A Full System of Invariants for Third-Order Linear Partial Differential Operators in General Form.

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Abstract. We find a full system of invariants with respect to gauge transformations $L \to g^{-1}Lg$ for third-order hyperbolic linear partial differential operators on the plane. The operators are considered in a normalized form, in which they have the symbol $\mathrm{Sym}_L = (pX + qY)XY$ for some non-zero bivariate functions p and q. For this normalized form, explicit formulae are given. The paper generalizes a previous result for the special, but important, case $p = q = 1$.

Key words: Linear Partial Differential Operators, Invariants, Gauge transformations

1 Introduction

For a second-order hyperbolic Linear Partial Differential Operators (LPDOs) on the plane in the normalized form

$$L = D_x \circ D_y + aD_x + bD_y + c \ , \tag{1}$$

where $a = a(x, y), b = b(x, y), c = c(x, y)$, it has been known for several centuries that the quantities

$$h = c - a_x - ab, \quad k = c - b_y - ab \tag{2}$$

are its invariants with respect to the gauge transformations $L \to g^{-1}Lg$. These two invariants were proved [2] to form together a full system of invariants for operators of the form (1). Thus, if two operators of the form (1) are known to have the same invariants h and k, then one may conclude that the operators are equivalent with respect to such transformations. Any other invariant of the operator, as well as all of its invariant properties, can be expressed in terms of h and k.

The case of operators of order two has been actively investigated. For example, we can note the classical Laplace hyperbolic second-order LPDOs, scalar hyperbolic non-linear LPDOs, and so on (sample references include [1, 3, 5]). For the case of hyperbolic operators of high orders, however, not much is known. A method for obtaining some invariants for a hyperbolic operator of arbitrary order was mentioned in [8]. In the paper [4] a method to compute some invariants for operators of order three was suggested.

Although the determination of some particular invariants is already important, there is an enormous area of applications for a full system of invariants. Whenever we have a full system of invariants for a certain class of LPDOs, we have an easy way to judge whether two operators of the class are equivalent, and it is possible to classify some of the corresponding partial differential equations in terms of their invariants. Thus, for example, classification has an immediate application to the integration of PDEs. Indeed, most integration methods work with operators given in some normalized form. Also a full system of invariants for a certain class of operators can be used for the description of all the invariant properties of the operators in terms of the invariants of the full system.

For third-order operators of the form

$$L = (D_x + D_y)D_x D_y + a_{20}D_x^2 + a_{11}D_{xy} + a_{02}D_y^2 + a_{10}D_x + a_{01}D_y + a_{00} \ , \quad (3)$$

where all the coefficients are functions in x and y, a full system of invariants was obtained in [6]. This is a special case — albeit an important one — of a general normalized form for a third-order hyperbolic bivariate LPDO. Indeed, the symbol of the normalized form of such operators has the form $(X + qY)XY$, where $q = q(x, y)$ is not zero.

Full systems of invariants have important applications, such as classification, integration algorithms, etc. So one needs them for as general a class of LPDOs as possible. In the present paper we establish a full system of invariants for operators of the form

$$L = (pD_x + qD_y)D_x D_y + a_{20}D_x^2 + a_{11}D_{xy} + a_{02}D_y^2 + a_{10}D_x + a_{01}D_y + a_{00} \ , \quad (4)$$

where $p = p(x, y)$ and $q = q(x, y)$ are not zero (Theorem 4).

2 Preliminaries

We consider a field K with a set $\Delta = \{\partial_1, \ldots, \partial_n\}$ of commuting derivations acting on it, and work with the ring of linear differential operators $K[D] = K[D_1, \ldots, D_n]$, where $D_1, \ldots, D_n$ correspond to the derivations $\partial_1, \ldots, \partial_n$, respectively.

Any operator $L \in K[D]$ is of the form

$$L = \sum_{|J| \le d} a_J D^J \ , \quad (5)$$

where $a_J \in K,\ J \in \mathbb{N}^n$ and $|J|$ is the sum of the components of J. Then we say that the polynomial

$$\mathrm{Sym}_L = \sum_{|J| = d} a_J X^J$$

is *the symbol* of L. Let K^* denotes the set of invertible elements in K. Then for $L \in K[D]$ and every $g \in K^*$ there is a gauge transformation

$$L \to g^{-1} L g \ .$$

We also can say that this is the operation of conjugation. Then an algebraic differential expression I in the coefficients appearing in L is invariant under the gauge transformations if it is unaltered under these transformations. Trivial examples of an invariant are coefficients of the symbol of the operator.

