

Algebraic Computation in Geometry *

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Abstract: Computations with algebraic curves and surfaces are very well suited for being treated with computer algebra. Many aspects of computer algebra need to be combined for successfully solving problems in this area, e.g. computations with algebraic coefficients, solution of algebraic equations and elimination theory, and derivation of power series approximations of branches. We will describe the application of computer algebra to problems arising in algebraic geometry. The program system CASA, which has been developed by the author and a group of students will be introduced.

Keywords: CASA, constructive algebraic geometry, computer algebra

1 Introduction

1.1 What is CASA?

Algebraic curves and surfaces will become more and more important in application areas such as geometric modeling and robotics, see e.g. [Baj92a] and [Hof89]. The system SHASTRA, which has been developed by Bajaj [Baj92b], makes extensive use of algebraic curves and surfaces in modeling physical objects.

Computations with algebraic curves and surfaces are very well suited for being treated with computer algebra. Many aspects of computer algebra need to be combined for successfully solving problems in this area, e.g. computations with algebraic coefficients, solution of systems of linear and nonlinear polynomial equations, factorization and gcd computation over algebraic extensions of the rationals, etc.

The program system CASA (Computer Algebra Software for constructive Algebraic geometry) can perform computations and reasoning about geometric objects defined by algebraic equations.

CASA has been developed at RISC-LINZ over the last years by a research group under the direction of the author. Other major contributors to CASA have been R. Gebauer, M. Kalkbrener, M. Mňuk, J.R. Sendra, B. Wall, D. Wang. Earlier versions of CASA have been described in [Geb91] and [Wal93a].

The operations available in CASA include

- ideal theoretic operations $+$, $*$, \cap , $/$,
- creating algebraic sets in different representations,
- generating curves of fixed multiplicities at given points,
- intersection, union, and difference of algebraic sets,
- computing tangent cones and tangent spaces,
- computation of the dimension of an algebraic set,
- decomposition into irreducible components,
- transformations of algebraic sets to hypersurfaces,
- computation of the genus of curves,
- rational parametrization of curves,
- implicitization of parametrically given algebraic sets,
- Puiseux series expansions of algebraic curves,
- plotting both explicitly and implicitly given curves and surfaces.

CASA is built on top of the Maple computer algebra system. Currently the system CASA 2.1 runs under Maple V.2. A manual is available as [Mňu93]. CASA can be obtained by sending e-mail to the author.

1.2 Representation of algebraic curves

Although CASA is able to deal with algebraic sets in higher dimension, let us explain some of its internal representations by looking at algebraic curves. A curve might be represented in CASA in the following ways.

(a) *Implicit representation:* the curve C is given by a defining polynomial, e.g. as the zeros of the equation

$$f(x, y) = x^6 + 3x^4y^2 - 4x^2y^2 + 3x^2y^4 + y^6 = 0.$$

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See Figure 1 for a plot of this curve in the affine plane over \mathbf{R} .

(b) *Parametric representation*: C is a rational curve, so it can be defined as (the Zariski closure of) the points (x, y) resulting from giving special values to the parameter t in the rational functions

$$x(t) = \frac{\sqrt{2}}{250} \cdot \frac{p_1(t)}{n(t)}, \quad y(t) = \frac{\sqrt{2}}{250} \cdot \frac{p_2(t)}{n(t)},$$

where

$$\begin{aligned} p_1(t) &= 2144374784t^6 - 2104338432t^5 \\ &= -297826560t^4 + 151338240t^3 \\ &= +6914160t^2 - 2844072t + 35721, \\ p_2(t) &= 778047488t^6 - 1619994624t^5 \\ &= +804314880t^4 + 70606080t^3 \\ &= -32017680t^2 + 795096t - 5103, \\ n(t) &= 20123648t^6 - 5326848t^5 \\ &= +2467584t^4 - 366336t^3 \\ &= +81648t^2 - 5832t + 729. \end{aligned}$$

