 0$), rotating about the x axis. It is statically balanced if and only its the centre of mass position of the linkage lies on the axis of rotation (i.e. the y and z components are 0). If only one of the moving bars (i.e. input crank, output crank or coupler) is antipodal, the only motion of the linkage is the self rotation of the antipodal bar, the linkage being stiff. So the only condition in this case would be that the centre of mass of the antipodal bar lies on its axis of rotation.

If at least two bodies are antipodal, at least one of the moving bars can freely rotate about itself. Thus, the linkage is statically balanced if this self motion does not change the centre of mass location (i.e. the centre of mass position of the antipodal bar is on the rotation axis of the self motion) and if the total centre of mass of the linkage lies on the x axis (i.e. y and z components are 0).

If at least one moving bar is antipodal, it is straightforward to prove that the linkage cannot be dynamically balanced since the self motion of the antipodal moving bar contributes to the angular momentum in an arbitrary way, since we can rotate it on itself as fast as we desire. This cannot be compensated with an independent motion of the other bodies. The only case left to consider is the case when only the base is antipodal, i.e. $\alpha = 0 \mod \pi$. But in this case, it corresponds to a triangle rotating on a sphere about the x axis, and the angular momentum cannot be zero.

Chapter 5

Bennett linkage

In 1903, G.T. Bennett[12] published a paper on a new linkage called the skew isogram, and shown that this linkage is movable if the opposite sides are equal. In 1914, Bennett[13] published a series of therorems on this linkage. In [39], Groeneveld supplied missing proofs of the skew isogram and gave a detailed description of the motion space of such linkage, which completed the work done by Krames [47]. Although there are only a few application of this linkage, it can be used to deploy structures [21] and interest arose recently by the fact that the cruciate ligament of the human-knee-joint is structurally very similar to a Bennett linkage [9]. Other references on the topic are [10, 22]. In the first part, we use the same approach as in Sections 3 and 4 to derive sufficient and necessary conditions for the static balancing of Bennett linkages. In the second part, we use a geometric approach to derive the same conditions. In this section, we do not address the dynamic balancing problem.

5.1 Algebraic approach

5.1.1 Model

Kinematic model

Since we assume the joint axes are not parallel (i.e. the planar case) nor concurrent (i.e. the spherical case), the relation between two consecutive joint axes have to be specified. This is achieved by using the twist of a link as defined in [39].

Definition A **screw** is a way of describing a displacement. It can be thought of as a rotation about an axis and a translation along that same axis. Any general body motion in 3D can be described by a screw.

Definition Let A_p and A_{p+1} be two consecutive joints in a skew polygon. Let h_p and h_{p+1} be their respective joint axes. The **twist** of the link A_pA_{p+1}

is the angle of rotation of a right screw-motion with A_pA_{p+1} as its axis which replaces h_p to h_{p+1}. This angle is equal to that of the right screw-motion with $A_{p+1}A_p$ as its axis which replaces h_{p+1} to h_p.

A Bennett linkage is shown in Fig. 5.1. The four links are connected by revolute joints, each of which has rotation axis perpendicular to the two adjacents links connected to it. The length of a link is defined as the distance between two neighbouring joints. The twist of a link is the skewed angle between the rotation axes of the two revolute joints attached to the link. The linkage is moveable if the following conditions are fulfilled:

1. Two adjacent links meet in a point on the common rotation axis
2. The opposite sides have equal lengths
3. The twists of the opposite links are equal
4. The ratio of the length of a link and the sine of its twist has the same value (or opposite value) for each link

For more details about the Bennett linkage and its properties, we refer the reader to [39, 84, 40, 54, 9]. Let p_1 and p_2 be the position of the fixed joints on the base and q_1 and q_2 be the coordinates of the moving joints on the coupler. The joint axes are denoted by A, B, C, D. Let β be the twist of the base link p_1p_2 and of the coupler q_1q_2, both of length b. Let α be the twist of the input crank p_1q_1 and of the output crank p_2q_2 of length a. We also have the following relation between the geometric design parameters a, b, α and β:

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \tag{5.1}$$

Let the angle $p_2p_1q_1$ be θ_1 and the angle $p_1p_2q_2$ be $\pi - \theta_2$. It is also shown in [39] that the opposite angles are equals, i.e. the angle $p_2q_2q_1$ is equal to θ_1 and the angle $p_1q_1q_2$ is equal to $\pi - \theta_2$.

