On Recurrences for Ising Integrals

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Abstract

We use WZ-summation methods to compute recurrences for the Ising-class integrals $C_{n,k}$. In this context, we describe an algorithmic approach to obtain homogeneous and inhomogeneous recurrences for a general class of multiple contour integrals of Barnes' type.

1 Introduction

This note gives an affirmative answer to one of the problems stated in Section 9 of [2] regarding recurrences in $k \geq 0$ for the members of the Ising-class integrals

$$C_{n,k} := \frac{1}{n!} \int_0^\infty \cdots \int_0^\infty \frac{dx_1 dx_2 \dots dx_n}{(\cosh x_1 + \cdots + \cosh x_n)^{k+1}}. \tag{1}$$

In [2], after transforming the member $C_{5,k}$ of the Ising-class integrals to a two-fold nested Barnes integral, it was asked whether, using this new transformed form of the integral, an already conjectured recurrence could be proven by means of computer algebra algorithms based on WZ-summation methods. As described in [2], this idea goes back to W. Zudilin.

Using Wegschaider's algorithm [7], which is an extension of multivariate WZ summation [9], we have obtained the conjectured recurrences in $k \geq 1$ for the integrals $C_{5,k}$ and $C_{6,k}$. Moreover, we will show that, in principle, one can obtain recurrences with respect to $k \geq 1$ for any integral of the form (1) with $n \in \mathbb{N}$, using the multivariate summation algorithm [7] after applying the above mentioned transformation method to the integral.

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In [3], J. M. Borwein and B. Salvy show the existence of linear recurrences with polynomial coefficients for the integrals (1) using the theory of D-finite series and a Bessel-kernel representation given in [1]. Also a very efficient algorithm to compute recurrences in $k \geq 1$ for the integrals $C_{n,k}$ for given $n \in \mathbb{N}$ is described in [3]. Our approach to obtain these recurrences is different; in particular, it gives a general algorithmic method to compute homogeneous and inhomogeneous recurrences for multiple nested Barnes type integrals.

2 The Problem

For the statement of the problem we invoke the renormalization

$$c_{n,k} := \frac{n!}{2^n} \Gamma(k+1) C_{n,k}$$

used in [2].

The idea of W. Zudilin presented in section 7 of [2] relies on the following analytic convolution theorem.

Theorem 1 *([2], Theorem 7) For complex k with $Re(k) > 0$ and $n, q \in \mathbb{N}$ such that $n \geq 1$ and $1 \leq q \leq n-1$ we have*

$$c_{n,k} = \frac{1}{2\pi i} \int_{\mathbf{C}} c_{n-q,k+s} c_{q,-1-s} ds$$

where the contour $\mathbf{C}$ runs over the vertical line $(-\lambda - i\infty, -\lambda + i\infty)$ with $\lambda \in \mathbb{R}$ such that $-1 - Re(k) < -\lambda < -1$.

Also in [2] the closed forms

$$C_{1,k} = \frac{2^k \Gamma\left(\frac{k+1}{2}\right)^2}{\Gamma(k+1)} \tag{2}$$

and

$$C_{2,k} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{k+1}{2}\right)^3}{2\Gamma\left(\frac{k}{2}+1\right)\Gamma(k+1)} \tag{3}$$

were computed.

Applying Theorem 1 twice, using the closed forms (2), (3), and making the substitution $(s,t) \to (2s, 2t)$, we are able to transform $C_{5,k}$ and $C_{6,k}$ for $k \geq 1$ into two-fold nested Barnes integrals

$$C_{5,k} = \frac{-1}{240\pi\Gamma(k+1)} \int_{\mathcal{C}_s}\int_{\mathcal{C}_t} \frac{\Gamma\left(s+\frac{k+1}{2}\right)^3 \Gamma(t-s)^3 \Gamma(-t)^2}{4^t \Gamma\left(s+\frac{k}{2}+1\right)\Gamma\left(t-s+\frac{1}{2}\right)} ds dt \tag{4}$$

and

$$C_{6,k} = \frac{-1}{720\sqrt{\pi}\Gamma(k+1)} \int_{\mathcal{C}_s}\int_{\mathcal{C}_t} \frac{\Gamma\left(s+\frac{k+1}{2}\right)^3 \Gamma(t-s)^3 \Gamma(-t)^3}{\Gamma\left(s+\frac{k}{2}+1\right)\Gamma\left(t-s+\frac{1}{2}\right)\Gamma\left(-t+\frac{1}{2}\right)} ds dt. \tag{5}$$