An operator $L \in K[D]$ is said to be hyperbolic if its symbol is completely factorable (all factors are of first order) and each factor has multiplicity one.

3 Obstacles to Factorizations and Their Invariance

In this section we briefly recapitulate a few results from [7], because they are essential to the next sections.

Definition 1. *Let $L \in K[D]$ and suppose that its symbol has a decomposition* $\mathrm{Sym}_L = S_1 \dots S_k$*. Then we say that the factorization*

$$L = F_1 \circ \ldots \circ F_k, \quad \textit{where} \quad \mathrm{Sym}_{F_i} = S_i \ , \ \forall i \in \{1, \ldots, k\}, \tag{6}$$

is of the factorization type $(S_1)(S_2) \ldots (S_k)$.

Definition 2. *Let $L \in K[D]$, $\mathrm{Sym}_L = S_1 \dots S_k$. An operator $R \in K[D]$ is called a* common obstacle *to factorization of the type $(S_1)(S_2) \ldots (S_k)$ if there exists a factorization of this type for the operator $L - R$ and R has minimal possible order.*

Remark 1. In general a common obstacle to factorizations of some factorization type is not unique.

Example 1. Consider a hyperbolic operator

$$L = D_{xy} - aD_x - bD_y - c,$$

where $a, b, c \in K$. An operator P_1 (in this particular case it is an operator of multiplication by a function) is a common obstacle to factorizations of the type $(X)(Y)$ if there exist $g_0, h_0 \in K$ such that

$$L - P_1 = (D_x - g_0) \circ (D_y - h_0).$$

Comparing the terms on the two sides of the equation, one gets $g_0 = b, h_0 = a$, and

$$P_1 = a_x - ab - c.$$

Analogously, we get a common obstacle to factorization of the type $(Y)(X)$:

$$P_2 = b_y - ab - c,$$

and the corresponding factorization for $(L - P_2)$: $L - P_2 = (D_x - a) \circ (D_y - b)$. Thus, the obtained common obstacles P_1 and P_2 are the Laplace invariants [2].

Theorem 1. *Consider a separable operator $L \in K[D_x, D_y]$ of order d, and the factorizations of L into first-order factors. Then*

1. *the order of common obstacles is less than or equal to $d-2$;*
2. *a common obstacle is unique for each factorization type;*
3. *there are $d!$ common obstacles;*
4. *if $d = 2$, then the common obstacles of order 0 are the Laplace invariants;*
5. *the symbol of a common obstacle is an invariant.*

Corollary 1. *For an LPDO of the form*

$$L = (pD_x + qD_y)D_xD_y + a_{20}D_x^2 + a_{11}D_{xy} + a_{02}D_y^2 + a_{10}D_x + a_{01}D_y + a_{00} \ , \quad (7)$$

where all the coefficients belong to K, and p, q are not zero, consider its factorizations into first-order factors. Then

1. *the order of common obstacles is zero or one;*
2. *a common obstacle is unique for each factorization type, and therefore, the corresponding obstacles consist of just one element;*
3. *there are 6 common obstacles to factorizations into exactly three factors;*
4. *the symbol of a common obstacle is an invariant with respect to the gauge transformations $L \to g^{-1}Lg$.*

4 Computing of Invariants

Consider the operator (7). Since the symbol of an LPDO does not change under the gauge transformations $L \to g^{-1}Lg$, then the symbol, and therefore the coefficients of the symbol, are invariants with respect to these transformations. Thus, p and q are invariants.

Now we use Corollary 1 to compute a number of invariants for the operator L. Suppose for a while that

$$p = 1 \ .$$

Denote the factors of the symbol $\mathrm{Sym}_L = (X + qY)XY$ of L by

$$S_1 = X, \ S_2 = Y, \ S_3 = X + qY \ .$$

Denote the common obstacle to factorizations of the type $(S_i)(S_j)(S_k)$ by Obst_{ijk}.

