Motives, Quantum Field Theory, and Pseudodifferential Operators

Boston University June 2-13, 2008

Difference Field Algorithms and Algebraic Independence of Nested Sums and Products

Carsten Schneider
RISC-Linz, Austria

9. June 2008

Starting point:
bonus problem 6.69 in “Concrete Mathematics”

FIND a closed form for

$$\sum_{k=1}^{n} k^2 \mathsf{H}_{n+k} = \text{?},$$

where $\mathsf{H}_n := \sum_{k=1}^{n} \frac{1}{k}$.

D.E. Knuth:

> *“It would be nice to automate the derivation of formulas such as this.”*

Telescoping

GIVEN $f(k) = k^2 H_{n+k}$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

Telescoping

GIVEN $f(k) = k^2 H_{n+k}$.

FIND $g(k)$:

$$f(k) = g(k+1) - g(k)$$

Sigma computes

$$g(k) = \frac{1}{36}(k(-4k^2 + (6n+3)k - 12n(n+1) + 1) + 6(2k^3 - 3k^2 + k + n(n+1)(2n+1))H_{n+k}).$$

Telescoping

GIVEN $f(k) = k^2 H_{n+k}$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

Summing the telescoping equation over k from 1 to n gives

$$\boxed{\sum_{k=1}^{n} k^2 \mathsf{H}_{n+k}} = \boxed{g(n+1) - g(1)}$$

$$= -\frac{1}{36} n(n+1)\left(10n + 6(2n+1)H_n - 12(2n+1)H_{2n} - 1\right).$$

Telescoping

GIVEN $f(k) = k^2 H_{n+k}$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

A difference field for the **summand:**

Take the rational function field

$(\mathbb{F}, \sigma)$ is a $\Pi\Sigma$-field

$$\mathbb{F} := \mathbb{Q}(n)(k)(h)$$

Karr 1981

and the field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),$$

$$\sigma(k) = k + 1, \qquad\qquad \mathcal{S}\, k = k + 1,$$

$$\sigma(h) = h + \frac{1}{n+k+1}, \qquad\qquad \mathcal{S}\, H_{n+k} = H_{n+k} + \frac{1}{n+k+1}.$$

Telescoping

GIVEN $f(k) = k^2 H_{n+k}$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

$$\downarrow$$

GIVEN $f := k^2 h \in \mathbb{F}$.

FIND $g \in \mathbb{F}$:

$$\boxed{f = \sigma(g) - g}$$

Telescoping

GIVEN $f(k) = k^2 H_{n+k}$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

$$\downarrow$$

GIVEN $f := k^2 h \in \mathbb{F}$.
FIND $g \in \mathbb{F}$:

$$\boxed{f = \sigma(g) - g}$$

$\downarrow$ Sigma

$$g = \frac{1}{36}(k(-4k^2 + (6n+3)k - 12n(n+1) + 1) + 6(2k^3 - 3k^2 + k + n(n+1)(2n+1))h)$$

Telescoping

GIVEN $f(k) = k^2 H_{n+k}$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

$\downarrow$

GIVEN $f := k^2 h \in \mathbb{F}$.

FIND $g \in \mathbb{F}$:

$$\boxed{f = \sigma(g) - g}$$

$\downarrow$ Sigma

h$\equiv H_{n+k}$ $\uparrow$

$$g = \frac{1}{36}(k(-4k^2 + (6n+3)k - 12n(n+1) + 1) + 6(2k^3 - 3k^2 + k + n(n+1)(2n+1))h)$$

The simple approach

GIVEN a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ such that $f \in \mathbb{F}$.

FIND $g \in \mathbb{F}$:

$$\sigma(g) - g = f.$$

Aspect 1: Refined difference field theory for summation

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ such that $f \in \mathbb{F}$.

2. FIND $g \in \mathbb{F}$:

$$\sigma(g) - g = f.$$

Aspect 1: Refined difference field theory for summation

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ such that $f \in \mathbb{F}$.

2. FIND an appropriate extension $\mathbb{E} > \mathbb{F}$ with $g \in \mathbb{E}$:

$$\sigma(g) - g = f.$$

Aspect 1: Refined difference field theory for summation

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F},\sigma)$ such that $f\in\mathbb{F}$.