The vertical contour $\mathcal{C}_s := (-\lambda - i\infty, -\lambda + i\infty)$ separates the poles of $\Gamma\left(s+\frac{k+1}{2}\right)$ from the poles of $\Gamma(t-s)$ and similarly $\mathcal{C}_t := (-\rho - i\infty, -\rho + i\infty)$ separates the poles of $\Gamma(t-s)$ from the poles of $\Gamma(-t)$.

For reasons that become clear in Section 5, we choose $\lambda, \rho \in \mathbb{R}$ such that the following conditions are satisfied:

$$-\frac{1+Re(k)}{2} < -\lambda < -\rho < -1.$$

Successively applying Theorem 1, we obtain an integral representation for arbitrary $C_{n,k}$ with $n, k \geq 1$:

$$C_{n,k} = \frac{2^n}{n!\,(2\pi i)^q}\frac{1}{\Gamma(k+1)} \int_{\mathbf{C}_{t_1}} \cdots \int_{\mathbf{C}_{t_q}} c_{2,k+t_1} \left(\prod_{j=1}^{q-1} c_{2,-1-t_j+t_{j+1}}\right) c_{\epsilon,-1-t_q} dt_1 \dots dt_q \tag{6}$$

where $q := \left\lceil \frac{n}{2} \right\rceil - 1$ and $\epsilon := n - 2q$.

Then we use the closed forms (2) and (3), and the substitutions $t_j \to 2t_j$ for all $1 \leq j \leq q$, to obtain from (6) the final representation of $C_{n,k}$ for arbitrary $k, n \geq 1$.

At last, we choose new integration contours $\mathcal{C}_{t_j} := (-\lambda_j - i\infty, -\lambda_j + i\infty)$ for all $1 \leq j \leq q$ which run over vertical lines separating the poles of gamma functions of the form $\Gamma(a+t_j)$ from the poles of gamma functions of the form $\Gamma(b-t_j)$. For reasons presented later, we choose these Barnes paths of integration such that the following conditions are satisfied:

$$-\frac{1+Re(k)}{2} < -\lambda_1 < -\lambda_2 < \cdots < -\lambda_q < -1. \tag{7}$$

3 Deriving Recurrences Algorithmically

Wegschaider's algorithm [7] is an extension of multivariate WZ summation [9], and in this context it is used to compute recurrences for sums of the form

$$Sum(\mu) = \sum_{\kappa_1 \in \mathcal{R}_1} \cdots \sum_{\kappa_r \in \mathcal{R}_r} \mathcal{F}(\mu, \kappa_1, \dots, \kappa_r). \tag{8}$$

Under some mild side conditions described in [7], it can be applied if the summands $\mathcal{F}(\mu,\kappa)$ are hypergeometric in all integer variables μ_i from $\mu=(\mu_1,\ldots,\mu_p)$ and in all summation variables κ_j from $\kappa=(\kappa_1,\ldots,\kappa_r)\in\mathcal{R}$ where $\mathcal{R}:=\mathcal{R}_1\times\cdots\times\mathcal{R}_r\subseteq\mathbb{Z}^r$ is the summation range.

Remark: Recall that an expression $\mathcal{F}(\mu,\kappa)$ is called hypergeometric [10, 9] if there exists a rational function $r(\mu,\kappa)$ such that $\frac{\mathcal{F}(\mu,\kappa)}{\mathcal{F}(\mu-m,\kappa-k)}=r(\mu,\kappa)$ at the points $m\in\mathbb{Z}^p$ and $k\in\mathbb{Z}^r$ where this ratio is defined.

The algorithm first finds a recurrence for the summand $\mathcal{F}(\mu,\kappa)$ called certificate recurrence. In non-elementary applications, this can be a time and space consuming problem. Therefore, Wegschaider's algorithm [7] is used after making an Ansatz about the structure of this recurrence, i.e., the set of shifts $\mathbb{S}\subset\mathbb{Z}^{p+r}$ which the recurrence should contain.