(c) *Approximation by Puiseux series*: locally a curve can be represented as a set of fractional power series (Puiseux series) approximations of the branches in the neighborhood of a point on the curve. So, for instance, the curve C can be expanded around $(0, 0)$ into the power series

$$\begin{aligned} (t, & \quad \frac{t^2}{2} + \frac{3t^4}{16} + \frac{39t^6}{256} + \frac{323t^8}{2048} + \dots), \\ (t, & \quad -\frac{t^2}{2} - \frac{3t^4}{16} - \frac{39t^6}{256} - \frac{323t^8}{2048} - \dots), \\ (\frac{t^2}{2}, & \quad t - \frac{3t^3}{16} - \frac{15t^5}{512} - \frac{77t^7}{8192} - \dots), \\ (-\frac{t^2}{2}, & \quad t - \frac{3t^3}{16} - \frac{15t^5}{512} - \frac{77t^7}{8192} - \dots) \end{aligned}$$

(d) *Projection onto a hypersurface*: a curve in 3-space can be mapped birationally to a plane algebraic curve. E.g., the space curve defined by

$$\begin{aligned} -y^2 + 2xy - z^2 + x + 1 &= 0, \\ 2y^4 - 8y^3z + 12y^2z^2 - 8yz^3 + 3x^4 \\ - 3y^2z + 6yz^2 - 5z^3 + z^2 &= 0 \end{aligned}$$

can be mapped birationally to the plane curve defined by

$$2u^4 + v^4 - 3u^2v - 2v^3 + v^2 = 0,$$

using the birational correspondence

$$x = u^2 - 1, \quad y = u + v, \quad z = v.$$

2 A CASA session

After having started Maple V.2, we invoke CASA.

```
> with(casa):
```

Welcome to CASA 2.1.

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For help type '?casa'.

We let the polynomial f define a curve C_1 .

C1im is the implicit representation of C_1 .

This curve is investigated in [Wal93b].

```
> read(f);
```

$$\begin{aligned} f := & -xy^4 + 2xy^5 + y^6x + 2y^8x - y^4x^3 - x^3y^2 \\ & + y^4x^4 + 4x^2y^4 - 2x^2y^3 - 2y^5x^2 + 3yx^5 + 7y^2x^4 \\ & - 4y^7x - 3y^6x^2 + 2y^5x^4 - 7yx^6 - 2y^3x^4 - 2y^2x^5 \\ & + 5x^3y^3 - 6y^5x^3 + 4y^4x^5 - 2y^8 + y^7 + y^9 - 2x^7 \\ & + 2x^8 \end{aligned}$$

```
> C1im:=mkImplAlgSet([f],[x,y]);
```

```
C1im := algebraic_set(
```

$$\begin{aligned} & [-xy^4 + 2xy^5 + y^6x + 2y^8x - y^4x^3 - x^3y^2 \\ & + y^4x^4 + 4x^2y^4 - 2x^2y^3 - 2y^5x^2 + 3yx^5 + 7y^2x^4 \\ & - 4y^7x - 3y^6x^2 + 2y^5x^4 - 7yx^6 - 2y^3x^4 - 2y^2x^5 \\ & + 5x^3y^3 - 6y^5x^3 + 4y^4x^5 - 2y^8 + y^7 + y^9 - 2x^7 \\ & + 2x^8], \end{aligned}$$

```
[x, y], attr)
```

Obviously the origin $(0,0)$ is a singularity

of C_1 . We expand the branches around $(0,0)$

into Puiseux series.

```
> C1plac:=convertRep(C1im,impl,plac,[0,0]);
```

```
C1plac := algebraic_set([
```

$$\begin{aligned} & [\text{proc}(\text{powparm}) \dots \text{end}, \text{proc}(\text{powparm}) \dots \text{end}], \\ & [\text{proc}(\text{powparm}) \dots \text{end}, \text{proc}(\text{powparm}) \dots \text{end}], \\ & [\text{proc}(\text{powparm}) \dots \text{end}, \text{proc}(\text{powparm}) \dots \text{end}], \\ & [\text{proc}(\text{powparm}) \dots \text{end}, \text{proc}(\text{powparm}) \dots \text{end}], \\ & [t], \text{attr}) \end{aligned}$$