2. FIND an appropriate extension $\mathbb{E}>\mathbb{F}$ with $g\in\mathbb{E}$:

$$\sigma(g)-g=f.$$

appropriate = sum representations with optimal nested depth

Example:

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{i}}{j}}{k}=\frac{1}{6}\left(\sum_{i=1}^{n}\frac{1}{i}\right)^3+\frac{1}{2}\left(\sum_{i=1}^{n}\frac{1}{i^2}\right)\left(\sum_{i=1}^{n}\frac{1}{i}\right)+\frac{1}{3}\sum_{i=1}^{n}\frac{1}{i^3}$$

depth 3 depth 1

Aspect 2: Proving algebraic independence of sums

- Let $(\mathbb{F}, \sigma)$ be a difference field with constant field

$$\text{const}_\sigma\mathbb{F} := \{k \in \mathbb{F} | \sigma(k) = k\}.$$

Aspect 2: Proving algebraic independence of sums

- Let $(\mathbb{F}, \sigma)$ be a difference field with constant field

$$\text{const}_\sigma\mathbb{F} := \{k \in \mathbb{F} | \sigma(k) = k\}.$$

- Let $(\mathbb{F}(t), \sigma')$ be a difference field s.t.
  1. $\mathbb{F}(t)$ is a field extension of $\mathbb{F}$,
  2. $\sigma'(h) = \sigma(h)$ for all $h \in \mathbb{F}$,
  3. $\sigma'(t) = t + f$ for some $f \in \mathbb{F}$.

Aspect 2: Proving algebraic independence of sums

- Let $(\mathbb{F}, \sigma)$ be a difference field with constant field

$$\text{const}_\sigma \mathbb{F} := \{k \in \mathbb{F} | \sigma(k) = k\}.$$

- Let $(\mathbb{F}(t), \sigma')$ be a difference field s.t.
  1. $\mathbb{F}(t)$ is a field extension of $\mathbb{F}$,
  2. $\sigma'(h) = \sigma(h)$ for all $h \in \mathbb{F}$,
  3. $\sigma'(t) = t + f$ for some $f \in \mathbb{F}$.

(Karr 1981) Then t is transcendental over $\mathbb{F}$ and $\text{const}_{\sigma'} \mathbb{F}(t) = \text{const}_\sigma \mathbb{F}$ iff

$$\nexists g \in \mathbb{F} : \quad \boxed{\sigma(g) - g = f}$$

Such an extension is called Σ^*-extension (the product case can be handled analogously with Π-extensions).

Aspect 2: Proving algebraic independence of sums

- Let $(\mathbb{F}, \sigma)$ be a difference field with constant field

$$\text{const}_\sigma \mathbb{F} := \{k \in \mathbb{F} | \sigma(k) = k\}.$$

- Let $(\mathbb{F}(t), \sigma')$ be a difference field s.t.
  1. $\mathbb{F}(t)$ is a field extension of $\mathbb{F}$,
  2. $\sigma'(h) = \sigma(h)$ for all $h \in \mathbb{F}$,
  3. $\sigma'(t) = t + f$ for some $f \in \mathbb{F}$.

(Karr 1981) Then t is transcendental over $\mathbb{F}$ and $\text{const}_{\sigma'} \mathbb{F}(t) = \text{const}_\sigma \mathbb{F}$ iff

$$\nexists g \in \mathbb{F} : \quad \boxed{\sigma(g) - g = f}$$

Example: The difference field $(\mathbb{Q}(n)(h_1)(h_2)\dots, \sigma)$ with $\sigma(n) = n + 1$ and

$$\sigma(h_1) = h_1 + \frac{1}{n+1}, \quad \sigma(h_2) = h_2 + \frac{1}{(n+1)^2}, \dots$$

can be constructed by an (infinite) tower of Σ^*-extensions.

The difference ring of sequences

- The set of sequences $\mathbb{K}^{\mathbb{N}}$ forms a commutative ring with component wise addition and multiplication:

$$(a_n)_{n\geq 0} + (b_n)_{n\geq 0} = (a_n + b_n)_{n\geq 0}, \quad (a_n)_{n\geq 0} \cdot (b_n)_{n\geq 0} = (a_n \cdot b_n)_{n\geq 0}.$$

The difference ring of sequences

- The set of sequences $\mathbb{K}^{\mathbb{N}}$ forms a commutative ring with component wise addition and multiplication:

$$(a_n)_{n\geq 0} + (b_n)_{n\geq 0} = (a_n + b_n)_{n\geq 0}, \quad (a_n)_{n\geq 0} \cdot (b_n)_{n\geq 0} = (a_n \cdot b_n)_{n\geq 0}.$$

- Problem: The shift is not inversive:

$$S((a_0, a_1, a_2, \dots)) = (a_1, a_2, \dots)$$
$$S^{-1}((a_1, a_2, \dots)) = (*, a_1, a_2, \dots)$$