Given such a set $\mathbb{S}$, also called structure set, together with the hypergeometric summand $\mathcal{F}(\mu,\kappa)$, the algorithm computes a certificate recurrence of the form

$$\sum_{(m,k)\in\mathbb{S}} a_{m,k}(\mu)\,\mathcal{F}(\mu-m,\kappa-k)=\sum_{j=1}^{r}\Delta_{\kappa_j}\left(\sum_{(m,k)\in\mathbb{S}_j} b_{m,k}(\mu,\kappa)\,\mathcal{F}(\mu-m,\kappa-k)\right), \tag{9}$$

where the polynomials $a_{m,k}(\mu)$, not all zero, and $b_{m,k}(\mu,\kappa)$ and also the sets $\mathbb{S}_j\subset\mathbb{Z}^{p+r}$ are determined algorithmically.

Note that the forward shift operators Δ_{κ_j} are defined as

$$\Delta_{\kappa_j}\mathcal{F}(\mu,\kappa):=\mathcal{F}(\mu,\kappa_1,\ldots,\kappa_j+1,\ldots,\kappa_r)-\mathcal{F}(\mu,\kappa).$$

Remark: The right hand side of (9) can always be rewritten as

$$\sum_{j=1}^{r}\Delta_{\kappa_j}\left(\sum_{(m,k)\in\mathbb{S}_j} b_{m,k}(\mu,\kappa)\,\mathcal{F}(\mu-m,\kappa-k)\right)=\sum_{j=1}^{r}\Delta_{\kappa_j}\left(r_j(\mu,\kappa)\,\mathcal{F}(\mu,\kappa)\right), \tag{10}$$

where r_j are rational functions of all variables from $\mu=(\mu_1,\ldots,\mu_p)$ and $\kappa=(\kappa_1,\ldots,\kappa_r)$.

Remark: In the certificate recurrence (9), the coefficients $a_{m,k}(\mu)$ are polynomials free of the summation variables κ_j from κ, while the coefficients $b_{m,k}(\mu,\kappa)$ of the delta-parts are polynomials in all the variables from μ and κ.

Finally, the recurrence for the multisum (8) is obtained by summing the certificate recurrence (9) over all variables from κ in the given summation range $\mathcal{R}$. Since it can be easily checked whether the summand $\mathcal{F}(\mu,\kappa)$ indeed satisfies the recurrence (9), the certificate recurrence also provides a proof of the recurrence for the multisum $Sum(\mu)$.

Two further remarks are in place. First, an algorithm to obtain a candidate structure set $\mathbb{S}$ is implemented in the Mathematica package MultiSum[1] (see also [4]), which also includes an implementation of Wegschaider's algorithm [7]. To use MultiSum within a Mathematica session, one calls it by:

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In[1]:= << MultiSum.m
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MultiSum Package by Kurt Wegschaider (enhanced by Axel Riese and Burkhard Zimmermann) – © RISC Linz – V2.02β (02/21/05)

Secondly, we remark that in many applications the function $\mathcal{F}(\mu, \kappa)$ has a finite support. In these cases, if we sum the recurrence (9) over a domain that is larger then the support of the function, the Δ-parts on the right hand side telescope and the values that are not in the support vanish. So, from the summand recurrence one obtains a homogeneous recurrence for the sum

$$\sum_{(m,k)\in\mathbb{S}} a_{m,k}(\mu)\, Sum\,(\mu - m) = 0. \tag{11}$$

This is not the case in general; i.e., in specific situations human inspection is still needed to pass from the recurrence (9) to a homogeneous or inhomogeneous recurrence for the sum (8). More information on this subject can be found in [9].

4 From Summation to Integration

In this section we will show how Wegschaider's algorithm [7] can be used to determine recurrences for multiple contour integrals of Barnes' type

$$Int\,(\mu) = \int_{\mathcal{C}_{\kappa_1}} \dots \int_{\mathcal{C}_{\kappa_r}} \mathcal{F}\,(\mu, \kappa_1, \dots, \kappa_r)\, d\kappa_1 \dots d\kappa_r, \tag{12}$$

where the integrands $\mathcal{F}\,(\mu, \kappa)$ are hypergeometric in all integer variables μ_i from $\mu = (\mu_1, \dots, \mu_p)$ and in all integration variables κ_j from $\kappa = (\kappa_1, \dots, \kappa_r) \in \mathbb{C}^r$.

