Combinatory Analysis 2008: Partitions, q-series, and Applications

Penn State University

George E. Andrews' Alternative Approach of Stembridge's TSPP Theorem

Carsten Schneider
RISC-Linz, Austria

December 7, 2008

Plane partitions

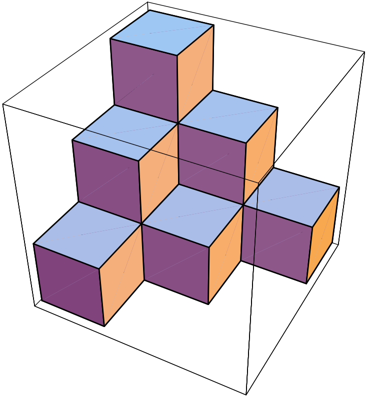
A plane partition with largest part $\leq n$ is

$$
\begin{array}{lccccccccc}
n \geq & & & & & & & & & \\
& a_{11} & \geq & a_{12} & \geq & a_{13} & \geq & \dots & a_{1r} & \\
& \vee\!| & & \vee\!| & & & & & \vee\!| & \\
& a_{21} & \geq & a_{22} & & & & \dots & a_{2r} & \\
& \vee\!| & & & & & & & \vee\!| & \\
& a_{31} & & & & & & & a_{3r} & \\
& \vdots & & & & & & & \vdots & \\
& & & & & & & & \vee\!| & \\
& a_{s1} & \geq & a_{s2} & \geq & a_{s3} & \geq & \dots \geq & a_{s,r} & \\
& & & & & & & & & \geq 0.
\end{array}
$$

Totally **S**ymmetric **P**lane **P**artition (TSPP)

Example:

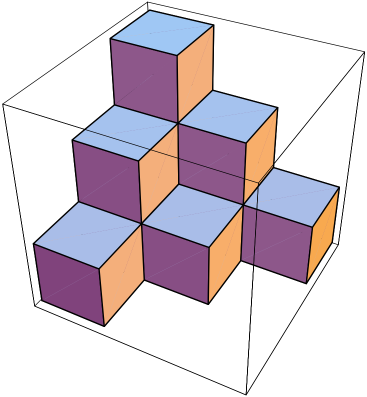
$$\begin{matrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{matrix} \longleftrightarrow$$



Totally Symmetric Plane Partition (TSPP)

Example:

$$\begin{matrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{matrix} \longleftrightarrow$$



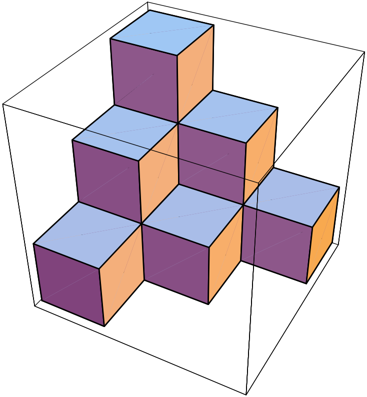
Conjecture (Andrews,Macdonald,Stanley; in 80th). Let T_n be the number of totally symmetric plane partitions with largest part $\leq n$. Then for $n \geq 1$

$$T_n = \prod_{1 \leq i \leq j \leq k \leq n} \frac{i+j+k-1}{i+j+k-2}.$$

Totally Symmetric Plane Partition (TSPP)

Example:

$$\begin{matrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{matrix} \quad \longleftrightarrow$$



Theorem (Stembridge, 1995). Let T_n be the number of totally symmetric plane partitions with largest part $\leq n$. Then for $n \geq 1$

$$T_n = \prod_{1 \leq i \leq j \leq k \leq n} \frac{i+j+k-1}{i+j+k-2}.$$

Okada's Theorem (1989). For $n \geq 3$,

$$T_{n-2}^2 = \begin{cases} \det\big(M(n)\big)x^{-1} & \text{if } n \text{ is odd,} \\ \det\big(M(n)\big) & \text{if } n \text{ is even,} \end{cases}$$

with $M(n) = \big(\mu_1(i,j)\big)_{0 \leq i,j \leq n-1}$

where

$$\mu(i,j) = \begin{cases} 0 & \text{if } j \leq i, \\ 2^{j-1} + (-1)^{j-1} & \text{if } i = 0,\ i < j, \\ (-1)^{j-i-1} + \sum_{s=i}^{j-1} \binom{i+j-2}{s} & \text{if } 0 < i < j, \end{cases}$$