The difference ring of sequences

- The set of sequences $\mathbb{K}^{\mathbb{N}}$ forms a commutative ring with component wise addition and multiplication:

$$(a_n)_{n\geq 0} + (b_n)_{n\geq 0} = (a_n + b_n)_{n\geq 0}, \quad (a_n)_{n\geq 0} \cdot (b_n)_{n\geq 0} = (a_n \cdot b_n)_{n\geq 0}.$$

- Problem: The shift is not inversive:

$$S((a_0, a_1, a_2, \dots)) = (a_1, a_2, \dots)$$
$$S^{-1}((a_1, a_2, \dots)) = (*, a_1, a_2, \dots)$$

- Trick: We identify two sequences, if they agree from a certain point on:

$$(a_n)_{n\geq 0} \sim (b_n)_{n\geq 0} \quad \Leftrightarrow \quad \exists \delta \geq 0 \; \forall n \geq \delta : a_n = b_n.$$

The difference ring of sequences

- The set of sequences $\mathbb{K}^{\mathbb{N}}$ forms a commutative ring with component wise addition and multiplication:

$$(a_n)_{n\geq 0} + (b_n)_{n\geq 0} = (a_n + b_n)_{n\geq 0}, \quad (a_n)_{n\geq 0} \cdot (b_n)_{n\geq 0} = (a_n \cdot b_n)_{n\geq 0}.$$

- Problem: The shift is not inversive:

$$S((a_0, a_1, a_2, \dots)) = (a_1, a_2, \dots)$$
$$S^{-1}((a_1, a_2, \dots)) = (*, a_1, a_2, \dots)$$

- Trick: We identify two sequences, if they agree from a certain point on:

$$(a_n)_{n\geq 0} \sim (b_n)_{n\geq 0} \quad \Leftrightarrow \quad \exists \delta \geq 0 \, \forall n \geq \delta : a_n = b_n.$$

Then $S : \mathbb{K}^{\mathbb{N}}/\sim \to \mathbb{K}^{\mathbb{N}}/\sim$ forms a ring automorphism.
$(\mathbb{K}^{\mathbb{N}}/\sim, S)$ is called the difference ring of sequences.

Embedding into the difference ring of sequences

Consider our $\Pi\Sigma^*$-field $(\mathbb{Q}(n)(h_1)(h_2)\ldots,\sigma)$ with $\sigma(n)=n+1$ and

$$\sigma(h_1)=h_1+\frac{1}{n+1},\quad \sigma(h_2)=h_2+\frac{1}{(n+1)^2},\ldots$$

1. There is an embedding τ of $(\mathbb{Q}(n)[h_1,h_2,\ldots],\sigma)$ into $(\mathbb{Q}^{\mathbb{N}}/\sim,S)$ with

$$h_1\longrightarrow\left\langle\sum_{i=1}^{n}\frac{1}{i}\right\rangle_{n\geq 0},\qquad h_2\longrightarrow\left\langle\sum_{i=1}^{n}\frac{1}{i^2}\right\rangle_{n\geq 0}\qquad\ldots$$

In particular,

$$(\mathbb{Q}(n)[h_1,h_2,\ldots],\sigma)\cong(\tau(\mathbb{Q}(n)[h_1,h_2,\ldots]),S)$$

Embedding into the difference ring of sequences

Consider our $\Pi\Sigma^*$-field $(\mathbb{Q}(n)(h_1)(h_2)\dots,\sigma)$ with $\sigma(n)=n+1$ and

$$\sigma(h_1)=h_1+\frac{1}{n+1},\quad \sigma(h_2)=h_2+\frac{1}{(n+1)^2},\dots$$

1. There is an embedding τ of $(\mathbb{Q}(n)[h_1,h_2,\dots],\sigma)$ into $(\mathbb{Q}^{\mathbb{N}}/\sim,S)$ with

$$h_1\longrightarrow\left\langle\sum_{i=1}^{n}\frac{1}{i}\right\rangle_{n\geq 0},\qquad h_2\longrightarrow\left\langle\sum_{i=1}^{n}\frac{1}{i^2}\right\rangle_{n\geq 0}\qquad\dots$$

$\Rightarrow$ The generalized harmonic numbers

$$H_n=\sum_{i=1}^{n}\frac{1}{i},\quad H_n^{(2)}=\sum_{i=1}^{n}\frac{1}{i^2},\quad H_n^{(3)}=\sum_{i=1}^{n}\frac{1}{i^3},\quad\dots$$

are algebraic independent over the rational sequences $\mathbb{Q}(n)$.