For instance, the integral representations obtained in Section 2 for $C_{n,k}$ for any $n, k \geq 1$ are of the considered form (12) if we distinguish between the even and odd values of $k \in \mathbb{N}$.

As in the case of the summation problem (8), the fundamental theorem of hypergeometric summation stated by Wilf and Zeilberger in [9] proves the existence of non-trivial certificate recurrences of the form (9) for the function $\mathcal{F}\,(\mu, \kappa)$. Using WZ-summation methods, Wegschaider's algorithm [7] delivers recurrences of the form (9); for the hypergeometric integrand from (12) and as remarked in Section

[1]available at http://www.risc.uni-linz.ac.at/research/combinat/software/

3, the coefficients on the left hand side of this recurrence are free of all integration variables $\kappa = (\kappa_1, \ldots, \kappa_r)$.

Moreover, although discrete functions are our main interest, one can evaluate the function $\mathcal{F}(\mu, \kappa)$ also for complex values of the variables μ_i and κ_j for all $1 \leq i \leq p$ and $1 \leq j \leq r$ except at certain poles. In our case, the singularities of the numerator gamma functions need to be excluded from the evaluation domain. The function $\mathcal{F}(\mu, \kappa)$ is then continuous on its evaluation domain and by taking limits it can be shown that the computed recurrences (9) hold in $\mathbb{C}^{p+r}$.

Therefore, after succesively integrating over the Barnes paths of integration $\mathcal{C}_{\kappa_j}$ for $1 \leq j \leq r$, (9) leads, in some cases, to a homogeneous recurrence for the integration problem (12), i.e.,

$$\sum_{(m,k)\in\mathbb{S}} a_{m,k}(\mu)\, Int(\mu - m) = 0. \tag{13}$$

However, again in analogy to the summation case, after integrating over the contours of integration $\mathcal{C}_{\kappa_j}$ for $1 \leq j \leq r$, it is not clear, in general, that we obtain a homogeneous equation of the type (13). Consequently, one needs to analyze the behavior of the contour integrals over the left hand side of (9).

For this purpose, we study the following integration problems:

$$I_j := \int_{\mathcal{C}_{\kappa_j}} \Delta_{\kappa_j} \mathcal{F}(\mu, \kappa)\, d\kappa_j = \int_{\mathcal{C}'_{\kappa_j}} \mathcal{F}(\mu, \kappa)\, d\kappa_j - \int_{\mathcal{C}_{\kappa_j}} \mathcal{F}(\mu, \kappa)\, d\kappa_j, \tag{14}$$

where the Barnes path $\mathcal{C}_{\kappa_j}$ runs vertically over $(c_j - i\infty, c_j + i\infty)$ while $\mathcal{C}'_{\kappa_j}$ denotes the path $(1 + c_j - i\infty, 1 + c_j + i\infty)$ for all $1 \leq j \leq r$.

For any $1 \leq j \leq r$, consider now the contour integral I_j^N over a rectangle with vertices at the points $c_j - iN$, $c_j + iN$, $1 + c_j + iN$ and $1 + c_j - iN$ with $N \in \mathbb{N}$; i.e.,

$$\begin{aligned} I_j^N = & \int_{c_j - iN}^{c_j + iN} \mathcal{F}(\mu, \kappa)\, d\kappa_j + \int_{c_j + iN}^{1 + c_j + iN} \mathcal{F}(\mu, \kappa)\, d\kappa_j \\ & - \int_{1 + c_j - iN}^{1 + c_j + iN} \mathcal{F}(\mu, \kappa)\, d\kappa_j + \int_{1 + c_j - iN}^{c_j - iN} \mathcal{F}(\mu, \kappa)\, d\kappa_j. \end{aligned} \tag{15}$$

If in any such rectangular region of integration, we have the asymptotic behavior

$$\mathcal{F}(\mu, \kappa) = \mathcal{O}\left(e^{-c|\kappa_j|}\right) \quad \text{as} \quad |\kappa_j| \to \infty \quad \text{with} \quad c > 0, \tag{16}$$

then $I_j^N \to I_j$ as $N \to \infty$. Since the function $\mathcal{F}(\mu, \kappa)$ is dominated by an exponential with negative exponent, it suffices to to analyze the integrals (14) instead of the integrals over the right hand side of (10).

