Computing the Algebraic Relations of C-finite Sequences and Multisequences

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Abstract

We present an algorithm for computing generators for the ideal of algebraic relations among sequences which are given by homogeneous linear recurrence equations with constant coefficients. Knowing these generators makes it possible to use Gröbner basis methods for carrying out certain basic operations in the ring of such sequences effectively. In particular, one can answer the question whether a given sequence can be represented in terms of other given sequences.

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1 Introduction

A *C-finite sequence* over a field k is a function $a\colon \mathbb{Z} \to k$ which satisfies a linear homogeneous recurrence with constant coefficients $c_0, c_1, \dots, c_s \in k$ with $c_0 \neq 0$ and $c_s \neq 0$,

$$c_0 a(n) + c_1 a(n+1) + \cdots + c_s a(n+s) = 0 \qquad (n \in \mathbb{Z});$$

(Zeilberger, 1990). C-finite sequences, also known as *recurrence sequences*, are well studied in the literature (Everest et al., 2003). The most famous C-finite sequence is the sequence of Fibonacci numbers satisfying $F_{n+2} = F_{n+1} + F_n$ and $F_0 = 0, F_1 = 1$.

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An *algebraic relation over k* among r sequences $a_1, \ldots, a_r \colon \mathbb{Z} \to k$ is a polynomial $f \in k[x_1, \ldots, x_r]$ such that $f(a_1(n), \ldots, a_r(n)) = 0$ for all $n \in \mathbb{Z}$. For instance, the polynomial $x_1 x_2 - x_3^2 - x_4$ is an algebraic relation over $\mathbb{Q}$ among the four sequences F_{n-1}, F_{n+1}, F_n and $(-1)^n$ by Cassini's identity $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$.

It is sometimes of interest to decide whether or not a given polynomial is an algebraic relation of given sequences. This is trivial for the case of C-finite (Nemes and Petkovšek, 1995) sequences and, nowadays, routine for holonomic sequences (Salvy and Zimmermann, 1994) and many other classes of sequences. However, finding the algebraic relations among given sequences in the first place is a completely different task. Note that the set of algebraic relations among sequences $a_1, \ldots, a_r$ forms an ideal of $k[x_1, \ldots, x_r]$. The aim of this paper is to give algorithms for computing generators for this ideal in the case of C-finite sequences (Section 4) and C-finite multisequences (Section 7).

Let $k[a_1, \ldots, a_r]$ be the smallest subring of $k^{\mathbb{Z}}$ that contains the sequences $a_1, \ldots, a_r$ and all constant sequences, and let I be the ideal of all algebraic relations among $a_1, \ldots, a_r$. A Gröbner basis (Buchberger, 1965; Adams and Loustaunau, 1994) of I allows us to compute in $k[a_1, \ldots, a_r]$ via the presentation by generators and relations

$$k[a_1, \ldots, a_r] \simeq k[x_1, \ldots, x_r]/I.$$

In particular, we can carry out addition, multiplication and canonical simplification effectively. Moreover, the question of whether a given C-finite sequence is representable in terms of other given C-finite sequences can be answered. The following is a typical example.

Example 1. *(Graham et al., 1994, Exercise 7.26).*

The second-order Fibonacci numbers $\mathfrak{F}_n$ are defined by the recurrence

$$\mathfrak{F}_n = \mathfrak{F}_{n-1} + \mathfrak{F}_{n-2} + F_n \quad (n \geq 2), \qquad \mathfrak{F}_0 = 0, \mathfrak{F}_1 = 1.$$

Express $\mathfrak{F}_n$ in terms of the usual Fibonacci numbers F_n and F_{n+1}.