Aspect 3: Algorithmic summation paradigms

GIVEN a definite sum

$$S(n) := \sum_{k=0}^{n} f(n, k).$$

FIND a recurrence for $S(n)$ in n (creative extension of telescoping).

Aspect 3: Algorithmic summation paradigms

GIVEN a definite sum

$$S(n) := \sum_{k=0}^{n} f(n, k).$$

FIND a recurrence for $S(n)$ in n (creative extension of telescoping).

$$\downarrow$$

GIVEN a recurrence

$$a_d(n)S(n+d) + \cdots + a_1(n)S(n+1) + a_0(n)S(n) = h(n).$$

FIND all solutions in $\Pi\Sigma$-extensions (d'Alembertian solutions).

Aspect 3: Algorithmic summation paradigms

GIVEN a definite sum

$$S(n) := \sum_{k=0}^{n} f(n,k).$$

FIND a recurrence for $S(n)$ in n (creative extension of telescoping).

↓

GIVEN a recurrence

$$a_d(n)S(n+d) + \cdots + a_1(n)S(n+1) + a_0(n)S(n) = h(n).$$

FIND all solutions in $\Pi\Sigma$-extensions (d'Alembertian solutions).

↓

FIND an alternative representation:

$$S(n) = \text{combined solutions}.$$

Example 1: A quadruple sum

A challenging email:

```
From: Doron Zeilberger
To: Robin Pemantle, Herbert Wilf
CC:Carsten Schneider



Robin and Herb,

I am willing to bet that Carsten Schneider's SIGMA package
for handling sums with harmonic numbers (among others)
can do it in a jiffy. I am Cc-ing this to Carsten.

Carsten: please do it, and Cc- the  answer to me.
             -Doron
```

The problem

```
From: Robin Pemantle [University of Pennsylvania]
To: herb wilf; doron zeilberger

Herb, Doron,

I have a sum that, when I evaluate numerically, looks suspiciously
like it comes out to exactly 1.
Is there a way I can automatically decide this?
The sum may be written in many ways, but one is:
```

$$\sum_{j,k=1}^{\infty} \frac{H_j(H_{k+1}-1)}{jk(k+1)(j+k)}; \qquad H_j := \sum_{i=1}^{j} \frac{1}{i}.$$

Simplify:

$$\sum_{k=1}^{\infty}\frac{H_{k+1}-1}{k(k+1)}\sum_{j=1}^{\infty}\frac{H_j}{j(j+k)}.$$

$$\sum_{j=1}^{a}\frac{H_j}{j(j+k)}=$$ FIND

Simplify:

$$\sum_{k=1}^{\infty} \frac{H_{k+1}-1}{k(k+1)} \sum_{j=1}^{\infty} \frac{H_j}{j(j+k)}.$$

$$\sum_{j=1}^{a} \frac{H_j}{j(j+k)} = \frac{kH_k^2 - 2H_k + kH_k^{(2)} + 2kH_a^{(2)}}{2k^2}$$

$$\underbrace{-\frac{(kH_a-1)}{k^2} \sum_{i=1}^{k} \frac{1}{a+i} - \frac{1}{k^1} \sum_{i=1}^{k} \frac{1}{i} \sum_{j=1}^{i} \frac{1}{a+j}}_{\text{Limits}}$$

Simplify:

$$\sum_{k=1}^{\infty}\frac{H_{k+1}-1}{k(k+1)}\sum_{j=1}^{\infty}\frac{H_j}{j(j+k)}.$$

$$\sum_{j=1}^{\infty}\frac{H_j}{j(j+k)} = \frac{kH_k^2-2H_k+kH_k^{(2)}+2k\zeta(2)}{2k^2}$$

Simplify:

$$\sum_{k=1}^{\infty}\frac{H_{k+1}-1}{k(k+1)}\sum_{j=1}^{\infty}\frac{H_j}{j(j+k)}.$$

$$\sum_{k=1}^{b}\frac{H_{k+1}-1}{k(k+1)}\sum_{j=1}^{\infty}\frac{H_j}{j(j+k)} = \sum_{k=1}^{b}\frac{H_{k+1}-1}{k(k+1)}\frac{kH_k^2-2H_k+kH_k^{(2)}+2k\zeta(2)}{2k^2}$$