Then the coefficient of Y in the symbol of the common obstacle Obst_{123} is

$$(a_{01}q^2 + a_{02}^2 - (3q_x + a_{11}q)a_{02} + q_xqa_{11} - \partial_x(a_{11})q^2 + q\partial_x(a_{02}) + 2q_x^2 - q_{xx})/q^2 \ .$$

By Theorem 1, this expression is invariant with respect to gauge transformations $L \to g^{-1}Lg$. Since the term $(2q_x^2 - q_{xx})/q^2$ and multiplication by q^2 does not influence the invariance property (because q is an invariant), the following expression is invariant also:

$$I_4 = a_{01}q^2 + a_{02}^2 - (3q_x + a_{11}q)a_{02} + q_xqa_{11} - \partial_x(a_{11})q^2 + q\partial_x(a_{02}) \ .$$

The coefficient of Y in the symbol of the common obstacle Obst_{213} is

$$(I_4 - (\partial_x(a_{20})q^2 - \partial_y(a_{02})q + a_{02}q_y)q + a_{02}q_y)q - q_x q_y q + q_{xy}q^2 + 2q_x^2 - q_{xx}q)/q^2 \ .$$

Again the expressions in q can be omitted, while I_4 is itself an invariant. Therefore,

$$I_2 = \partial_x(a_{20})q^2 - \partial_y(a_{02})q + a_{02}q_y$$

is an invariant.

Similarly, we obtain the invariants

$$\begin{aligned} I_1 &= 2a_{20}q^2 - a_{11}q + 2a_{02} \ , \\ I_3 &= a_{10} + a_{20}(qa_{20} - a_{11}) + \partial_y(a_{20})q - \partial_y(a_{11}) + 2a_{20}q_y \ . \end{aligned}$$

Generally speaking, by Corollary 1, there are six different obstacles to factorizations into exactly three factors. In fact, all the coefficients of the symbols of the common obstacles can be expressed in terms of four invariants

$$I_1, I_2, I_3, I_4 \ .$$

Denote the symbol of the common obstacle Obst_{ijk} by Sym_{ijk}. Direct computations justify the following theorem:

Theorem 2.

$$\begin{aligned}
q^2\mathrm{Sym}_{123} &= (q^2 I_3 + I_2 - q_{xy}q + q_{yy}q^2 + q_x q_y)D_x &&+ (I_4 + 2q_x^2 - q_{xx})D_y \ , \\
q^2\mathrm{Sym}_{132} &= (i_2 + I_2)D_x &&+ (I_4 + 2q_x^2 - q_{xx})D_y \ , \\
q^2\mathrm{Sym}_{213} &= (q^2 I_3 + q^2 q_{yy})D_x &&+ i_3 D_y \ , \\
q^2\mathrm{Sym}_{231} &= (q^2 I_3 + q^2 q_{yy})D_x &&+ i_1 D_y \ , \\
q^2\mathrm{Sym}_{312} &= (i_2 + I_2)D_x &&+ (i_1 + I_2 q)D_y \ , \\
q^2\mathrm{Sym}_{321} &= i_2 D_x &&+ i_1 D_y \ ,
\end{aligned}$$

where

$$\begin{aligned} i_1 &= I_4 - 2\partial_x(I_1)q + 4q_x I_1 - 2I_2 q \ , \\ i_2 &= q^2 I_3 - 2\partial_y(I_1)q + 2I_1 q_y + I_2 \ , \\ i_3 &= I_4 - I_2 q - q_x q_y q + q_{xy}q^2 + 2q_x^2 - q_{xx}q \ . \end{aligned}$$

Note that neither of the obtained invariants I_1, I_2, I_3, I_4 depends on the "free" coefficient a_{00} of the operator L, and, therefore, we need at least one more invariant.