It is an easy matter to compute the recurrence

$$\mathfrak{F}_{n+4} = 2\mathfrak{F}_{n+3} + \mathfrak{F}_{n+2} - 2\mathfrak{F}_{n+1} - \mathfrak{F}_n \quad (n \geq 0);$$

we use this recurrence as the "C-finite definition" of the second order Fibonacci numbers $\mathfrak{F}_n$. Using the algorithm for problem RatRep, it is a matter of less than a second to prove that $\mathfrak{F}_n$ cannot be represented as a rational function in F_n and F_{n+1} alone; and the algorithm for problem AlgRep tells us that $\mathfrak{F}_n$ cannot even be represented by an algebraic function in F_n and F_{n+1}. However, $\mathfrak{F}_n$ can be expressed as a polynomial in F_n, F_{n+1} and n, and the algorithm for *problem PolyRep finds the representation* $\mathfrak{F}_n = \frac{1}{5}(2(n+1)F_n + nF_{n+1})$*; see Section 8 for details. No other algorithm is known to us which provides both the negative and the positive answers.* □

Countless identities in the literature on Fibonacci numbers (Hoggatt, 1979) are algebraic relations among C-finite sequences of several arguments; Catalan's identity

$$F_n^2 - F_{n+m}F_{n-m} = (-1)^{n-m}F_m^2, \tag{1}$$

a typical example. With Algorithm 3 (Section 7) all such identities can be found – and proved – automatically.

2 Problem Specification

In this section, we give a concrete description of the problem that we are dealing with. The *shift operator* E is defined on univariate sequences $a\colon \mathbb{Z} \to k$ by

$$(E \cdot a)(n) = a(n+1) \qquad (n \in \mathbb{Z}).$$

Polynomials in $k[E]$ represent linear constant coefficient recurrence operators. For instance, $(E^2 - E - 1) \cdot F = 0$ is the recurrence $F_{n+2} - F_{n+1} - F_n = 0$ in operator notation. The *i-th partial shift operator* E_i is defined on multisequences $a\colon \mathbb{Z}^d \to k$ by

$$(E_i \cdot a)(n_1, \ldots, n_i, \ldots, n_d) := a(n_1, \ldots, n_i + 1, \ldots, n_d) \qquad (n_1, \ldots, n_d \in \mathbb{Z}).$$

Following Zeilberger (1990), we define:

Definition 2 (C-finite sequences and multisequences)**.** A sequence $a\colon \mathbb{Z} \to k$ is *C-finite over* k iff it is annihilated by some nonzero operator $P \in k[E]$:

$$P \cdot a = 0, \quad P \in k[E], \quad P \neq 0.$$

A multisequence $a\colon \mathbb{Z}^d \to k$ is *C-finite over* k iff for each i with $1 \leq i \leq d$ there is a nonzero operator P_i in $k[E_i]$ such that

$$P_i \cdot a = 0.$$

If $a\colon \mathbb{Z} \to k$ is a C-finite sequence and $\alpha_1, \ldots, \alpha_d$ are integers, then

$$b(n_1, \ldots, n_d) = a(\alpha_1 n_1 + \cdots + \alpha_d n_d)$$

is a C-finite multisequence.

Definition 3 (Algebraic Relations)**.** Let $k \subseteq K$ be fields and let S be a set. The *ideal of algebraic relations over* k among functions $a_1, \ldots, a_r\colon S \to K$

is the kernel of the ring map $\varphi\colon k[x_1,\dots,x_r] \to K^S$ which maps x_i to a_i for $1 \le i \le r$ and which maps elements of k to corresponding constant functions. We denote it by $I(a_1,\dots,a_r;k)$. Algebraic relations among sequences and multisequences are defined by taking $S = \mathbb{Z}$ and $S = \mathbb{Z}^d$ respectively.

By Hilbert's basis theorem, $I(a_1,\dots,a_r;k)$ is finitely generated. The aim of this paper is to give an algorithm for computing generators for $I(a_1,\dots,a_r;\mathbb{Q})$ in the case where $a_1,\dots,a_r\colon \mathbb{Z}^d \to \mathbb{Q}$ are C-finite multisequences:

Problem MCRels.
(Algebraic Relations among C-finite Multisequences)
Input: C-finite multisequences $a_1,\dots,a_r\colon \mathbb{Z}^d \to \mathbb{Q}$, where each sequence is given by d recurrences (one for each argument) and sufficiently many initial values.
Output: A set $\{g_1,\dots,g_m\} \subseteq \mathbb{Q}[x_1,\dots,x_r]$ such that

$$I(a_1,\dots,a_r;\mathbb{Q}) = \langle g_1,\dots,g_m\rangle$$

Although we focus on sequences in $\mathbb{Q}$, all our results generalize immediately to sequences in algebraic number fields.