$= $ Sigma

Simplify:

$$\sum_{k=1}^{\infty}\frac{H_{k+1}-1}{k(k+1)}\sum_{j=1}^{\infty}\frac{H_j}{j(j+k)}.$$

$$\sum_{k=1}^{b}\frac{H_{k+1}-1}{k(k+1)}\sum_{j=1}^{\infty}\frac{H_j}{j(j+k)}=\sum_{k=1}^{b}\frac{H_{k+1}-1}{k(k+1)}\frac{kH_k^2-2H_k+kH_k^{(2)}+2k\zeta(2)}{2k^2}$$

$$=\frac{1}{2(b+1)^2}\Big(6H_b+4bH_b+4H_b^2+3bH_b^2+H_b^3+bH_b^3-6b\zeta(2)$$

$$+2H_b\zeta(2)+2bH_b\zeta(2)-2H_b^{(2)}-7bH_b^{(2)}+H_bH_b^{(2)}+bH_bH_b^{(2)}\Big)$$

$$-\frac{2b^2}{(b+1)^2}\Big(\zeta(2)+H_b^{(2)}\Big)$$

$$+(\zeta(2)-1)\sum_{i=1}^{b}\frac{H_i}{i^2}-\sum_{i=1}^{b}\frac{H_i^2}{i^3}+\frac{1}{2}\sum_{i=1}^{b}\frac{H_i^3}{i^2}+\frac{1}{2}\sum_{i=1}^{b}\frac{H_iH_i^{(2)}}{i^2}$$

ζ-relations

This gives

$$S = \sum_{j,k=1}^{\infty} \frac{H_j(H_{k+1}-1)}{jk(k+1)(j+k)}$$

$$= -4\zeta(2) + (\zeta(2)-1)\sum_{i=1}^{\infty}\frac{H_i}{i^2} - \sum_{i=1}^{\infty}\frac{H_i^2}{i^3} + \frac{1}{2}\sum_{i=1}^{\infty}\frac{H_i^3}{i^2} + \frac{1}{2}\sum_{i=1}^{\infty}\frac{H_iH_i^{(2)}}{i^2}.$$

ζ-relations

This gives

$$S = \sum_{j,k=1}^{\infty} \frac{H_j(H_{k+1}-1)}{jk(k+1)(j+k)}$$

$$= -4\zeta(2) + (\zeta(2)-1)\sum_{i=1}^{\infty}\frac{H_i}{i^2} - \sum_{i=1}^{\infty}\frac{H_i^2}{i^3} + \frac{1}{2}\sum_{i=1}^{\infty}\frac{H_i^3}{i^2} + \frac{1}{2}\sum_{i=1}^{\infty}\frac{H_iH_i^{(2)}}{i^2}.$$

E.g., in J.M. Borwein, Girgensohn. Evaluation of triple Euler sums, 1996.

Flajolet, Salvy. Euler sums and contour integral representations, 1998.

Granville. A decomposition of Riemann's zeta-function. 1997.

Zagier. Values of zeta functions and their applications. 1994.

we find

$$\sum_{i=1}^{\infty}\frac{H_i}{i^2} = 2\zeta(3), \qquad \sum_{i=1}^{\infty}\frac{H_i^2}{i^3} = -\zeta(2)\zeta(3) + \frac{7}{2}\zeta(5),$$

$$\sum_{i=1}^{\infty}\frac{H_i^3}{i^2} = \zeta(2)\zeta(3) + 10\zeta(5), \qquad \sum_{i=1}^{\infty}\frac{H_iH_i^{(2)}}{i^2} = \zeta(2)\zeta(3) + \zeta(5).$$

Theorem (Sigma 2005).

$$S = \sum_{j,k=1}^{\infty} \frac{H_j(H_{k+1}-1)}{jk(k+1)(j+k)}$$
$$= -4\zeta(2) - 2\zeta(3) + 4\zeta(2)\zeta(3) + 2\zeta(5) = 0.999222\ldots$$

Similarly, we obtain simplifications like

$$\sum_{i,j=1}^{\infty}\frac{S_1(i)S_1(i+j+N)}{i(i+j)(j+N)} = 6\frac{S_1(N)}{N}\zeta_3 + \zeta_2\Big(2\frac{S_1^2(N)}{N} + \frac{S_2(N)}{N}\Big)$$
$$+\frac{S_1^4(N)}{6N} + \frac{S_1^2(N)S_2(N)}{N} - \frac{S_2^2(N)}{N} + 4\frac{S_{2,1,1}(N)}{N}$$
$$+S_1(N)\Big(-3\frac{S_{2,1}(N)}{N} + 4\frac{S_3(N)}{3N}\Big) - 2\frac{S_{3,1}(N)}{N} - \frac{S_4(N)}{2N}.$$

where

$$S_a(N) = \sum_{i=1}^{N}\frac{1}{i^a} \qquad S_{2,1}(N) = \sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{1}{j^1}}{i^2}$$

$$S_{3,1}(N) = \sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{1}{j^1}}{i^3} \qquad S_{2,1,1}(N) = \sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{1}{k^1}}{j}}{i^2}$$

Two–Loop Massive Operator Matrix Elements for Unpolarized Heavy Flavor Production to $O(\epsilon)$ (I. Bierenbaum, J. Blümlein, S. Klein, C.S.)