The result consists of procedures for determining

the coefficients of these power series, which can

be displayed by the show-algebraic-set command.

```
> shAlgSet(C1plac,7);
```

The algebraic set is known to have the following properties:

It is given by the following places:

$$\begin{aligned} & [[t, t^2 - 2t^4 + 17t^6 + O(t^7)], \\ & [t, 2t^2 + 16t^4 + 112t^6 + O(t^7)], \\ & [t^2, -t^2 - t^3 - 1/2t^5 - 3/2t^6 + O(t^7)], \\ & [t^3, t + 1/3t^5 + O(t^7)]] \end{aligned}$$

CASA does reliable plots of implicitly defined

algebraic sets, in contrast to the plotting

routines implemented in Maple V.2.

```
> plotAlgSet(C1im,x=-2..2,y=-1..3,
```

```
numpoints=200);
```

The result is given in Figure 2.

Let us try to decompose C_1 into irreducible

components.

```
> decompose(C1im,absolute);
```

```
algebraic_set(
```

$$[x^4 - x^3 - 2x^2y - xy^2 + 2xy^4 + y^5], [x, y], \text{attr}),$$

```
algebraic_set(
```

$$[x^4 - 3/2x^2y + 1/2y^2 - y^3 + 1/2y^4], [x, y], \text{attr})$$

So C_1 is the union of two curves, C_2 and C_3 .

C_3 turns out to have genus 0, so it can be

rationally parametrized.

```
> C2im:="[1];
```

```
C2im := algebraic_set(
```

$$[x^4 - x^3 - 2x^2y - xy^2 + 2xy^4 + y^5], [x, y], \text{attr})$$

```
> C3im:=""[2];
```

```

C3im := algebraic_set(
  [x^4 - 3/2x^2y + 1/2y^2 - y^3 + 1/2y^4], [x, y], attr)
> genus(C2im);
3
> genus(C3im);
0
# We convert the implicit representation of C1
# into a parametric representation ...
> C3par:=convertRep(C3im,impl,para);
C3par := algebraic_set(
  [
    (t^3 - 6t^2 + 9t - 2) / (2t^4 - 16t^3 + 40t^2 - 32t + 9),
    (t^2 - 4t + 4) / (2t^4 - 16t^3 + 40t^2 - 32t + 9),
    [t], attr)
# ... and check the result.
> xpar:=represent(C3par)[1]:
> ypar:=represent(C3par)[2]:
> f3:=represent(C3im)[1]:
> simplify(subs({x=xpar,y=ypar},f3));
0
# Now let us consider a space curve C4 and
# a surface S.
> read(eqns);
> eqns:=[-y^2z^2 + 10y^2z - yz^2 - 25y^2
  + 6yz + x - 5y + 1,
  y^3z^2 - 10y^3z + 25y^3 - y^2z + 5y^2
  - yz + y]
> C4im:=mkImplAlgSet(eqns);
C4im := algebraic_set(
  [-y^2z^2 + 10y^2z - yz^2 - 25y^2 + 6yz + x - 5y + 1,
  y^3z^2 - 10y^3z + 25y^3 - y^2z + 5y^2 - yz + y],
  [x, y, z], attr)
> read(g);
g:= x^2y + y^4 - yz^2
> Sim:=mkImplAlgSet([g]);
Sim := algebraic_set([x^2y + y^4 - yz^2], [x, y, z], attr)
> plotAlgSet(Sim,x=-1..1,y=-1..1,z=-1..1,
  numpoints=40);
# The result is given in Figure 3.
> C5im:=implIntersect(C4im,Sim);
C5im := algebraic_set(
  [x^2y + y^4 - yz^2,
  -y^2z^2 + 10y^2z - yz^2 - 25y^2 + 6yz + x - 5y + 1,
  y^3z^2 - 10y^3z + 25y^3 - y^2z + 5y^2 - yz + y],
  [z, x, y], attr)
> dimension(C5im);
1
# So the intersection of C4 and S is a curve C5.
> decompose(C4im);
algebraic_set(
  [-1064y^2 - 16x^2y + 65z - 44x - 438y - 197yx