$$\mu_1(i,j) = \begin{cases} x & \text{if } i = j = 0, \\ (-1)^{j-1} & \text{if } i = 0,\ j > 0, \\ (-1)^{i} & \text{if } j = 0,\ i > 0, \\ 0 & \text{if } i = j > 0, \\ \mu(i-1, j-1) & \text{if } j > i \geq 1, \\ -\mu(j-1, i-1) & \text{if } 1 \leq j < i. \end{cases}$$

Okada's Theorem (1989). For $n \geq 3$,

$$T_{n-2}^2 = \begin{cases} \det\big(M(n)\big)x^{-1} & \text{if } n \text{ is odd,} \\ \det\big(M(n)\big) & \text{if } n \text{ is even,} \end{cases}$$

with $M(n) = \big(\mu_1(i,j)\big)_{0 \leq i,j \leq n-1}$

where

$$\mu(i,j) = \begin{cases} 0 & \text{if } j \leq i, \\ 2^{j-1} + (-1)^{j-1} & \text{if } i = 0,\ i < j, \\ (-1)^{j-i-1} + \displaystyle\sum_{s=i}^{j-1} \binom{i+j-2}{s} & \text{if } 0 < i < j, \end{cases}$$

$$\mu_1(i,j) = \begin{cases} x & \text{if } i = j = 0, \\ (-1)^{j-1} & \text{if } i = 0,\ j > 0, \\ (-1)^i & \text{if } j = 0,\ i > 0, \\ 0 & \text{if } i = j > 0, \\ \mu(i-1, j-1) & \text{if } j > i \geq 1, \\ -\mu(j-1, i-1) & \text{if } 1 \leq j < i. \end{cases}$$

PROBLEM: Evaluate $\det(M)$.

Okada's Theorem (1989). For $n \geq 3$,

$$T_{n-2}^2 = \begin{cases} \det\big(M(n)\big)x^{-1} & \text{if } n \text{ is odd,} \\ \det\big(M(n)\big) & \text{if } n \text{ is even,} \end{cases}$$

with $M(n) = \big(\mu_1(i,j)\big)_{0 \leq i,j \leq n-1}$

G. Andrews guessed (1990)

$$W = \begin{pmatrix} * & \dots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \quad \text{with} \quad \det(W) = 1, \quad MW = \begin{pmatrix} * & & 0 \\ \vdots & \ddots & \\ * & \dots & * \end{pmatrix} =: U.$$

($M = W^{-1}U$ is "LU-decomposition")

Okada's Theorem (1989). For $n \geq 3$,

$$T_{n-2}^2 = \begin{cases} \det\big(M(n)\big)x^{-1} & \text{if } n \text{ is odd,} \\ \det\big(M(n)\big) & \text{if } n \text{ is even,} \end{cases}$$

with $M(n) = \big(\mu_1(i,j)\big)_{0 \leq i,j \leq n-1}$

G. Andrews guessed (1990)

$$W = \begin{pmatrix} * & \dots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \quad \text{with} \quad \det(W) = 1, \quad MW = \begin{pmatrix} * & & 0 \\ \vdots & \ddots & \\ * & \dots & * \end{pmatrix}.$$

THEN

$$\begin{aligned} \det(M) &= \det(M)\det(W) = \det(M\,W) \\ &= \text{product of diagonal elements of } M\,W \end{aligned}$$

Define

$$\left\{ \begin{matrix} x \\ n \end{matrix} \right\} = \frac{1}{2}\left(\binom{x}{n} + \binom{x-1}{n}\right);$$

$$t_1(n) = \begin{cases} 1 & \text{if } n = 0, \\ \frac{(n+1)(n+3)\dots(3n-1)}{(n)_n} & \text{if } n > 0; \end{cases}$$

$$t(n) = \begin{cases} 1 & \text{if } n = 0, \\ \frac{t_1(n)}{t_1(n-1)} & \text{if } n > 0. \end{cases}$$

$$r_3(s,j) = 4^{-s} \sum_{k=0}^{s} \frac{(j-k)(j)_k(-3j-1)_k}{jk!(-2j+\frac{1}{2})_k},$$