Example 2: In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon)=\int_0^1 dx_1\dots\int_0^1 dx_7\sum_{i=1}^{K}\frac{p_i(x_1,x_2,\dots,x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1,x_2,\dots,x_7)^{\dots+s_i\varepsilon+\dots}}$$

where $K\in\mathbb{N}$, $r_i,s_i\in\mathbb{Q}$, and p_i,q_i are polynomials in $x_1,\dots,x_7$.

Example 2: In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon)=\int_0^1 dx_1\dots\int_0^1 dx_7\sum_{i=1}^{K}\frac{p_i(x_1,x_2,\dots,x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1,x_2,\dots,x_7)^{\dots+s_i\varepsilon+\dots}}$$

$$=F_{-3}(n)\varepsilon^{-3}+F_{-2}(n)\varepsilon^{-2}+\boxed{F_{-1}(n)}\varepsilon^{-1}+\dots$$

The 3-loop anomalous dimensions can be derived from the single pole part of $F(n,\varepsilon)$. The other poles are needed for the renormalization.

Vermaseren, Moch: 3-5 CPU years (2004)

Example 2: In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon)=\int_0^1 dx_1\dots\int_0^1 dx_7\sum_{i=1}^{K}\frac{p_i(x_1,x_2,\dots,x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1,x_2,\dots,x_7)^{\dots+s_i\varepsilon+\dots}}$$

$$=F_{-3}(n)\varepsilon^{-3}+F_{-2}(n)\varepsilon^{-2}+\boxed{F_{-1}(n)}\varepsilon^{-1}+\dots$$

$$\downarrow$$

Initial values $F_{-1}(i)$, $i=1,\dots,450$

Example 2: In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon) = \int_0^1 dx_1 \dots \int_0^1 dx_7 \sum_{i=1}^{K} \frac{p_i(x_1,x_2,\dots,x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1,x_2,\dots,x_7)^{\dots+s_i\varepsilon+\dots}}$$

$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + \boxed{F_{-1}(n)}\varepsilon^{-1} + \dots$$

$$\downarrow$$

Initial values $F_{-1}(i)$, $i = 1,\dots,450$

$\downarrow$ Recurrence finder (M. Kauers)

$$a_0(n)F_{-1}(n) + a_1(n)F_{-1}(n+1) + \dots + a_7(n)F_{-1}(n+7) = 0$$

Example 2: In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon)=\int_0^1 dx_1\ldots\int_0^1 dx_7\sum_{i=1}^{K}\frac{p_i(x_1,x_2,\ldots,x_7)^{n+\cdots+r_i\varepsilon+\ldots}}{q_i(x_1,x_2,\ldots,x_7)^{\cdots+s_i\varepsilon+\ldots}}$$

$$=F_{-3}(n)\varepsilon^{-3}+F_{-2}(n)\varepsilon^{-2}+\boxed{F_{-1}(n)}\varepsilon^{-1}+\ldots$$

$\downarrow$

Initial values $F_{-1}(i)$, $i=1,\ldots,450$

$\downarrow$ Recurrence finder (M. Kauers)

$$a_0(n)F_{-1}(n)+a_1(n)F_{-1}(n+1)+\cdots+a_7(n)F_{-1}(n+7)=0$$

$\downarrow$ Sigma

CLOSED FORM

BIG

Appendix

- Integration
- $\Pi\Sigma$-fields
- Padé approximation
- Creative telescoping

Sigma's algorithmic summation paradigms

GIVEN a definite sum

$$S(n) := \sum_{k=0}^{n} f(n, k).$$

FIND a recurrence for $S(n)$ in n (Z's creative extension of telescoping).

Sigma's algorithmic summation paradigms

GIVEN a definite sum

$$S(n) := \sum_{k=0}^{n} f(n,k).$$

FIND a recurrence for $S(n)$ in n (Z's creative extension of telescoping).

$$\downarrow$$

GIVEN a recurrence

$$a_d(n)S(n+d) + \cdots + a_1(n)S(n+1) + a_0(n)S(n) = h(n).$$

FIND all solutions in $\Pi\Sigma$-extensions (d'Alembertian solutions).