Let $R_A(\theta)$ be the rotation matrix representing the rotation of θ about the unit vector A. The orientation of the joint axes are given by the following unit vectors

$$\begin{aligned} A &= [0,0,1]^T \\ B &= R_x(\beta)\,A \\ D &= R_z(\theta_1)\,R_x(\alpha)\,A \end{aligned} \tag{5.2}$$

We ommited C since it is not required in our computation.

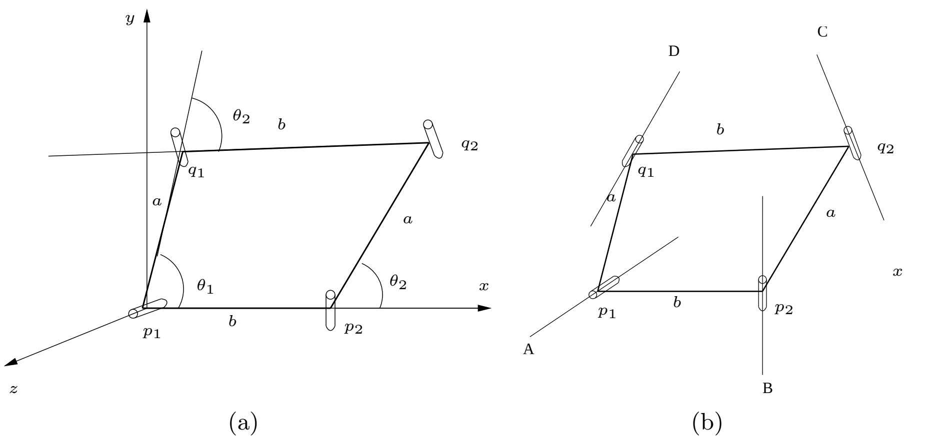


Figure 5.1: Representation of the Bennett linkage (algebraic approach)

The position of the joints are given by:

$$\begin{aligned} p_1 &= [0,0,0]^T \\ p_2 &= [b,0,0]^T \\ q_1 &= R_z(\theta_1)\,[a,0,0]^T \\ q_2 &= p_2 + R_B(\theta_2)\,[a,0,0]^T \quad (5.3) \\ &= p_2 + R_x(\beta)\,R_z(\theta_2)\,R_x(\beta)^T\,[a,0,0]^T \quad (5.4) \end{aligned}$$

Alternatively, the position of q_2 can be computed through the input crank and coupler:

$$\begin{aligned} q_2' &= q_1 + \frac{b}{a} R_D(\theta_2)\, q_1 \\ &= \left(E + \frac{b}{a} R_D(\theta_2) \right) q_1 \end{aligned} \quad (5.5)$$

where E is a 3×3 identity matrix.

The loop closure equation (geometric constraint) is expressed as $g = 0$, where g is given by

$$\begin{aligned} g &:= \| q_1 - q_2 \|_2^2 - b^2 \\ &= \left\| \begin{bmatrix} b + a(\cos\theta_2 - \cos\theta_1) \\ a(\cos\beta\sin\theta_2 - \sin\theta_1) \\ a\sin\beta\sin\theta_2 \end{bmatrix} \right\|_2^2 - b^2 \quad (5.6) \\ &= 2a\left[a(1 - \cos\theta_1\cos\theta_2 - \cos\beta\sin\theta_1\sin\theta_2) + b(\cos\theta_2 - \cos\theta_1)\right] \quad (5.7) \end{aligned}$$

Since (5.7) is homogeneous in terms of a and b, $a \neq 0$, $\frac{b}{a} = \frac{\sin\beta}{\sin\alpha}$ and $g = 0$, we can rewrite Eq. (5.7) as

$$\begin{aligned} g &= (1 - \cos\theta_1 \cos\theta_2 - \cos\beta \sin\theta_1 \sin\theta_2) + \frac{b}{a}(\cos\theta_2 - \cos\theta_1) \\ &= (1 - \cos\theta_1 \cos\theta_2 - \cos\beta \sin\theta_1 \sin\theta_2) + \frac{\sin\beta}{\sin\alpha}(\cos\theta_2 - \cos\theta_1) \\ &= 0 \end{aligned} \tag{5.8}$$