By "sufficiently many" initial values, we mean that the sequences should be determined uniquely by the recurrence equations and the initial values. To be precise, if $a\colon \mathbb{Z}^d \to \mathbb{Q}$ is defined by d recurrences having the orders $s_1,\dots,s_d \in \mathbb{N}$, respectively, then the specification of all values $a(n_1,\dots,n_d)$ for $0 \le n_i < s_i$ $(i = 1,\dots,d)$ would be needed in order to uniquely define a.

For solving Problem MCRels in full generality, we solve special cases of it first: The algorithm for the C-finite multisequences calls an algorithm for C-finite univariate sequences. That algorithm, in turn, calls an algorithm for the case of univariate geometric sequences. In summary, the problem reductions are:

$$\text{GeoRels (Section 3)} \longleftarrow \text{CRels (Section 4)} \longleftarrow \text{MCRels (Section 7)}$$

3 Relations among Geometric Sequences

Let $\bar{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ and $\bar{\mathbb{Q}}^\times = \bar{\mathbb{Q}} \setminus \{0\}$. It is well-known that any C-finite sequence over $\mathbb{Q}$ can be represented in terms of various geometric sequences $n \mapsto \zeta^n$ with $\zeta \in \bar{\mathbb{Q}}^\times$ and the sequence $n \mapsto n$. (For the Fibonacci numbers, Binet's formula (7) gives such a representation.) We study the algebraic relations among such sequences.

Problem GeoRels.
(Algebraic Relations among Geometric Sequences)
Input: $\alpha \in \bar{\mathbb{Q}}^\times$, given by $q \in \mathbb{Q}[x]\setminus\{0\}$ with $q(\alpha) = 0$, and $\zeta_1, \ldots, \zeta_r \in \mathbb{Q}(\alpha)^\times$
Output: A set $\{g_1, \ldots, g_m\} \subseteq \mathbb{Q}(\alpha)[x_0, x_1, \ldots, x_r]$ such that

$$I(n, \zeta_1^n, \ldots, \zeta_r^n; \bar{\mathbb{Q}}) = \langle g_1, \ldots, g_m \rangle$$

where x_0 corresponds to the arithmetic sequence $n \mapsto n$, and x_i corresponds to the geometric sequence $n \mapsto \zeta_i^n$, for $i = 1, \ldots, r$.

Multiplicative relations among the numbers $\zeta_1, \ldots, \zeta_r$ immediately imply corresponding relations among the geometric sequences $\zeta_1^n, \ldots, \zeta_r^n$: A trivial calculation shows that

$$\prod_{i=1}^{r} (\zeta_i^n)^{a_i} - \prod_{i=1}^{r} (\zeta_i^n)^{b_i} = 0 \qquad (n \in \mathbb{Z}), \tag{2}$$

for any integers $a_1, \ldots, a_r$ and $b_1, \ldots, b_r$ satisfying

$$\prod_{i=1}^{r} \zeta_i^{a_i - b_i} = 1. \tag{3}$$

Observe that the logarithmic map $\zeta \mapsto \log \zeta$ turns a multiplicative dependence $\prod_i \zeta_i^{m_i} = 1$ into a $\mathbb{Z}$-linear dependence $\sum_i m_i \log \zeta_i = 0$. We recall the following usual definitions (Ge, 1993; Sturmfels et al., 1995).