Sigma's algorithmic summation paradigms

GIVEN a definite sum

$$S(n) := \sum_{k=0}^{n} f(n,k).$$

FIND a recurrence for $S(n)$ in n (Z's creative extension of telescoping).

↓

GIVEN a recurrence

$$a_d(n)S(n+d) + \cdots + a_1(n)S(n+1) + a_0(n)S(n) = h(n).$$

FIND all solutions in $\Pi\Sigma$-extensions (d'Alembertian solutions).

↓

FIND an alternative representation:

$$S(n) = \text{combined solutions}.$$

ΠΣ-Fields

Telescoping Appendix

- Let $\mathbb{K}(t_1, t_2, \dots, t_e)$ be a rational function field.
- Let σ be a field automorphism of $\mathbb{K}(t_1)\dots(t_e)$ with

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

ΠΣ-Fields

Telescoping | Appendix

- Let $\mathbb{K}(t_1, t_2, \ldots, t_e)$ be a rational function field.
- Let σ be a field automorphism of $\mathbb{K}(t_1)\ldots(t_e)$ with

$$\sigma(c) = c \qquad \forall c \in \mathbb{K}$$

$$\sigma(t_1) = \alpha_1\, t_1 + \beta_1, \qquad\qquad \alpha_1, \beta_1 \in \mathbb{K}$$

ΠΣ-Fields

▸ Telescoping ▸ Appendix

- Let $\mathbb{K}(t_1, t_2, \dots, t_e)$ be a rational function field.
- Let σ be a field automorphism of $\mathbb{K}(t_1)\dots(t_e)$ with

$$\sigma(c) = c \qquad \forall c \in \mathbb{K}$$
$$\sigma(t_1) = \alpha_1\, t_1 + \beta_1, \qquad \alpha_1, \beta_1 \in \mathbb{K}$$
$$\sigma(t_2) = \alpha_2\, t_2 + \beta_2, \qquad \alpha_2, \beta_2 \in \mathbb{K}(t_1)$$

ΠΣ-Fields

▸ Telescoping ▸ Appendix

- Let $\mathbb{K}(t_1, t_2, \dots, t_e)$ be a rational function field.
- Let σ be a field automorphism of $\mathbb{K}(t_1)\dots(t_e)$ with

$$\begin{aligned}
\sigma(c) &= c \qquad \forall c \in \mathbb{K} \\
\sigma(t_1) &= \alpha_1\, t_1 + \beta_1, & \alpha_1, \beta_1 &\in \mathbb{K} \\
\sigma(t_2) &= \alpha_2\, t_2 + \beta_2, & \alpha_2, \beta_2 &\in \mathbb{K}(t_1) \\
&\vdots \\
\sigma(t_e) &= \alpha_e\, t_e + \beta_e, & \alpha_e, \beta_e &\in \mathbb{K}(t_1, \dots, t_{e-1}).
\end{aligned}$$

ΠΣ-Fields

Telescoping | Appendix

- Let $\mathbb{K}(t_1, t_2, \ldots, t_e)$ be a rational function field.
- Let σ be a field automorphism of $\mathbb{K}(t_1)\ldots(t_e)$ with

$$\begin{aligned}
\sigma(c) &= c \qquad \forall c \in \mathbb{K} \\
\sigma(t_1) &= \alpha_1\, t_1 + \beta_1, && \alpha_1, \beta_1 \in \mathbb{K} \\
\sigma(t_2) &= \alpha_2\, t_2 + \beta_2, && \alpha_2, \beta_2 \in \mathbb{K}(t_1) \\
&\vdots \\
\sigma(t_e) &= \alpha_e\, t_e + \beta_e, && \alpha_e, \beta_e \in \mathbb{K}(t_1, \ldots, t_{e-1}).
\end{aligned}$$

- Then $(\mathbb{K}(t_1, \ldots, t_e), \sigma)$ is a ΠΣ-field if the set of constants is $\mathbb{K}$, i.e.,

$$\{c \in \mathbb{K}(t_1, \ldots, t_e) | \sigma(c) = c\} = \mathbb{K}.$$

Constructive difference field theory

Summation and Integration

▸ Telescoping ▸ Appendix

Indefinite summation

GIVEN $f(k)$
FIND $g(k)$:

$$g(k+1) - g(k) = f(k)$$

$$\downarrow$$

$$g(b+1) - g(a) = \sum_{k=a}^{b} f(k)$$

Integration

GIVEN $f(x)$
FIND $g(x)$:

$$D_x g(x) = f(x)$$

$$\downarrow$$

$$g(b) - g(a) = \int_a^b f(x)dx$$

Example from quadratic Padé approximation

Appendix

For all $m \geq 0$,

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}^3\Big[3(H_{m-k}-H_k)^2+H_{m-k}^{(2)}+H_k^{(2)}\Big]=0$$

where

$$H_k=\sum_{i=1}^{k}\frac{1}{i},\quad H_k^{(2)}=\sum_{i=1}^{k}\frac{1}{i^2}.$$

Example from quadratic Padé approximation

Appendix

Theorem (Sigma; 2002)