$$f_1(c,j) =$$
$$(-1)^c \sum_{s=0}^{\lfloor \frac{c}{2} \rfloor} \frac{(-1)^s \binom{j-1-s}{c-2s} (j)_s (-3j+1)_s (3j-3s-1)}{4^s s! (-2j+\frac{3}{2})_s (3j-1)},$$

$$f_2(c,j) = (-1)^c \sum_{s=0}^{\lfloor \frac{c}{2} \rfloor} (-1)^s \left\{ \begin{matrix} j-s \\ c-2s \end{matrix} \right\} r_3(s,j),$$

$$r_2(j) = \begin{cases} \frac{t_1(j-1)}{2} & \text{if } j \text{ even}, \\ \frac{t_1(j-1)}{2} + \frac{f_2(j-2, \frac{j-1}{2})}{2} & \text{if } j \text{ odd}, \end{cases}$$

$$r_1(j) = \begin{cases} t_1(j-1) & \text{if } j \text{ even}, \\ 0 & \text{if } j \text{ odd}, \end{cases}$$

$$e_1(i,j) = \begin{cases} 0 & \text{if } i > j, \\ 1 & \text{if } i = j, \\ r_1(j) & \text{if } i = 0,\ i < j, \\ r_2(j) & \text{if } i = 1,\ i < j, \\ f_1(j-i, \frac{j}{2}) & \text{if } 2 \le i < j,\ j \text{ even}, \\ f_2(j-i, \frac{j-1}{2}) & \text{if } 2 \le i < j,\ j \text{ odd}, \end{cases}$$

$$e(i,j) = \begin{cases} 0 & \text{if } i > j, \\ 1 & \text{if } i = j, \\ \begin{matrix} e_1(i,j) - t(j-1)\times \\ \times x^{(-1)^j} e_1(i,j-1) \end{matrix} & \text{if } i < j, \end{cases}$$

$$W(n) = \big(e(i,j)\big)_{0 \le i,j \le n-1}$$

Conjecture (G. Andrews, 1990) For each $n \geq 1$,

$$M(n)W(n) = \begin{pmatrix} x & 0 & 0 & 0 & \dots & 0 \\ A_{10} & \frac{t_1(0)^2}{x} & 0 & 0 & \dots & 0 \\ A_{20} & A_{21} & t_1(1)^2 x & 0 & \dots & 0 \\ \vdots & & & & \ddots & \vdots \\ A_{n-1,0} & A_{n-1,1} & \dots & \dots & \dots & t_1(n-2)^2 x^{\pm 1} \end{pmatrix}.$$

Define

$$\left\{ \begin{matrix} x \\ n \end{matrix} \right\} = \frac{1}{2}\left(\binom{x}{n} + \binom{x-1}{n}\right);$$

$$t_1(n) = \begin{cases} 1 & \text{if } n = 0, \\ \frac{(n+1)(n+3)\dots(3n-1)}{(n)_n} & \text{if } n > 0; \end{cases}$$

$$t(n) = \begin{cases} 1 & \text{if } n = 0, \\ \frac{t_1(n)}{t_1(n-1)} & \text{if } n > 0. \end{cases}$$

$$r_3(s,j) = 4^{-s} \sum_{k=0}^{s} \frac{(j-k)(j)_k(-3j-1)_k}{jk!(-2j+\frac{1}{2})_k},$$

$$f_1(c,j) =$$
$$(-1)^c \sum_{s=0}^{\lfloor \frac{c}{2} \rfloor} \frac{(-1)^s \binom{j-1-s}{c-2s} (j)_s (-3j+1)_s (3j-3s-1)}{4^s s! (-2j+\frac{3}{2})_s (3j-1)},$$

$$f_2(c,j) = (-1)^c \sum_{s=0}^{\lfloor \frac{c}{2} \rfloor} (-1)^s \left\{ \begin{matrix} j-s \\ c-2s \end{matrix} \right\} r_3(s,j),$$

$$r_2(j) = \begin{cases} \frac{t_1(j-1)}{2} & \text{if } j \text{ even}, \\ \frac{t_1(j-1)}{2} + \frac{f_2(j-2,\frac{j-1}{2})}{2} & \text{if } j \text{ odd}, \end{cases}$$