Multiplying (5.8) by $\sin\alpha$, we obtain the following constraint:

$$g' = \sin\alpha\,(1 - \cos\theta_1\cos\theta_2 - \cos\beta\sin\theta_1\sin\theta_2) + \sin\beta\,(\cos\theta_2 - \cos\theta_1) = 0 \tag{5.9}$$

Eq. (5.9) does not depends on a and b anymore, but only on the design parameters α and β. This trick to remove the length of the links (a and b) from the geometric constraint is described for example in [43][1] and [54][2]. Using trigonometric identities, Hunt [43] showed that Eq. 5.8 can be decomposed into two components:

$$g' = \left(\frac{\tan\frac{1}{2}\theta_2}{\tan\frac{1}{2}\theta_1} + \frac{\sin\frac{1}{2}(\beta+\alpha)}{\sin\frac{1}{2}(\beta-\alpha)}\right)\left(\frac{\tan\frac{1}{2}\theta_2}{\tan\frac{1}{2}\theta_1} - \frac{\cos\frac{1}{2}(\beta+\alpha)}{\cos\frac{1}{2}(\beta-\alpha)}\right) = 0 \tag{5.10}$$

However, we prefer to use a different formulation. Replace θ_1 and θ_2 by the complex variables $\mathbf{z}_1$ and $\mathbf{z}_2$ using (2.4) and the angles α and β by c and d using (2.6) in (5.7). Then, g can be written as a Laurent polynomial in the variables $\mathbf{z}_1$ and $\mathbf{z}_2$ as

$$\overline{g} = \frac{(cd\mathbf{z}_2\mathbf{z}_1 + \mathbf{z}_1 - \mathbf{z}_2 - cd)(d\mathbf{z}_1\mathbf{z}_2 + c\mathbf{z}_1 - c\mathbf{z}_2 - d)}{(1+c^2)(1+d^2)\mathbf{z}_1\mathbf{z}_2} \tag{5.11}$$

Denote

$$\begin{aligned} g_1 &= (cd\mathbf{z}_2\mathbf{z}_1 + \mathbf{z}_1 - \mathbf{z}_2 - cd) \\ g_2 &= (d\mathbf{z}_1\mathbf{z}_2 + c\mathbf{z}_1 - c\mathbf{z}_2 - d) \end{aligned} \tag{5.12}$$

The denominator is always different than zero, therefore, this expression vanishes iff

$$\overline{g}' := g_1 g_2 = (cd\mathbf{z}_2\mathbf{z}_1 + \mathbf{z}_1 - \mathbf{z}_2 - cd)(d\mathbf{z}_1\mathbf{z}_2 + c\mathbf{z}_1 - c\mathbf{z}_2 - d) = 0 \tag{5.13}$$

Therefore, we obtain two different kinematic modes but only one is valid using our convention. We can check which component is valid by comparing q_2 and q_2' from equation (5.4) and (5.5) respectively which should be equal.

$$q_2' - q_2 = \frac{g_2\,(\mathbf{z}_1\mathbf{z}_2 + cd\mathbf{z}_1 + cd\mathbf{z}_2 + 1)}{(1+c^2)(1+d^2)\mathbf{z}_1\mathbf{z}_2}\begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.14}$$

Therefore, we must have $g_2 = 0$.

[1] Section 10, p.285

[2] p.205 eq. (9.71) and (9.72)

Static model

Let M be the total mass of the linkage ($M = m_1 + m_2 + m_3$). The position of the centre of mass of the linkage $\mathbf{rS}$ is:

$$\mathbf{rS} = \frac{1}{M}\left(\mathbf{rS}_1 + \mathbf{rS}_2 + \mathbf{rS}_3\right) \tag{5.15}$$

where $\mathbf{rS}_1$, $\mathbf{rS}_2$ and $\mathbf{rS}_3$ are the positions of the centre of mass of the three moving links expressed in the *reference frame*:

$$\begin{aligned}\mathbf{rS}_1 &= R_z\left(\theta_1\right)\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_1)c_1 - \sin(\theta_1)c_2 \\ \sin(\theta_1)c_1 + \cos(\theta_1)c_2 \\ c_3 \end{bmatrix}\end{aligned} \tag{5.16}$$