We guess the form of the fifth invariant by analyzing the structure of invariant

$$I_5 = a_{00} - a_{01}a_{20} - a_{10}a_{02} + a_{02}a_{20}a_{11} + (2a_{02} - a_{11} + 2a_{20})\partial_x(a_{20}) + \partial_{xy}(a_{20} - a_{11} + a_{02})$$

of the case $p = 1$, $q = 1$, considered in [6], and then perform some elimination. One of the difficulties here lies in the handling of large expressions, which appear during such manipulations. Naturally, a computer algebra system is needed, and we used Maple running our own package for linear partial differential operators with parametric coefficients. Thus, we get several candidates to be the fifth invariant. The most convenient of them has the form

$$I_5 = a_{00} - \frac{1}{2}\partial_{xy}(a_{11}) + q_x\partial_y(a_{20}) + q_{xy}a_{20} + \\ \left(2qa_{20} + \frac{2}{q}a_{02} - a_{11} + q_y\right)\partial_x(a_{20}) - \frac{1}{q}a_{02}a_{10} - a_{01}a_{20} + \frac{1}{q}a_{20}a_{11}a_{02} \ .$$

5 A Full System of Invariants for Third Order LPDOs

Here we prove that the obtained five invariants together form a full system of invariants for the case of operators with the symbol $(X + qY)XY$, and then, as the consequence, obtain a full system of invariants for operators with the symbol $(pX + qY)XY$.

One can prove that invariants I_1, I_2, I_3, I_4, I_5 form a full system in a similar way to that which was done for invariants of operators with the symbol $(X + Y)XY$ [6]. Below we suggest a simplification of such a way of proving, even though we consider a more general case.

Theorem 3. *For some non-zero $q \in K$, consider the operators of the form*

$$L = (D_x + qD_y)D_xD_y + a_{20}D_x^2 + a_{11}D_{xy} + a_{02}D_y^2 + a_{10}D_x + a_{01}D_y + a_{00} \ , \quad (8)$$

where the coefficients belong to K. Then the following is a full system of invariants of such an operator with respect to the gauge transformations $L \to g^{-1}Lg$:

$$\begin{aligned} I_1 &= 2a_{20}q^2 - a_{11}q + 2a_{02} \ , \\ I_2 &= \partial_x(a_{20})q^2 - \partial_y(a_{02})q + a_{02}q_y \ , \\ I_3 &= a_{10} + a_{20}(qa_{20} - a_{11}) + \partial_y(a_{20})q - \partial_y(a_{11}) + 2a_{20}q_y \ , \\ I_4 &= a_{01}q^2 + a_{02}^2 - (3q_x + a_{11}q)a_{02} + q_xqa_{11} - \partial_x(a_{11})q^2 + q\partial_x(a_{02}) \ , \\ I_5 &= a_{00} - \frac{1}{2}\partial_{xy}(a_{11}) + q_x\partial_y(a_{20}) + q_{xy}a_{20} + \\ &\quad \left(2qa_{20} + \frac{2}{q}a_{02} - a_{11} + q_y\right)\partial_x(a_{20}) - \frac{1}{q}a_{02}a_{10} - a_{01}a_{20} + \frac{1}{q}a_{20}a_{11}a_{02} \ . \end{aligned}$$

Thus, an operator $L' \in K[D]$

$$L' = (D_x + qD_y)D_xD_y + b_{20}D_x^2 + b_{11}D_xD_y + b_{02}D_y^2 + b_{10}D_x + b_{01}D_y + b_{00} \quad (9)$$

is equivalent to L (with respect to the gauge transformations $L \to g^{-1}Lg$) if and only if their corresponding invariants I_1, I_2, I_3, I_4, I_5 are equal.

Remark 2. Since the symbol of an LPDO L does not alter under the gauge transformations $L \to g^{-1}Lg$, we consider the operators with the same symbol.

Proof. 1. The direct computations show that the five expressions from the statement of the theorem are invariants with respect to the gauge transformations $L \to g^{-1}Lg$. One just has to check that these expressions do not depend on g, when calculate them for the operator $g^{-1}Lg$. Basically, we have to check the fifth expression I_5 only, since the others are invariants by construction.

2. Prove that these five invariants form a complete set of invariants, in other words, the operators L and L' are equivalent (with respect to the gauge transformations $L \to g^{-1}Lg$) if and only if their corresponding invariants are equal.