$$r_1(j) = \begin{cases} t_1(j-1) & \text{if } j \text{ even}, \\ 0 & \text{if } j \text{ odd}, \end{cases}$$

$$e_1(i,j) = \begin{cases} 0 & \text{if } i > j, \\ 1 & \text{if } i = j, \\ r_1(j) & \text{if } i = 0,\ i < j, \\ r_2(j) & \text{if } i = 1,\ i < j, \\ f_1(j-i, \frac{j}{2}) & \text{if } 2 \le i < j,\ j \text{ even}, \\ f_2(j-i, \frac{j-1}{2}) & \text{if } 2 \le i < j,\ j \text{ odd}, \end{cases}$$

$$e(i,j) = \begin{cases} 0 & \text{if } i > j, \\ 1 & \text{if } i = j, \\ e_1(i,j) - t(j-1) \times \\ \quad \times x^{(-1)^j} e_1(i,j-1) & \text{if } i < j, \end{cases}$$

$$\boxed{W(n) = \big(e(i,j)\big)_{0 \le i,j \le n-1}}$$

Theorem (Andrews, Paule, Sigma, 2005) For each $n \geq 1$,

$$M(n)W(n) = \begin{pmatrix} x & 0 & 0 & 0 & \dots & 0 \\ A_{10} & \frac{t_1(0)^2}{x} & 0 & 0 & \dots & 0 \\ A_{20} & A_{21} & t_1(1)^2 x & 0 & \dots & 0 \\ \vdots & & & & \ddots & \vdots \\ A_{n-1,0} & A_{n-1,1} & \dots & \dots & \dots & t_1(n-2)^2 x^{\pm 1} \end{pmatrix}.$$

Reduction to summation problems (P. Paule):

E.g., define

$$h(k,m) := \sum_{s=0}^{\lfloor \frac{2m-k}{2} \rfloor -1} \frac{k}{m-s} \binom{m-s}{2m-2s-k} \frac{(-1)^{s+k}}{2m\, 4^s} \sum_{r=0}^{s} \frac{(m-r)(m)_r(-3m-1)_r}{r!(\frac{1}{2}-2m)_r}$$

and

FIND Rec

$$A_0(i,m) := \sum_{k=0}^{2m} \binom{i+k-3}{i-2} h(k,m),$$

$$A_2(i,m) := \sum_{k=i}^{2m} (-1)^k h(k,m).$$

Theorem. For all $m \geq 1$ and $3 \leq i \leq 2m+1$,

$$2h(i-2,m) - 5h(i-1,m) - A_0(i,m) + 6(-1)^i A_2(i,m) - 3(-1)^i \prod_{s=1}^{2m-1} \frac{2(m+s-1)}{2m+s-2} = 0.$$

Reduction to summation problems (P. Paule):

E.g., define

$$h(k,m) := \sum_{s=0}^{\lfloor\frac{2m-k}{2}\rfloor-1} \frac{k}{m-s}\binom{m-s}{2m-2s-k}\frac{(-1)^{s+k}}{2m\,4^s}\sum_{r=0}^{s}\frac{(m-r)(m)_r(-3m-1)_r}{r!(\frac{1}{2}-2m)_r}$$

and

$$A_0(i,m) := \sum_{k=0}^{2m}\binom{i+k-3}{i-2}h(k,m),$$

$$A_2(i,m) := \sum_{k=i}^{2m}(-1)^k h(k,m).$$

Theorem. For all $m \geq 1$ and $3 \leq i \leq 2m+1$, ADD

$$2h(i-2,m) - 5h(i-1,m) - A_0(i,m) + 6(-1)^i A_2(i,m) - 3(-1)^i \prod_{s=1}^{2m-1}\frac{2(m+s-1)}{2m+s-2} = 0.$$

Summation over holonomic sequences

TASK: Find a recurrence for

$$S(n) = \sum_{k=0}^{n} f(n, k)$$

Summation over holonomic sequences

TASK: Find a recurrence for

$$S(n) = \sum_{k=0}^{n} f(n,k)$$

$\downarrow$

creative telescoping ansatz Zeilberger (1990)

Summation over holonomic sequences

TASK: Find a recurrence for

$$S(n) = \sum_{k=0}^{n} f(n,k)$$

creative telescoping ansatz   Chyzak (2000)