```

```

-306y^2x + 2zx + zx^2 - 8y^2x^2 - x^3y + 2x^3y^2
-108,
-11y^3x - 76y^3 + x^2y^3 - 15y^2x - 15y^2 - 3yx
-3y - x - 1,
4z^2 + zx - 23z - 2y^2x^2 + 22y^2x + 152y^2 + 41yx
+106y + 16x + 36 - x^2y,
20yz + zx + 5z - 152y^2 - x^2y - 54y - 19yx
-22y^2x + 2y^2x^2 - 4],
[x, y, z], attr),
algebraic_set([y, x + 1], [x, y, z], attr)
# Actually C5 is the union of a line and a curve C6.
> C6im:=mkImplAlgSet(Groebnerbasis("[1],
  plex));
C6im := algebraic_set(
  [x - yz^2 - z + 5yz + 2,
  25y^2 - z + y^2z^2 + 5y - 10y^2z - yz + 1],
  [x, y, z], attr)
# We convert C6 into projected representation,
# i.e. as a plane curve and a rational map into 3-space.
> C6proj:=convertRep(C6im,impl,proj);
C6proj := algebraic_set(
  [ [-174z + 91 - 59z^3 + 126z^2 - z^5 + 13z^4
  + (-114z - 3z^3 + 34z^2 + 95)x
  + (25 + z^2 - 10z)x^2 ],
  [
    (z^2x - 5zx - z + 2) / (z^2 - 5z - 1),
    (z - 2 - x) / (z^2 - 5z - 1),
    z],
  [x, z], attr)
# Converting back to implicit representation we get C6im.
> proj2impl(C6proj);
algebraic_set([2 - x3 + 5x3x2 - x3^2x2 + x1,
  1 - x3 + 5x2 - x3x2 - 10x3x2^2 + x3^2x2^2 + 25x2^2],
  [x1, x2, x3], attr)
# Now let us demonstrate some arithmetic operations
# on algebraic sets.
# We start from the curve C3.
> plotAlgSet(C3im,x=-2..2,y=-1..3,
  numpoints=200);
# The result is given in Figure 4.
# We consider three sets of points, P1, P2, P3.
# We add P3 to the algebraic set C3 and compute
# the implicit representation of this union V1.
# Then we subtract P2 from V1 and compute the
# Zariski closure V2 of the result.
# Finally we subtract P3 from V2 and compute the
# Zariski closure V3 of the result.
# V3 turns out to be the original curve C3.
> P1im:=mkImplAlgSet([x, (y-2)*(y+2)], [x, y]);
P1im := algebraic_set([x, (y-2) (y+2)], [x, y], attr)
> V1:=implUnionLCM(C7im,P1im);
V1 := algebraic_set(
  [-4y^2 + 8y^3 - 3y^4 + 12x^2y - 8x^4
  - 2y^5 + y^6 - 3x^2y^3 + 2x^4y^2,
  y^2x - 2y^3x + y^4x - 3x^3y + 2x^5],
  [x, y], attr)
> P2im:=mkImplAlgSet([x,y*(y-2)], [x, y]);

```

```

P2im := algebraic_set([x, y (y - 2)], [x, y], attr)
> V2:=implIdealQuo(V,P2im);
V2 := algebraic_set(
  [y5 - 3y3 + 2y2 - 3y2x2 + 2x4y - 6x2y + 4x4,
  y2x - 2y3x + y4x - 3x3y + 2x5 ],
  [x, y], attr)
> P3im:=mkImplAlgSet([x,y*(y+2)], [x,y]);
P3im := algebraic_set([x, y (y + 2)], [x, y], attr)
> V3:=implIdealQuo(V1,P3im);
V3 := algebraic_set(
  [2x4 - 3x2y + y2 - 2y3 + y4 ], [x, y], attr)

```

3 References

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Figure 2

Figure 3

Figure 4

Figure 1