$$\begin{aligned}\mathbf{rS}_2 &= \left(p_2 + R_B\left(\theta_2\right)\begin{bmatrix} c_4 \\ c_5 \\ c_6 \end{bmatrix}\right) \\ &= \left(p_2 + R_x\left(\beta\right)R_z\left(\theta_2\right)R_x\left(\beta\right)^T\begin{bmatrix} c_4 \\ c_5 \\ c_6 \end{bmatrix}\right)\end{aligned} \tag{5.17}$$

$$\begin{aligned}\mathbf{rS}_3 &= \left(q_1 + R_D\left(\theta_2\right)R_z\left(\theta_1\right)\begin{bmatrix} c_7 \\ c_8 \\ c_9 \end{bmatrix}\right) \\ &= \left(q_1 + R_z\left(\theta_1\right)R_x\left(\alpha\right)R_z\left(\theta_2\right)R_x\left(\alpha\right)^T R_z\left(\theta_1\right)^T R_z\left(\theta_1\right)\begin{bmatrix} c_7 \\ c_8 \\ c_9 \end{bmatrix}\right) \\ &= \left(q_1 + R_z\left(\theta_1\right)R_x\left(\alpha\right)R_z\left(\theta_2\right)R_x\left(\alpha\right)^T\begin{bmatrix} c_7 \\ c_8 \\ c_9 \end{bmatrix}\right)\end{aligned} \tag{5.18}$$

Therefore, the position of the centre of mass $\mathbf{rS}$ can be written in terms of the following 12 static parameters:

$$\{m_1, m_2, m_3, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9\} \tag{5.19}$$

Denote as rS_x, rS_y and rS_z the x,y and z components of $\mathbf{rS}$, i.e.

$$\mathbf{rS} = \left[\mathrm{rS}_x, \mathrm{rS}_y, \mathrm{rS}_z\right]^T \tag{5.20}$$

5.1.2 Static balancing

A Bennett linkage is statically balanced if and only if there exists Laurent polynomials $h_x(\mathbf{z}_1, \mathbf{z}_2)$, $h_y(\mathbf{z}_1, \mathbf{z}_2)$, $h_z(\mathbf{z}_1, \mathbf{z}_2)$ and arbitrary constants c_x, c_y, c_z (i.e. depending on the design parameters but independent of the joint variables $\mathbf{z}_1$ and $\mathbf{z}_2$) such that

$$\begin{aligned} \mathrm{rS}_x(\mathbf{z}_1, \mathbf{z}_2) =& h_x(\mathbf{z}_1, \mathbf{z}_2) g_2(\mathbf{z}_1, \mathbf{z}_2) + c_x \\ \mathrm{rS}_y(\mathbf{z}_1, \mathbf{z}_2) =& h_y(\mathbf{z}_1, \mathbf{z}_2) g_2(\mathbf{z}_1, \mathbf{z}_2) + c_y \\ \mathrm{rS}_z(\mathbf{z}_1, \mathbf{z}_2) =& h_z(\mathbf{z}_1, \mathbf{z}_2) g_2(\mathbf{z}_1, \mathbf{z}_2) + c_z \end{aligned} \tag{5.21}$$

We apply the toric polynomial division and obtain the following four constraints:

$$\begin{aligned} m_1 b c_1 + m_3 a (b - c_7) &= 0 \\ m_2 b c_4 + m_3 a c_7 &= 0 \\ -m_1 b c_2 + m_3 a \left(c_9 \sin\gamma + c_8 \cos\gamma\right) &= 0 \\ m_2 b \left(c_5 \cos\beta + c_6 \sin\beta\right) + m_3 a \left(c_8 \cos\alpha + c_9 \sin\alpha\right) &= 0 \end{aligned} \tag{5.22}$$

where $\gamma = \alpha + \beta$.