For all $m \geq 0$,

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k}^3 \left[3(H_{m-k} - H_k)^2 + H_{m-k}^{(2)} + H_k^{(2)}\right] = 0$$

where

$$H_k = \sum_{i=1}^{k} \frac{1}{i}, \quad H_k^{(2)} = \sum_{i=1}^{k} \frac{1}{i^2}.$$

Proof.

Sigma

□

Zeilberger's creative telescoping paradigm

▸ Paradigms ▸ Appendix

GIVEN

$$\mathsf{SUM}(m) := \sum_{k=0}^{m} \underbrace{(-1)^k \binom{m}{k}^3 \left[3(H_{m-k} - H_k)^2 + H^{(2)}_{m-k} + H^{(2)}_k\right]}_{=:\, f(m,k)}.$$

FIND $c_0(m)$, $c_1(m)$, $c_2(m)$, and $g(m,k)$:

$$\boxed{g(m,k+1) - g(m,k)} = \boxed{c_0(m)\, f(m,k) + c_1(m)\, f(m+1,k) + c_2(m)\, f(m+2,k)}$$

for all $0 \leq k \leq m$ and all $m \geq 0$.

Zeilberger's creative telescoping paradigm

▸ Paradigms ▸ Appendix

GIVEN

$$\mathsf{SUM}(m) := \sum_{k=0}^{m} \underbrace{(-1)^k \binom{m}{k}^3 \left[3(H_{m-k} - H_k)^2 + H^{(2)}_{m-k} + H^{(2)}_k\right]}_{=:\, f(m,k)}.$$

FIND $c_0(m)$, $c_1(m)$, $c_2(m)$, and $g(m,k)$:

$$\boxed{g(m,k+1) - g(m,k)} = \boxed{c_0(m)\, f(m,k) + c_1(m)\, f(m+1,k) + c_2(m)\, f(m+2,k)}$$

for all $0 \le k \le m$ and all $m \ge 0$.

Sigma:

$$c_0(m) := 3(3m+2)(3m+4)(3m+8), \quad c_1(m) := 0,$$
$$c_2(m) := (m+2)^2(3m+8),$$

$$g(m,k) := (-1)^k \binom{m}{k}^3 \frac{p_1(k,m,H_k,H^{(2)}_k,H_{m-k},H^{(2)}_{m-k})}{(m-k+1)^5(m-k+2)^5},$$

$$g(m,k+1) := (-1)^k \binom{m}{k}^3 \frac{p_2(k,m,H_k,H^{(2)}_k,H_{m-k},H^{(2)}_{m-k})}{(m-k+1)^5}.$$

Zeilberger's creative telescoping paradigm

Paradigms Appendix

GIVEN

$$\mathsf{SUM}(m) := \sum_{k=0}^{m} \underbrace{(-1)^k \binom{m}{k}^3 \Big[3(H_{m-k} - H_k)^2 + H^{(2)}_{m-k} + H^{(2)}_k\Big]}_{=:\, f(m,k)}.$$

GIVEN $c_0(m)$, $c_1(m)$, $c_2(m)$, and $g(m,k)$:

$$\boxed{g(m,k+1) - g(m,k)} = \boxed{c_0(m)\, f(m,k) + c_1(m)\, f(m+1,k) + c_2(m)\, f(m+2,k)}$$

for all $0 \leq k \leq m$ and all $m \geq 0$.

Summing this equation over k from 0 to m gives:

$$\boxed{g(m,m+1) - g(m,0)}$$

$$= \boxed{\begin{array}{l} c_0(m)\,\mathsf{SUM}(m) + \\ c_1(m)\,\big[\mathsf{SUM}(m+1) - f(m+1,m+1)\big] \\ c_2(m)\,\big[\mathsf{SUM}(m+2) - f(m+2,m+1) - f(m+2,m+2)\big]. \end{array}}$$

Back

Indefinite nested sum expression:

$$\sum_{k=0}^{a}\left(\sum_{i=0}^{k}\binom{n}{i}\right)$$

Definite nested sum expression:

$$\sum_{k=0}^{a}\left(\sum_{i=0}^{k}\binom{n+k}{i}\right)$$