$\downarrow$ Gröbner basis

solve coupled linear recurrence system

Summation over holonomic sequences

TASK: Find a recurrence for

$$S(n) = \sum_{k=0}^{n} f(n,k)$$

creative telescoping ansatz — Chyzak (2000)

↓ Gröbner basis

solve coupled linear recurrence system

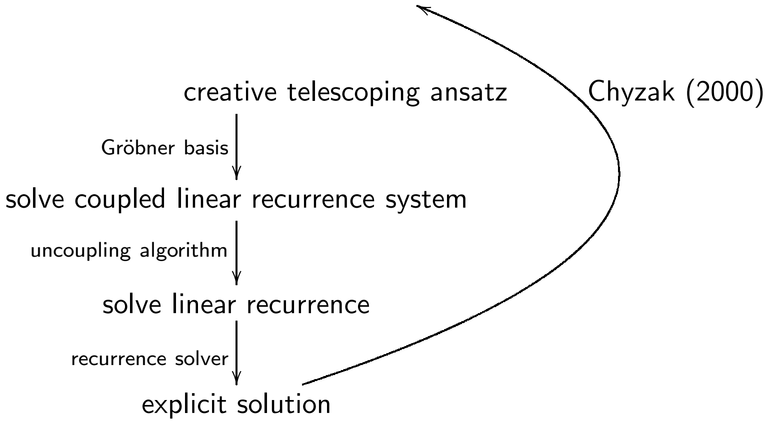
↓ uncoupling algorithm

solve linear recurrence

Summation over holonomic sequences

TASK: Find a recurrence for

$$S(n) = \sum_{k=0}^{n} f(n,k)$$



Summation over holonomic sequences

TASK: Find a recurrence for

$$S(n)=\sum_{k=0}^{n} f(n,k)$$

slightly restricted creative telescoping ansatz

explicit formula

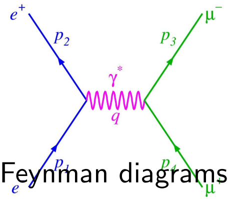
solve linear recurrence

efficient recurrence solver

explicit solution

Sigma (2005)

Follow up project: Evaluation of Feynman Integrals

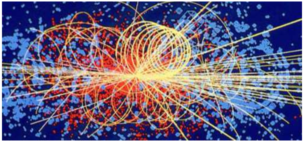


Feynman diagrams

$$\int \Phi(x)dx$$

Feynman integrals

Reduction



$$\sum_{k=1}^{\infty} \frac{\Gamma(N)\Gamma(k)}{\Gamma(N+k)} \frac{\left(\sum_{i=1}^{k} \frac{1}{i}\right)\left(\sum_{i=1}^{N+k} \frac{1}{i}\right)}{k+N+2}$$

Feynman sums

Sigma

$$-2\frac{\sum_{k=1}^{N} \frac{1}{k^3} - \zeta_3}{N+2} + \text{etc.}$$

Simplified sums

Zeilberger's Opinion 65:

Seeing all the details, (that nowadays can (and should!) be easily relegated to the computer), even if they are extremely hairy, is a hang-up that traditional mathematicians should learn to wean themselves from. A case in point is the excellent but unnecessarily long-winded recent article

Andrew, Paule, Schneider. Plane Partitions VI: Stembridge's TSPP Theorem. 2005.

Zeilberger's Opinion 65:

Seeing all the details, (that nowadays can (and should!) be easily relegated to the computer), even if they are extremely hairy, is a hang-up that traditional mathematicians should learn to wean themselves from. A case in point is the excellent but unnecessarily long-winded recent article

Andrew, Paule, Schneider. Plane Partitions VI: Stembridge's TSPP Theorem. 2005.

Zeilberger's Opinion 65:

Seeing all the details, (that nowadays can (and should!) be easily relegated to the computer), even if they are extremely hairy, is a hang-up that traditional mathematicians should learn to wean themselves from. A case in point is the excellent but unnecessarily long-winded recent article

Andrew, Paule, Schneider. Plane Partitions VI: Stembridge's TSPP Theorem. 2005.

You don't need 30 pages, and frankly all this EXPLICIT LANGUAGE of hairy computer output is almost pornographic.