On the other hand, we can calculate the integrals (15) by considering the residues of the function $\mathcal{F}(\mu, \kappa)$ at the poles lying inside the closed rectangular contours. Therefore, if, for all $1 \leq j \leq r$, the Barnes paths of integration $\mathcal{C}_{\kappa_j}$ can be chosen such that the function $\mathcal{F}(\mu, \kappa)$ has no poles inside these rectangular regions, then the integrals (14) will be zero. This is why conditions (7) are imposed on the integral representation of $C_{n,k}$ for $n, k \geq 1$.

Under these restrictions, we obtain from the certificate recurrence (9) a homogeneous recurrence (13) for the multiple Barnes type integral (12). Note that, by a different choice of the integration contours, this method will lead to inhomogeneous recurrences for multiple Barnes integrals which satisfy the asymptotic condition (16).

5 Homogeneous Recurrences for the Integrals $C_{n,k}$

After distinguishing between the odd and the even values of the parameter k, for an arbitrary Ising-class integral $C_{n,k}$, $n, k \geq 1$, one obtains two representations of the form

$$C_{n,\mu} = \frac{2^{n+2q}}{n!\,(2\pi i)^q} \frac{1}{\Gamma(\mu+1)} \int_{\mathcal{C}_{t_1}} \cdots \int_{\mathcal{C}_{t_q}} \Psi(\mu, t_1, \ldots, t_q)\, dt_1 \ldots dt_q, \tag{17}$$

where $\mu = \frac{k}{2}$, respectively, $\mu = \frac{k-1}{2}$ such that $\mu \in \mathbb{N}$. In both cases the integrand $\Psi(\mu, \mathbf{t})$ then is hypergeometric in $\mu \geq 0$ and in all integration variables t_j from $\mathbf{t} = (t_1, \ldots, t_q)$.

Therefore, Wegschaider's algorithm [7] can be applied to deliver a certificate recurrence of the form (9), that can always be rewritten as

$$\sum_{(m,\tau)\in\mathbb{S}} a_{m,\tau}(\mu)\, \Psi(\mu - m, \mathbf{t} - \tau) = \sum_{j=1}^{r} \Delta_{t_j}\left(\phi_j(\mu, \mathbf{t})\, \Psi(\mu, \mathbf{t})\right), \tag{18}$$

where $\mathbb{S}$ is a pre-computed structure set, where the coefficients $a_{m,\tau}(\mu)$ are polynomials, not all zero, free of all integration variables, and where the $\phi_j(\mu, \mathbf{t})$ are rational functions.

We will now show that for all $1 \leq j \leq q$ and $k \geq 1$ we have

$$\int_{\mathcal{C}_{t_j}} \Delta_{t_j}\left(\phi_j(k, \mathbf{t})\, \Psi(k, \mathbf{t})\right) = 0, \tag{19}$$

where the contours $\mathcal{C}_{t_j} := (-\lambda_j - i\infty, -\lambda_j + i\infty)$ satisfy the conditions (7).

Because of the iterative construction of the integral representation of $C_{n,k}$, computed in Section 2 for $k, n \geq 1$, it suffices to study the behavior of the following four contour integrals of Barnes' type

$$I_1 := \int_{\mathcal{C}_{t_1}} \Delta_{t_1} \left(\frac{\Gamma\left(\frac{k+1}{2}+t_1\right)^3}{\Gamma\left(\frac{k}{2}+t_1+1\right)} \frac{\Gamma\left(-t_1\right)^2}{4^{t_1}} \right) dt_1, \tag{20}$$

$$I_2 := \int_{\mathcal{C}_{t_1}} \Delta_{t_1} \left(\frac{\Gamma\left(\frac{k+1}{2}+t_1\right)^3}{\Gamma\left(\frac{k}{2}+t_1+1\right)} \frac{\Gamma\left(-t_1+t_2\right)^3}{\Gamma\left(-t_1+t_2+\frac{1}{2}\right)} \right) dt_1, \tag{21}$$