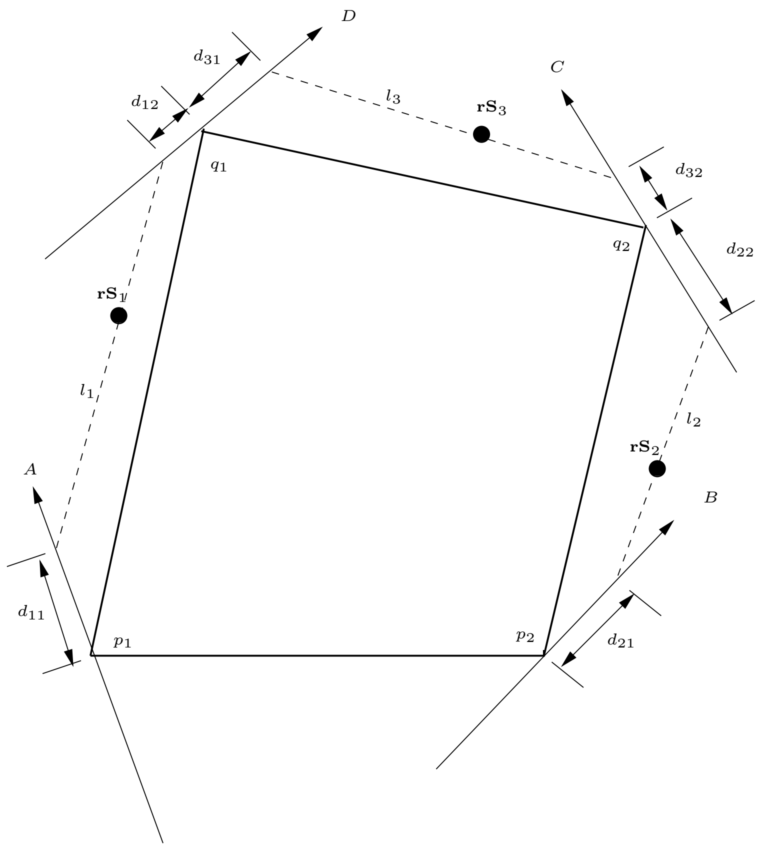


Figure 5.2: Static balancing of Bennett linkage using line geometry

5.2 Geometric approach

5.2.1 Model

Let $\mathbf{rS}_1$, $\mathbf{rS}_2$, $\mathbf{rS}_3$ be the position of the centre of mass of the input crank, output crank and coupler respectively. Let $\overline{A}$, $\overline{B}$, $\overline{C}$ and $\overline{D}$ be the lines corresponding to the axis A, B, C and D respectively. We make a clear distinction between the lines (which do not pass necessarily through the origin) and the axis of rotation (which define only the orientation of the line) to avoid any confusion.

For the input crank, consider the unique line, denoted l_1, going through $\mathbf{rS}_1$ and intersecting the lines A and D. This line is unique if and only if $\mathbf{rS}_1$ is not on one of the axis [3]. So we assume that $\mathbf{rS}_1$ does not lie on A or D. Denote by d_{11} the "oriented" distance between p_1 and the intersection

[3]Given two skew lines L_1 and L_2 and a point p not on L_1 and not on L_2, there exists a unique line through p intersecting both L_1 and L_2. To see this consider P_1 the plane formed by L_1 and p and the plane P_2 formed by L_2 and p. These two planes intersect in a unique line going through p.

point of the line l_1 and axis A. Similarly, let d_{12} be the "oriented" distance between q_1 and the intersection point of the line l_1 and the axis D. Let α_{11}, α_{12} be real values, with $\alpha_{11} + \alpha_{12} = 1$. We can write

$$\mathbf{rS}_1 = \alpha_{11}(p_1 + d_{11}A) + \alpha_{12}(q_1 + d_{12}D) \tag{5.23}$$

We can use the same argumentation to write the centre of mass position of the output crank and coupler. The notation and symbols are illustrated in Fig. 5.2.

$$\mathbf{rS}_2 = \alpha_{21}(p_2 + d_{21}B) + \alpha_{22}(q_2 + d_{22}C) \tag{5.24}$$

$$\mathbf{rS}_3 = \alpha_{31}(q_1 + d_{31}D) + \alpha_{32}(q_2 + d_{32}C) \tag{5.25}$$

The idea is to distribute and transfer the mass contribution of the coupler (i.e. $m_3\mathbf{rS}_3$) to the input and output crank, more specifically on the axes D and C. First we split the mass of the coupler into two masses: $m_3' = m_3\alpha_{31}$ and $m_3'' = m_3\alpha_{32}$ with $\alpha_{31} + \alpha_{32} = 1$. We position the mass m_3' at $\mathbf{rS}_3' = q_1 + d_{31}D$ and the mass m_3'' at the position $\mathbf{rS}_3'' = q_2 + d_{32}C$. Clearly, the total mass is the same

$$m_3' + m_3'' = m_3\alpha_{31} + m_3\alpha_{32} = m_3(\alpha_{31} + \alpha_{32}) = m_3 \tag{5.26}$$

and $m_3\mathbf{rS}_3 = m_3'\mathbf{rS}_3' + m_3''\mathbf{rS}_3''$.