Definition 4. A *lattice* is a submodule of the $\mathbb{Z}$-module $\mathbb{Z}^r$. The *exponent lattice* of nonzero elements $\zeta_1, \ldots, \zeta_r$ of a field is given by

$$L(\zeta_1, \ldots, \zeta_r) := \left\{ (m_1, \ldots, m_r) \in \mathbb{Z}^r : \prod_{i=1}^{r} \zeta_i^{m_i} = 1 \right\}.$$

The *lattice ideal* $I(L)$ of a lattice $L \subseteq \mathbb{Z}^r$ is the ideal

$$I(L) := \left\langle \left\{ \prod_{i=1}^{r} x_i^{a_i} - \prod_{i=1}^{r} x_i^{b_i} : a \in \mathbb{N}^r, b \in \mathbb{N}^r, \text{ and } a - b \in L \right\} \right\rangle$$

of $\bar{\mathbb{Q}}[x_1, \ldots, x_r]$.

These definitions allow us to state (2)–(3) concisely as

$$I(\zeta_1^n, \ldots, \zeta_r^n; \bar{\mathbb{Q}}) \supseteq I(L(\zeta_1, \ldots, \zeta_r)). \tag{4}$$

In fact, equality holds true in (4), and throwing in the linear sequence $n \mapsto n$ does not introduce any new relations:

Proposition 5. *The relations among the $r + 1$ sequences $n, \zeta_1^n, \ldots, \zeta_r^n$ over $\bar{\mathbb{Q}}$ form the ideal of $R := \bar{\mathbb{Q}}[x_0, x_1, \ldots, x_r]$ generated by the lattice ideal of the* *exponent lattice of* $\zeta_1, \ldots, \zeta_r$*:*

$$I(n, \zeta_1^n, \ldots, \zeta_r^n; \bar{\mathbb{Q}}) = R\, I(L(\zeta_1, \ldots, \zeta_r))$$

Proof. Let $I := I(n, \zeta_1^n, \ldots, \zeta_r^n; \bar{\mathbb{Q}})$ and $J := R\, I(L(\zeta_1, \ldots, \zeta_r))$. We already know that $I \supseteq J$ by (2)–(3). It remains to show $I \subseteq J$. Let G be a Gröbner basis of J with respect to some fixed term order $\prec$. We show that we can reduce any $f \in I$ to 0 by G. Let $f \in I$ be arbitrary. Assume that f is totally reduced by G. We have to show that $f = 0$. Write f as

$$f = \sum_{a \in S} f_a(x_0) \prod_{i=1}^{r} x_i^{a_i}$$

with a minimal $S \subseteq \mathbb{Z}^r$, i.e., with $f_a \neq 0$ for $a \in S$. Since $f \in I$,

$$\sum_{a \in S} f_a(n) \left(\prod_{i=1}^{r} \zeta_i^{a_i} \right)^n = 0 \tag{5}$$

for all integers n. In (5), the bases $\prod_{i=1}^r \zeta_i^{a_i}$ of the geometric sequences are pairwise distinct. (Suppose, to the contrary, that $\prod_{i=1}^r \zeta_i^{a_i} = \prod_{i=1}^r \zeta_i^{b_i}$ for $a \neq b$ with $a \in S$ and $b \in S$. Then f would involve monomials $x_0^{a_0} \prod_{i=1}^r x_i^{a_i}$ and $x_0^{b_0} \prod_{i=1}^r x_i^{b_i}$ with $\prod_{i=1}^r x_i^{a_i} - \prod_{i=1}^r x_i^{b_i} \in J$, contradicting the assumption that f is totally reduced with respect to G.) Geometric sequences over a field k with pairwise distinct bases are linearly independent over $k[n]$ (for a proof of this well-known fact, see, for instance, Milne-Thomson, 1933, Section 13.0). Therefore, (5) implies that $f_a = 0$ for all $a \in S$. But we assumed $f_a \neq 0$ for all $a \in S$. So $S = \emptyset$, which means that $f = 0$. □

Algorithm 1 is a straightforward implementation of Proposition 5.