$$I_3 := \int_{\mathcal{C}_{t_2}} \Delta_{t_2} \left(\frac{\Gamma\left(-t_1+t_2\right)^3}{\Gamma\left(-t_1+t_2+\frac{1}{2}\right)} \frac{\Gamma\left(-t_2\right)^2}{4^{t_2}} \right) dt_2, \tag{22}$$

$$I_4 := \int_{\mathcal{C}_{t_2}} \Delta_{t_2} \left(\frac{\Gamma\left(-t_1+t_2\right)^3}{\Gamma\left(-t_1+t_2+\frac{1}{2}\right)} \frac{\Gamma\left(-t_2+t_3\right)^3}{\Gamma\left(-t_2+t_3+\frac{1}{2}\right)} \right) dt_2. \tag{23}$$

We will prove here that $I_2 = 0$, for the other integals one can proceed analogously. Let us first denote the integrand of I_2

$$F\left(k, t_1\right) := \frac{\Gamma\left(\frac{k+1}{2}+t_1\right)^3}{\Gamma\left(\frac{k}{2}+t_1+1\right)} \frac{\Gamma\left(-t_1+t_2\right)^3}{\Gamma\left(-t_1+t_2+\frac{1}{2}\right)}. \tag{24}$$

Using this notation and the transformation $t_1 + 1 \to t_1$ we can write

$$I_2 = \int_{\mathcal{C}'_{t_1}} F\left(k, t_1\right) dt_1 - \int_{\mathcal{C}_{t_1}} F\left(k, t_1\right) dt_1,$$

where the Barnes path of integration $\mathcal{C}_{t_1}$ is the vertical line $(-\lambda_1 - i\infty, -\lambda_1 + i\infty)$ separating the poles of $\Gamma\left(\frac{k+1}{2}+t_1\right)$ from the poles of $\Gamma\left(-t_1+t_2\right)$ such that

$$-\frac{Re(k)+1}{2} < -\lambda_1 < -1$$

and where the contour $\mathcal{C}'_{t_1}$ runs vertically on the line $(1 - \lambda_1 - i\infty, 1 - \lambda_1 + i\infty)$.

Next we define an integral of the form (15),

$$\begin{aligned} I_2^N := &\int\limits_{-\lambda_1-iN}^{-\lambda_1+iN} F\left(k, t_1\right) dt_1 + \int\limits_{-\lambda_1+iN}^{1-\lambda_1+iN} F\left(k, t_1\right) dt_1 \\ &- \int\limits_{1-\lambda_1-iN}^{1-\lambda_1+iN} F\left(k, t_1\right) dt_1 - \int\limits_{-\lambda_1-iN}^{1-\lambda_1-iN} F\left(k, t_1\right) dt_1, \end{aligned}$$

where $N > 0$ is an arbitrary integer.

If the conditions (7) are fulfilled, then there are no poles of the function $F(k, t_1)$ within the closed rectangular contour of integration. Therefore I_2^N is zero for any integer $N \in \mathbb{N}$.

It only remains to show that $I_2^N \to I_2$ as $N \to \infty$. For this we need to prove that the integrals

$$J_1^N := \int_{-\lambda_1+iN}^{1-\lambda_1+iN} F(k, t_1)\, dt_1 \quad \text{and} \quad J_2^N := \int_{-\lambda_1-iN}^{1-\lambda_1-iN} F(k, t_1)\, dt_1$$

tend to zero as $N \to \infty$.

The following asymptotic representation of the function $\log \Gamma(z)$ for large $|z|$ in the region where $|\arg(z)| \le \pi - \delta$ and $|\arg(z+a)| \le \pi - \delta$ with $\delta > 0$,

$$\log \Gamma(z+a) = (z + a - \frac{1}{2}) \log z - z + \mathcal{O}(1) \tag{25}$$

can be found in [8], section 13.6. Note that an expression is $\mathcal{O}(1)$ if it is bounded, in our case for $|z| \to \infty$.