We therefore obtain only two bodies fow which to consider the mass: the augmented input crank with centre of mass position $\mathbf{rS}_1'$ with mass m_1' and the augmented output crank which centre of mass position is given by $\mathbf{rS}_2'$ with mass m_2':

$$m_1' = m_1 + m_3' = m_1 + m_3\alpha_{31} \tag{5.27}$$

$$m_1'\mathbf{rS}_1' = m_1\alpha_{11}(p_1 + d_{11}A) + m_1\alpha_{12}(q_1 + d_{12}D) + m_3\alpha_{31}(q_1 + d_{31}D) \tag{5.28}$$

$$m_2' = m_2 + m_3'' = m_2 + m_3\alpha_{32} \tag{5.29}$$

$$m_2'\mathbf{rS}_2' = m_2\alpha_{21}(p_2 + d_{21}B) + m_2\alpha_{22}(q_2 + d_{22}C) + m_3\alpha_{32}(q_2 + d_{32}C) \tag{5.30}$$

We want that the position of the centre of mass of the linkage, resulting from the motion of these two centres of mass be fixed at all time. Since these two masses are moving on two circles which do not lie in parallel planes, they cannot cancel each other. Therefore, each of these two masses must be fixed for any motion. This is possible if and only if m_1' is located on the axis of rotation A and m_2' is on the axis B. Therefore, for some constant e_1 and e_2, we must have

$$\mathbf{rS}_1' = p_1 + e_1A \tag{5.31}$$

$$\mathbf{rS}_2' = p_2 + e_2B \tag{5.32}$$

For the augmented input crank, combining (5.28) with (5.31), we obtain the following condition:

$$(m_1\alpha_{11})p_1 + (m_1\alpha_{12} + m_3\alpha_{31})q_1 + (m_1\alpha_{12}d_{12} + m_3\alpha_{31}d_{31})D = m_1'p_1 + e_1'A \tag{5.33}$$

with

$$e_1' = m_1'(e_1 - m_1\alpha_{11}d_{11}) \tag{5.34}$$

Choosing p_1 as the origin, we obtain:

$$(m_1\alpha_{12} + m_3\alpha_{31})q_1 + (m_1\alpha_{12}d_{12} + m_3\alpha_{31}d_{31})D = e_1'A \tag{5.35}$$

Since q_1, D and A are linearly independent, we must have that the coefficients vanish, i.e.

$$\begin{aligned} m_1\alpha_{12} + m_3\alpha_{31} &= 0 \\ m_1\alpha_{12}d_{12} + m_3\alpha_{31}d_{31} &= 0 \end{aligned} \tag{5.36}$$

Since $\alpha_{12} \neq 0$ and $\alpha_{31} \neq 0$ (otherwise the mass of a body would be on the axis of rotation), the solution is

$$d_{12} = d_{31}, m_1\alpha_{12} + m_3\alpha_{31} = 0 \tag{5.37}$$

Using the same computation for the augmented output crank (or simply by symmetry), we obtain that either

$$d_{22} = d_{32}, m_2\alpha_{22} + m_3\alpha_{32} = 0 \tag{5.38}$$

In the general case, we have

$$\begin{aligned} d_{12} - d_{31} &= 0 \\ m_1\alpha_{12} + m_3\alpha_{31} &= 0 \\ d_{22} - d_{32} &= 0 \\ m_2\alpha_{22} + m_3\alpha_{32} &= 0 \end{aligned} \tag{5.39}$$

Therefore, we have the following set of 15 static parameters:

$$\{m_1, m_2, m_3, \alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}, \alpha_{31}, \alpha_{32}, d_{11}, d_{12}, d_{21}, d_{22}, d_{31}, d_{32},\} \tag{5.40}$$

subject to the following 8 constraints:

1. We can scale the mass, for example by setting $m_3 = 1$.

2. We have the following three equations: $\alpha_{i1} + \alpha_{i2} = 1$, for $i = 1, 2, 3$.

3. Equations (5.39).

Therefore we have 7 $(15-8)$ degrees of freedom to choose the static parameters to obtain a statically balanced Bennett linkage.