Algorithm 1 (solving Problem GeoRels)**.**
Input: $\alpha \in \bar{\mathbb{Q}}^\times$, given by $q \in \mathbb{Q}[x] \setminus \{0\}$ with $q(\alpha) = 0$, and $\zeta_1, \ldots, \zeta_r \in \mathbb{Q}(\alpha)^\times$
Output: A set $\{g_1, \ldots, g_m\} \subseteq \mathbb{Q}(\alpha)[x_0, x_1, \ldots, x_r]$ such that

$$I(n, \zeta_1^n, \ldots, \zeta_r^n; \bar{\mathbb{Q}}) = \langle g_1, \ldots, g_m \rangle\,.$$

```
function GeoRels(ζ_1, ..., ζ_r)
    L := ExponentLattice(ζ_1, ..., ζ_r; α)
    I := LatticeIdeal(L)
    return I
```

It builds on two procedures LatticeIdeal and ExponentLattice, which solve the following problems:

Problem ExponentLattice.
Input: $\alpha \in \bar{\mathbb{Q}}^\times$ and $\zeta_1, \ldots, \zeta_r \in \mathbb{Q}(\alpha)^\times$

Output: A set $\{v_1, \ldots, v_t\} \subseteq \mathbb{Z}^r$ such that

$$L(\zeta_1, \ldots, \zeta_r) = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_t.$$

Problem LatticeIdeal.
Input: A finite set $\{v_1, \ldots, v_t\}$ of vectors from $\mathbb{Z}^r$.
Output: A set $\{g_1, \ldots, g_m\} \subseteq \bar{\mathbb{Q}}[x_1, \ldots, x_r]$ such that

$$I(\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_t) = \langle g_1, \ldots, g_m \rangle .$$

Ge (1993) gives an efficient algorithm for solving Problem ExponentLattice. Algorithms for Problem LatticeIdeal can be found, for instance, in (Sturmfels et al., 1995).

Example 6. *What are the algebraic relations among n, ζ_+^n, ζ_-^n, and $(-1)^n$ over $\bar{\mathbb{Q}}$, where $\zeta_+ = (1+\sqrt{5})/2$ and $\zeta_- = (1-\sqrt{5})/2$? Ge's algorithm for Problem ExponentLattice delivers*

$$L(\zeta_+, \zeta_-, -1) = (1,1,1)\mathbb{Z} + (0,0,2)\mathbb{Z}$$

corresponding to $\zeta_+\zeta_- = -1$ and $(-1)^2 = 1$. Calling LatticeIdeal *on that lattice gives*

$$I(n, \zeta_+^n, \zeta_-^n, (-1)^n; \bar{\mathbb{Q}}) = \langle y_1y_2 - y_3, y_3^2 - 1\rangle$$

which means that all algebraic relations among n, ζ_+^n, ζ_-^n and $(-1)^n$ are consequences of $\zeta_+^n\zeta_-^n - (-1)^n = 0$ and $((-1)^n)^2 - 1 = 0$. □

4 Relations among C-finite Sequences over $\mathbb{Q}$

A fundamental and well known fact is that every C-finite sequence $a\colon \mathbb{Z} \to k$ can be written as a linear combination of geometric sequences with polynomial coefficients. If a satisfies the recurrence

$$c_0a(n) + c_1a(n+1) + \cdots + c_{s-1}a(n+s-1) + a(n+s) = 0 \quad (n \in \mathbb{Z})$$

then it has a representation of the form

$$a(n) = p_1(n)\zeta_1^n + \cdots + p_\ell(n)\zeta_\ell^n \quad (n \in \mathbb{Z}) \tag{6}$$

where $\zeta_1, \ldots, \zeta_\ell$ are the disctinct roots of the *characteristic polynomial*

$$c(z) = c_0 + c_1z + \cdots + c_{s-1}z^{s-1} + z^s$$

and $p_i(n)$ is a polynomial in n whose degree is less than the multiplicity of the root ζ_i $(i = 1, \ldots, \ell)$. As we may assume $c_0 \neq 0$ without loss of generality, we can assume that all roots ζ_i be different from 0. Representation (6) allows us to reduce the problem of finding all relations among C-finite sequences (Problem CRels) to the problem of finding all relations among geometric sequences $\zeta_1^n, \dots, \zeta_\ell^n$ and the arithmetic sequence n (Problem GeoRels).