We use (25) to study the behavior of $\log F(k, t_1)$ when $|t_1| \to \infty$. Since $\left|e^{iy}\right| = 1$ for any real y, we will denote with PI the pure imaginary terms involved in the exponent of $e^{\log F(k,t_1)}$. We can also write $\log(z) = \ln|z| + i \arg(z)$, where $\ln(x)$ is the real natural logarithm and where $-\pi < \arg(z) < \pi$.

Therefore, one obtains when $|t_1| \to \infty$,

$$\log F(k, t_1) = \mathrm{Re}\,(k + 2t_2 - 2) \ln|t_1| + 2\mathrm{Im}(t_1)\,[-\arg(t_1) + \arg(-t_1)] + PI + \mathcal{O}(1),$$

which is equivalent to

$$F(k, t_1) = \mathcal{O}\left(|t_1|^{\mathrm{Re}(k+2t_2-2)}\, e^{2\mathrm{Im}(t_1)[-\arg(t_1)+\arg(-t_1)]}\right).$$

Here we distinguish two cases, either $\mathrm{Im}(t_1) > 0$ or $\mathrm{Im}(t_1) < 0$, and in any of these cases the function $F(k, t_1)$ fulfills the condition (16). Also because of this asymptotic behavior, we can now prove that (19) holds for $j = 1$ and $n \ge 4$.

6 The Recurrence for the Integral $C_{5,k}$

In [2], the following recurrence for the integral $C_{5,k}$ was conjectured

$$\begin{aligned} &-(k+1)^5 C_{5,k} + (k+2)\left(35k^4 + 280k^3 + 882k^2 + 1288k + 731\right) \\ &\quad C_{5,k+2} - (k+2)(k+3)(k+4)\left(259k^2 + 1554k + 2435\right) C_{5,k+4} \\ &+225(k+2)(k+3)(k+4)(k+5)(k+6) C_{5,k+6} = 0. \end{aligned} \tag{26}$$

To prove that the integral (1) for $n = 5$ satisfies the above recurrence, we use the representation (4). We consider the integrand in (4) as a function of $k \geq 0$ and $s, t \in \mathbb{C}$, and input this function in Mathematica

In[2]:= $$F[k\_, s\_, t\_] := \frac{\left(\Gamma\left[\frac{k+1}{2}+s\right]\Gamma[t-s]\right)^3\Gamma^2[-t])}{4^t\Gamma\left[\frac{k}{2}+s+1\right]\Gamma\left[\frac{1}{2}+t-s\right]\Gamma[k+1]}.$$

In the first part of the proof we want to apply Wegschaider's algorithm [7] that was already introduced in Section 3, to obtain a certificate recurrence for $F(k, s, t)$. For this we need the function to be hypergeometric not only with respect to the integration variables s, t but also with respect to the additional parameter $k \geq 0$. This leads to a case distinction between even and odd values of k.

In each of the two cases, we introduce a new variable $K \geq 0$ such that the function $\mathcal{F}(K, s, t) := F(2K, s, t)$, respectively $\mathcal{F}(K, s, t) := F(2K+1, s, t)$, is hypergeometric with respect to all the variables; i.e., the $C_{5,k}$ are expressed as integrals of the form

$$C_{5,2K+\epsilon} = \frac{-1}{240\pi}\int_{\mathcal{C}_s}\int_{\mathcal{C}_t}\mathcal{F}(K, s, t)dsdt, \tag{27}$$

with $K \geq 0$ and where $\epsilon = 0$ or $\epsilon = 1$.

In each of the two cases the setting of (12) applies, and the conditions to apply Wegschaider's algorithm [7] are fulfilled. After computing a certificate recurrence for $\mathcal{F}(K, s, t)$ the algorithm delivers a homogeneous recurrence for the Barnes type integral (27).

In the case $k = 2K$ and $K \geq 0$ we obtain the recurrence

$$\begin{aligned}&(2K+1)^5\mathrm{SUM}[K] - 2(K+1)\left(560K^4 + 2240K^3 + 3528K^2 + 2576K + 731\right)\\ &\mathrm{SUM}[K+1] + 4(K+1)(K+2)(2K+3)\left(1036K^2 + 3108K + 2435\right)\mathrm{SUM}[K+2]\\ &- 1800(K+1)(K+2)(K+3)(2K+3)(2K+5)\mathrm{SUM}[K+3] = 0,\end{aligned} \tag{28}$$

where $\mathrm{SUM}[K]$ denotes $C_{5,2K}$.