5.2.2 Comparison

The goal of this section is to prove that the geometric and the algebraic method yield the same constraints. This should give also some information about the structure of the equations. The goal is to explicitely derived values for the c_i in terms of $\alpha_{ij}, \alpha.\beta$ and b. For this, we will compare the value of $\mathbf{rS}_1$, $\mathbf{rS}_2$ and $\mathbf{rS}_3$ for both methods and derive expression for the c_i.

$\mathbf{rS}_1$ depends only on the variable θ_1 and can be written in the following form:

$$\mathbf{rS}_1 = \begin{bmatrix} e_{sx}\sin(\theta_1) + e_{cx}\cos(\theta_1) \\ e_{sy}\sin(\theta_1) + e_{cy}\cos(\theta_1) \\ e_{1z} \end{bmatrix} \tag{5.41}$$

We obtain

$$\begin{aligned} e_{sx} - c_2 &= \alpha_{12}\sin(\alpha)d_{12} \\ e_{cx}c_1 &= \alpha_{12}a \\ e_{sy}c_1 &= \alpha_{12}a \\ e_{sy}c_2 &= \alpha_{12}\sin(\alpha)d_{12} \\ e_{1z}c_3 &= \alpha_{11}d_{11} + \alpha_{12}\cos(\alpha)d_{12} \end{aligned} \tag{5.42}$$

We therefore obtain the values for c_1, c_2 and c_3.

For the output crank, $\mathbf{rS}_2$ depends only on the variable θ_2 and can be written in the following form:

$$\mathbf{rS}_2 = \begin{bmatrix} e_{sx}\sin(\theta_2) + e_{cx}\cos(\theta_2) \\ e_{sy}\sin(\theta_2) + e_{cy}\cos(\theta_2) + e_{1y} \\ e_{sz}\sin(\theta_2) + e_{cz}\cos(\theta_2) + e_{1z} \end{bmatrix} \tag{5.43}$$

We obtain

$$\begin{aligned} e_{sx} &:= -c_5\cos(\beta) - c_6\sin(\beta) = \alpha_{22}\sin(\alpha)d_{22} \\ e_{cx} &:= c_4 = \alpha_{22}a \\ e_{1x} &:= b = b \\ e_{sy} &:= \cos(\beta)c_4 = \cos(\beta)\alpha_{22}a \\ e_{cy} &:= \cos(\beta)(c_5\cos(\beta) + c_6\sin(\beta)) = -\cos(\beta)\alpha_{22}d_{22}\sin(\alpha) \\ e_{1y} &:= \sin(\beta)(c_5\sin(\beta) - c_6\cos(\beta)) = \sin(\beta)(-\alpha_{22}d_{22}\cos(\alpha) - \alpha_{21}d_{21}) \\ e_{sz} &:= \sin(\beta)c_4 = \sin(\beta)\alpha_{22}a \\ e_{cz} &:= \sin(\beta)(\cos(\beta)c_5 + c_6\sin(\beta)) = -\sin(\beta)\alpha_{22}d_{22}\sin(\alpha) \\ e_{1z} &:= \cos(\beta)(-\sin(\beta)c_5 + \cos(\beta)c_6) = \cos(\beta)(\alpha_{22}d_{22}\cos(\alpha) + \alpha_{21}d_{21}) \end{aligned} \tag{5.44}$$

and therefore:

$$\begin{aligned} c_4 &= \alpha_{22}a \\ c_5\cos(\beta) + c_6\sin(\beta) &= -\alpha_{22}d_{22}\sin(\alpha) \\ c_5\sin(\beta) - c_6\cos(\beta) &= -\alpha_{22}d_{22}\cos(\alpha) - \alpha_{21}d_{21} \end{aligned} \tag{5.45}$$

Combining the last 2 equations, we can easily eliminate either c_5 or c_6. We obtain

$$\begin{aligned}
c_4 &= \alpha_{22} a \\
c_5 &= -\alpha_{21} d_{21} \sin(\beta) - \alpha_{22} d_{22} (\sin(\beta)\cos(\alpha) + \cos(\beta)\sin(\alpha)) \\
c_6 &= \alpha_{21} d_{21} \cos(\beta) + \alpha_{22} d_{22} (\cos(\beta)\cos(\alpha) - \sin(\beta)\sin(\alpha))
\end{aligned} \tag{5.46}$$