Problem CRels.
(Algebraic Relations among C-finite Sequences)
Input: C-finite sequences $a_1, \dots, a_r \colon \mathbb{Z} \to \mathbb{Q}$, where each sequence is given by a recurrence and sufficiently many initial values.
Output: A set $\{g_1, \dots, g_m\} \subseteq \mathbb{Q}[x_1, \dots, x_r]$ such that

$$I(a_1, \dots, a_r; \mathbb{Q}) = \langle g_1, \dots, g_m \rangle .$$

Algorithm 2 receives recurrences for $a_1, \dots, a_r$ as input, and starts by expressing them in terms of suitable geometric sequences ζ_i^n and the arithmetic sequence n (line 2). Next, it computes a set A of generators for the ideal $J := I(n, \zeta_1^n, \dots, \zeta_\ell^n; \bar{\mathbb{Q}}) \subseteq \bar{\mathbb{Q}}[y_0, y_1, \dots, y_\ell]$ of relations among these helper sequences (line 4) by calling Algorithm 1. Since $a_j(n) = \sum_{i=1}^{\ell} p_{ij}(n)\zeta_i^n$, the ideal $I(a_1, \dots, a_r; \bar{\mathbb{Q}})$ is the kernel of the ring map $\psi \colon \bar{\mathbb{Q}}[x_1, \dots, x_r] \to \bar{\mathbb{Q}}[y_0, y_1, \dots, y_\ell]/J$ given by

$$\psi(x_j) := \sum_{i=1}^{s} p_{ij}(y_0) y_i + J, \qquad \psi(c) = c + J \text{ for } c \in \bar{\mathbb{Q}}.$$

A set G of generators for this kernel is computed by elimination using a Gröbner basis (line 5 – line 8) with respect to a suitable elimination ordering; the technique used is based on (Adams and Loustaunau, 1994, Theorem 2.4.2).

Algorithm 2 (solving Problem CRels)**.**
Input: C-finite sequences $a_1, \dots, a_r$ over $\mathbb{Q}$. Each sequence is given by a recurrence and initial values.
Output: A set $\{g_1, \dots, g_m\} \subseteq \mathbb{Q}[x_1, \dots, x_r]$ such that

$$I(a_1, \dots, a_r; \mathbb{Q}) = \langle g_1, \dots, g_m \rangle .$$

1 **function** CRels$(a_1, \dots, a_r)$
2 Compute $\zeta_i \in \bar{\mathbb{Q}}^\times$ and $p_{ij} \in \bar{\mathbb{Q}}[y_0]$ for $i = 1, \dots, \ell$ and $j = 1, \dots, r$ such that $a_j(n) = \sum_{i=1}^{\ell} p_{ij}(n)\zeta_i^n$ for $j = 1, \dots, r$ and every $n \in \mathbb{Z}$.
3 $\alpha :=$ PrimitiveElement$(\zeta_1, \dots, \zeta_\ell)$
4 $A :=$ GeoRels$(\zeta_1, \dots, \zeta_\ell; \alpha)$ as an ideal of $\bar{\mathbb{Q}}[y_0, \dots, y_\ell]$
5 $B := \{x_j - \sum_{i=1}^{s} p_{ij}(y_0)\, y_i : j = 1, \dots, r\}$
6 Endow $R := \bar{\mathbb{Q}}[y_0, y_1, \dots, y_\ell, x_1, \dots, x_r]$ with an elimination order $\prec$ that has $y_0, y_1, \dots, y_\ell$ higher than $x_1, \dots, x_r$.