In the second case when $k = 2K+1$ and $K \geq 0$ we obtain for $\mathrm{SUM}[K] := C_{5,2K+1}$ the recurrence

$$\begin{aligned}&-8(K+1)^5\mathrm{SUM}[K] + 4(2K+3)\left(35K^4 + 210K^3 + 483K^2 + 504K + 201\right)\\ &\mathrm{SUM}[K+1] - 2(K+2)(2K+3)(2K+5)\left(259K^2 + 1036K + 1062\right)\mathrm{SUM}[K+2]\\ &+225(K+2)(K+3)(2K+3)(2K+5)(2K+7)\mathrm{SUM}[K+3] = 0.\end{aligned} \tag{29}$$

As already pointed out, one can reduce the running time of the algorithm by first making an Ansatz for a small structure set of the recurrence. For example, before computing a certificate recurrence for $\mathcal{F}(K, s, t)$, we find a structure set with the command

In[3]:= **FindStructureSet** [$\boldsymbol{F}$[**2**$\boldsymbol{K}$**,** $\boldsymbol{s}$**,** $\boldsymbol{t}$]**,** $\boldsymbol{K}$**,** {$\boldsymbol{s}$**,** $\boldsymbol{t}$} **,** {**1, 1**} **, 1**]

which gives us 3 candidates for a structure set. Using the first candidate structure set one already succeeds in finding a certificate recurrence by

In[4]:= **FindRecurrence** [$\boldsymbol{F}$[**2**$\boldsymbol{K}$**,** $\boldsymbol{s}$**,** $\boldsymbol{t}$]**,** $\boldsymbol{K}$**,** {$\boldsymbol{s}$**,** $\boldsymbol{t}$} **,** %[[**1**]]**, 1, WZ** → True] **.**

Finally, recurrence (28) is the output of the following command

In[5]:= **SumCertificate** [%] **.**

This corresponds to integrating with respect to s and t over the previous output which was the certificate recurrence.

The last step of the proof consists in obtaining the recurrence for the sequence of integrals $C_{5,k}$ with $k \geq 0$. To this end, we utilize the fact that the sequences $C_{5,2K}$ and $C_{5,2K+1}$ defined for all $K \geq 0$ are P-recursive (also called holonomic [6, 11]); i.e., they satisfy linear recurrences with polynomial coeffecents.

To compute the recurrence for $C_{n,k}$ from those of $C_{n,2K}$ and $C_{n,2K+1}$, we load the Mathematica package

In[6]:= **<< GeneratingFunctions.m**

GeneratingFunctions Package by Christian Mallinger – © RISC Linz – V 0.68 (07/17/03)

The command **REInterlace** computes a recurrence that is satisfied by the sequence obtained by interlacing the input recurrences (see [5] for more details). This means, we input the recurrences (28) and (29) satisfied by $(C_{5,2K})_{K\geq 0}$ and $(C_{5,2K+1})_{K\geq 0}$, respectively, and we obtain a polynomial recurrence for the sequence $C_{5,k}$ with $k \geq 0$. The computed recurrence is exactly (26) and herewith the proof is complete.

7 Conclusion

This algorithmic method to prove and compute recurrences for members of the Ising-class integrals using Wegschaider's algorithm [7] delivers the recurrence conjectured in [2] for $C_{6,k}$ in completely analogous manner. Neglecting practical issues like computation time, this method applies to all $n \geq 1$.

Though, we need to remark that the algorithm [7] determines recurrences, after making an Ansatz about their structure set (i.e., fixing the set of shifts that they contain), by solving a large system of equations over a field of rational functions. Therefore, if the input of the algorithm is too involved, computations might become time consuming.

Basic ingredients of the approach are the representation of the Ising integrals $C_{n,k}$ for $k, n \geq 1$ as nested Barnes type integrals and the convolution theorem stated in Section 7 of [2], ideas going back W. Zudilin.

In addition, the method briefly explained in Section 4 of this paper is more general and has a wider range of applications that deserve to be explored further.

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