The direction "$\Rightarrow$" is implied from 1. Prove the direction "$\Leftarrow$". Let

$$I'_1, I'_2, I'_3, I'_4, I'_5$$

be the invariants computed from the coefficients of the operator L' by the formulas from the statement of the theorem, and

$$I_i = I'_i, \ i = 1, 2, 3, 4, 5 \ . \tag{10}$$

Look for a function $g = e^f, \ f, g \in K$, such that

$$g^{-1}Lg = L' \ . \tag{11}$$

Equate the coefficients of D_{xx}, D_{yy} on both sides of (11), and get

$$\partial_y(f) = b_{20} - a_{20} \ , \tag{12}$$
$$\partial_x(f) = (b_{02} - a_{02})/q \ . \tag{13}$$

In addition, the assumption $I_2 = I'_2$ implies

$$(b_{20} - a_{20})_x = ((b_{02} - a_{02})/q)_y .$$

Therefore, there is only one (up to a multiplicative constant) function f, which satisfies the conditions (12) and (13).

Consider such a function f. Then substitute the expressions

$$b_{20} = a_{20} + f_y \ , \tag{14}$$
$$b_{02} = a_{02} + qf_x \ . \tag{15}$$

for b_{20}, b_{02} in (11), and prove that it holds for $g = e^f$.

Subtracting the coefficients of D_{xy} in $g^{-1}Lg$ from that in L' we get

$$b_{11} - a_{11} - 2f_x - 2qf_y \ ,$$

which equals

$$2q(I_1 - I'_1) \ ,$$

which is zero by the assumption (10). Now we can substitute

$$b_{11} = a_{11} + 2f_x + 2qf_y \ .$$

Analogously, subtracting the coefficients of D_x,D_y in $g^{-1}Lg$ from those in L', correspondingly, we get

$$b_{10} - a_{10} - 2a_{20}f_x - a_{11}f_y - 2f_{xy} - 2f_xf_y - qf_{yy} - qf_y^2 = I_3' - I_3 = 0 \ ,$$
$$b_{01} - a_{01} - 2a_{02}f_y - a_{11}f_x - 2qf_{xy} - 2qf_xf_y - f_{xx} - f_x^2 = I_4' - I_4 = 0 \ .$$

Now we can express b_{10} and b_{01}. Now, subtracting the "free" coefficient of $g^{-1}Lg$ from that of L', we get

$$b_{00} - a_{00} - a_{10}f_x - a_{01}f_y - a_{20}(f_{xx} + f_x^2) - a_{11}(f_{xy} + f_xf_y) - a_{02}(f_{yy} + f_y^2) - f_{xxy} - 2f_{xy}f_x - f_yf_{xx} - f_yf_x^2 - qf_xf_{yy} - qf_xf_y^2 - qf_{xyy} - 2qf_yf_{xy} = I_5' - I_5 = 0 \ .$$

Thus, we proved that for the chosen function f, the equality (11) holds, and therefore, the operators L and L' are equivalent.

Remark 3. The Theorem 3 is a generalization of the result of [6], where the case $q = 1$ is considered.

Thus, a full system of invariants for the case $p = 1$ has been found. Now we give the formulae for the general case.

Theorem 4. *For some non-zero $p, q \in K$ consider the operators of the form*

$$L = (pD_x + qD_y)D_xD_y + a_{20}D_x^2 + a_{11}D_{xy} + a_{02}D_y^2 + a_{10}D_x + a_{01}D_y + a_{00} \ , \quad (16)$$

where the coefficients belong to K. Then the following is a full system of invariants of such an operator with respect to the gauge transformations $L \to g^{-1}Lg$:

$$I_1 = 2a_{20}q^2 - a_{11}pq + 2a_{02}p^2 \ ,$$
$$I_2 = \partial_x(a_{20})pq^2 - \partial_y(a_{02})p^2q + a_{02}p^2q_y - a_{20}q^2p_x \ ,$$
$$I_3 = a_{10}p^2 - a_{11}a_{20}p + 2a_{20}q_yp - 3a_{20}qp_y + a_{20}^2q - \partial_y(a_{11})p^2 + a_{11}p_yp + \partial_y(a_{20})pq \ ,$$
$$I_4 = a_{01}q^2 - a_{11}a_{02}q + 2a_{02}qp_x - 3a_{02}pq_x + a_{02}^2p - \partial_x(a_{11})q^2 + a_{11}q_xq + \partial_x(a_{02})pq \ ,$$
$$\begin{aligned} I_5 = {} & a_{00}p^3q - p^3a_{02}a_{10} - p^2qa_{20}a_{01} + \\ & (pI_1 - pq^2p_y + qp^2q_y)a_{20x} + (qq_xp^2 - q^2p_xp)a_{20y} \\ & + (4q^2p_xp_y - 2qp_xq_yp + qq_{xy}p^2 - q^2p_{xy}p - 2qq_xpp_y)a_{20} \\ & + (\frac{1}{2}p_{xy}p^2q - p_xp_ypq)a_{11} - \frac{1}{2}p^3qa_{11xy} + \frac{1}{2}a_{11x}p_yp^2q + \frac{1}{2}a_{11y}p_xp^2q \\ & + p^2a_{02}a_{20}a_{11} + pqp_xa_{20}a_{11} - 2p_xq^2a_{20}^2 - 2p^2p_xa_{20}a_{02} \ . \end{aligned}$$

Proof. Since $p \neq 0$ we can multiply (16) by p^{-1} on the right, and get some new operator

$$L_1 = (D_x + \frac{q}{p}D_y)D_xD_y + \frac{a_{20}}{p}D_x^2 + \frac{a_{11}}{p}D_{xy} + \frac{a_{02}}{p}D_y^2 + \frac{a_{10}}{p}D_x + \frac{a_{01}}{p}D_y + \frac{a_{00}}{p} \ .$$

The invariants of the operator L and L_1 are the same. We compute the invariants of the operator L_1 by the formulae of Theorem 3, and get the invariants of the statement of the current theorem up to multiplication by integers and p, q.

Example 2. For some $p, q, c \in K$ consider the simple operator

$$L = (pD_x + qD_y)D_xD_y + c \ . \tag{17}$$

Compute the system of invariants of Theorem 4 for L:

$$\begin{aligned} 0 = I_1 = I_2 = I_3 = I_4 \ , \\ I_5 = p^3qc \ . \end{aligned}$$

Thus, every LPDO in $K[D_x, D_y]$ with the symbol $XY(pX + qY)$ that has the same set of invariants is equivalent to the simple operator (17). In fact, LPDOs that are equivalent to the operator (17) are not always trivial looking. Such operators have the form

$$\begin{aligned} L = {} & (pD_x + qD_y)D_xD_y + pf_yD_x^2 + (2pf_x + 2qf_y)D_{xy} + qf_xD_y^2 + \\ & (2pf_{xy} + 2pf_xf_y + qf_{yy} + qf_yf_y)D_x + (pf_{xx} + pf_xf_x + 2qf_{xy} + 2qf_xf_y)D_y + \\ & c + pf_{xxy} + 2pf_{xy}f_x + pf_yf_{xx} + pf_yf_x^2 + qf_xf_{yy} + qf_xf_y^2 + qf_{xyy} + 2qf_yf_{xy} \ , \end{aligned}$$

for some $f \in K$.

6 Conclusion

For operators of the form

$$L = (pD_x + qD_y)D_xD_y + a_{20}D_x^2 + a_{11}D_{xy} + a_{02}D_y^2 + a_{10}D_x + a_{01}D_y + a_{00} \ , \tag{18}$$

where all the coefficients belong to K, we have found five invariants with respect to the gauge transformations $L \to g^{-1}Lg$ and proved that together they form a full system of operators.

In fact, Theorem 1 provides a way to find a number of invariants for hyperbolic bivariate LPDOs of arbitrary order, rather than just for those of order three. One of the difficulty lies in very large expressions, which appear already for third-order operators. Moreover, even if one manages to compute them, in general one gets a number of very large expressions. Then a challenge is to extract some nice looking invariants out of those large ones, so that these nice looking invariants generate the obtained ones. Thus, for the case of third-order LPDOs, we extracted four invariants out of twelve ones.

Another problem is that for applications one rather needs a full system of invariants. Thus, for the considered operators (18) we had to find a fifth invariant. However, even in this case it was not easy. Also for operators of high order, one needs to find more than one invariants so that they together with the obtained from obstacles ones form a full system.

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