$G :=$ MonicReducedGröbnerBasis$(A \cup B)$ in R with respect to $\prec$
return $G \cap \mathbb{Q}[x_1, \ldots, x_r]$

Example 7. *What are the algebraic relations among F_n, F_{n+1}, and $(-1)^n$ over $\mathbb{Q}$, where F_n is the sequence of Fibonacci numbers?*

Factorization of the characteristic polynomial $z^2 - z - 1$ and consideration of initial values gives Binet's formula

$$F_n = \frac{1}{\sqrt{5}}\zeta_+^n - \frac{1}{\sqrt{5}}\zeta_-^n, \quad F_{n+1} = \frac{1+\sqrt{5}}{2\sqrt{5}}\zeta_+^n - \frac{1-\sqrt{5}}{2\sqrt{5}}\zeta_-^n \qquad (n \in \mathbb{Z}), \tag{7}$$

where $\zeta_\pm = (1 \pm \sqrt{5})/2$ as in Example 6. There we got the result

$$I(n, \zeta_+^n, \zeta_-^n, (-1)^n; \bar{\mathbb{Q}}) = \langle y_1 y_2 - y_3, y_3^2 - 1\rangle.$$

By elimination via Buchberger's algorithm,

$$\begin{aligned}
&I(F_n, F_{n+1}, (-1)^n; \bar{\mathbb{Q}}) \\
&= \Big\langle x_1 - \frac{1}{\sqrt{5}}y_1 + \frac{1}{\sqrt{5}}y_2,\ x_2 - \frac{1+\sqrt{5}}{2\sqrt{5}}y_1 + \frac{1-\sqrt{5}}{2\sqrt{5}}y_2,\ x_3 - y_3, \\
&\qquad y_1 y_2 - y_3,\ y_3^2 - 1\Big\rangle \cap \bar{\mathbb{Q}}[x_1, x_2, x_3] \\
&= \langle x_1^2 + x_1 x_2 - x_2^2 + x_3,\ x_3^2 - 1\rangle.
\end{aligned}$$

The generators of this ideal correspond to the identities

$$F_n^2 + F_n F_{n+1} - F_{n+1}^2 + (-1)^n = 0 \text{ and } ((-1)^n)^2 - 1 = 0;$$

all other polynomial identities among F_n, F_{n+1}, and $(-1)^n$ are consequences of those two. □

By construction, Algorithm 2 returns a set of generators $G \subseteq \bar{\mathbb{Q}}[x_1, \ldots, x_r]$ for the ideal $I(a_1, \ldots, a_r; \bar{\mathbb{Q}})$ of $\bar{\mathbb{Q}}[x_1, \ldots, x_r]$. However, Problem CRels asks for generators $G \subseteq \mathbb{Q}[x_1, \ldots, x_r]$ for the ideal $I(a_1, \ldots, a_r; \mathbb{Q})$ of $\mathbb{Q}[x_1, \ldots, x_r]$. For proving Algorithm 2 correct in that sense (Theorem 10 below), we need two lemmata.

Lemma 8. *Let $f \in K[x_1, \ldots, x_r]$ be an algebraic relation of some sequences $a_1, \ldots, a_r \colon \mathbb{Z} \to k$ where K is an extension field of k. Then f is a linear combination of algebraic relations whose coefficients are in k.*

Proof. As K is an extension field of k, we can write f as

$$f = \alpha_1 f_1 + \cdots + \alpha_m f_m \tag{8}$$

with $f_1, \dots, f_m \in k[x_1, \dots, x_r]$ and coefficients $\alpha_1, \dots, \alpha_m \in K$ which are linearly independent over k. We show that $f_1, \dots, f_m$ are algebraic relations of $a_1, \dots, a_r$. Fix an arbitrary $n \in \mathbb{Z}$. As f is an algebraic relation, it follows by (8) that

$$\alpha_1 f_1(a_1(n), \dots, a_r(n)) + \dots + \alpha_k f_k(a_1(n), \dots, a_r(n)) = 0.$$

Note that $f_i(a_1(n), \dots, a_r(n)) \in k$ for $i = 1, \dots, m$. As $\alpha_1, \dots, \alpha_m$ are linearly independent over k, it follows that $f_i(a_1(n), \dots, a_r(n)) = 0$ for $i = 1, \dots, m$. Therefore, $f_1, \dots, f_m$ are algebraic relations of $a_1, \dots, a_r$. $\square$

Lemma 9. *Let $I \subseteq K[x_1, \dots, x_r]$ be the ideal of algebraic relations over K among sequences $a_1, \dots, a_r$ that take values in a subfield k of K. Then I has a finite set of generators in $k[x_1, \dots, x_r]$, i.e., I is defined over k.*

Proof. By Hilbert's Basis Theorem, I is generated by finitely many elements of $K[x_1, \dots, x_r]$.