For $\mathbf{rS}_3$, it is more difficult to directly compare the equation and using coefficient comparison for finding the values of c_7, c_8 and c_9. The reason is that the x and y components of $\mathbf{rS}_3$ depends on both variables θ_1 and θ_2 which are related through the geometric constraint. However, the z component (which depends solely on θ_2) can be used to derive the solutions.

$$\mathbf{rS}_{3z} = e_{sz} \sin(\theta_2) + e_{cz} \cos(\theta_2) + e_{1z} \tag{5.47}$$

with

$$\begin{aligned}
e_{sz} &:= \sin(\alpha) c_7 = \sin(\alpha) b \alpha_{32} \\
e_{cz} &:= \sin\alpha (c_9 \sin(\alpha) + c_8 \cos(\alpha)) = -\sin(\alpha) \alpha_{32} d_{32} \sin(\beta) \\
e_{1z} &:= \cos(\alpha)(c_9 \cos(\alpha) - c_8 \sin(\alpha)) = -\cos(\alpha)(-\alpha_{31} d_{31} - \alpha_{32} d_{32} \cos(\beta))
\end{aligned} \tag{5.48}$$

Again, combining the last 2 equations, we obtain:

$$\begin{aligned}
c_7 &= \alpha_{32} b \\
c_8 &= -\alpha_{32} d_{32} \sin(\beta)\cos(\alpha) - \sin(\alpha)\alpha_{31} d_{31} - \sin(\alpha)\alpha_{32} d_{32} \cos(\beta) \\
c_9 &= -\alpha_{32} d_{32} \sin(\beta)\sin(\alpha) + \alpha_{31} d_{31} \cos(\alpha) + \alpha_{32} d_{32} \cos(\beta)\cos(\alpha)
\end{aligned} \tag{5.49}$$

Therefore, we obtain the following mapping:

$$\begin{aligned}
c_1 &= \alpha_{12} a \\
c_2 &= -\alpha_{12} \sin(\alpha) d_{12} \\
c_3 &= \alpha_{11} d_{11} + \alpha_{12} \cos(\alpha) d_{12} \\
c_4 &= \alpha_{22} a \\
c_5 &= -\alpha_{21} d_{21} \sin(\beta) - \alpha_{22} d_{22} (\sin(\beta)\cos(\alpha) + \cos(\beta)\sin(\alpha)) \\
c_6 &= \alpha_{21} d_{21} \cos(\beta) + \alpha_{22} d_{22} (\cos(\beta)\cos(\alpha) - \sin(\beta)\sin(\alpha)) \\
c_7 &= \alpha_{32} b \\
c_8 &= -\alpha_{32} d_{32} \sin(\beta)\cos(\alpha) - \sin(\alpha)\alpha_{31} d_{31} - \sin(\alpha)\alpha_{32} d_{32} \cos(\beta) \\
c_9 &= -\alpha_{32} d_{32} \sin(\beta)\sin(\alpha) + \alpha_{31} d_{31} \cos(\alpha) + \alpha_{32} d_{32} \cos(\beta)\cos(\alpha)
\end{aligned} \tag{5.50}$$

Replacing these values in the constraints obtained using the algebraic method (5.22), we obtain (5.39).

Chapter 6

Conclusion

In this thesis, a novel systematic approach has been proposed to find sufficient and necessary conditions for the static and dynamic balancing of 1 degree of freedom parallel mechanisms. The main contributions of this method are:

1. A clear algebraic formulation of the problem as the intersection of parametric complex curves.

2. A clear understanding on the fundamental importance of studying the kinematic modes of the linkage, each kinematic mode leading to different balancing conditions.

3. A toric polynomial division algorithm to obtain balancing conditions in a simplified form.

The method has been applied on the balancing of 4R linkages and the main results are:

1. A complete classification of dynamically balanced planar 4R linkages. Although all cases were known, it was not clear that the classification was complete.

2. Sufficient and necessary conditions for the static balancing of spherical 4R linkages and a formal proof that spherical 4R linkages cannot be dynamically balanced.

3. Sufficient and necessary conditions for the static balancing of Bennett linkages.

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