In that ideal basis, we can replace each element $f \in K[x_1, \dots, x_r]$ by elements $f_1, \dots, f_m \in I \cap k[x_1, \dots, x_r]$ according to Lemma 8. $\square$

Theorem 10. *Algorithm 2 is correct. Its output G satisfies*

(1) $G \subseteq \mathbb{Q}[x_1, \dots, x_r]$,
(2) G generates the ideal $I(a_1, \dots, a_r; \mathbb{Q})$ of $\mathbb{Q}[x_1, \dots, x_r]$.

Proof. 1. By Lemma 9 with $k = \mathbb{Q}$, $K = \bar{\mathbb{Q}}$ there is an $A \subseteq \mathbb{Q}[x_1, \dots, x_r]$ that generates $I(a_1, \dots, a_r; \bar{\mathbb{Q}})$ over $\bar{\mathbb{Q}}$. Let B be the monic reduced Gröbner basis of A. As computing a Gröbner basis involves only field operations on the coefficient level, $B \subseteq \mathbb{Q}[x_1, \dots, x_r]$, too. By construction, both G and B are monic reduced Gröbner bases of $I(a_1, \dots, a_r; \bar{\mathbb{Q}})$. Since the monic reduced Gröbner basis of an ideal is unique, $G = B$, and $G \subseteq \mathbb{Q}[x_1, \dots, x_r]$ follows.

2. Let $f \in I(a_1, \dots, a_r; \mathbb{Q})$ be arbitrary. As $G = \{g_1, \dots, g_m\}$ generates $I(a_1, \dots, a_r; \bar{\mathbb{Q}})$ over $\bar{\mathbb{Q}}$, we can find, by reduction, cofactors $u_1, \dots, u_m$ in $\bar{\mathbb{Q}}[x_1, \dots, x_r]$ such that

$$f = u_1 g_1 + \dots + u_m g_m. \tag{9}$$

But, in fact, $u_1, \dots, u_m \in \mathbb{Q}[x_1, \dots, x_r]$: Both f and $g_1, \dots, g_m$ have coefficients in $\mathbb{Q}$, and reduction involves only rational operations on the coefficient level. By way of (9), G generates $I(a_1, \dots, a_r; \mathbb{Q})$ over $\mathbb{Q}$. $\square$

5 Separation of C-finite Multisequences

We say that a multisequence $a\colon \mathbb{Z}^d \to k$ is *quasiunivariate* if $a(n_1,\dots,n_d)$ depends only on one of its d arguments, i.e., if there is an index i and a sequence $b\colon \mathbb{Z} \to k$ such that $a(n_1,\dots,n_d) = b(n_i)$ for all $n_1,\dots,n_d \in \mathbb{Z}$. In this section we show that any C-finite multisequence can be expressed as a polynomial in quasiunivariate C-finite multisequences (Theorem 13). We call such a representation *separated*. While this result is almost trivial, it is the key for reducing Problem MCRels to Problem CRels in Section 7. Note that separated representations are particular to C-finite multisequences; P-finite multisequences in general do not admit them.

Example 11. *The well-known addition theorem for the Fibonacci numbers*

$$F_{m+n} = F_{m+1}F_n + F_mF_{n+1} - F_mF_n$$

gives a separated representation for F_{m+n}. □

The sequences annihilated by a fixed recurrence operator $P \in k[E]$ of order r form an r-dimensional vector space over k. The sequences $e_{P,0},\dots,e_{P,r-1}\colon \mathbb{Z} \to k$ defined by the recurrence $P \cdot e_{P,i} = 0$ and the "canonical" initial values

$$e_{P,i}(n) = \begin{cases} 1 & \text{if } n = i \\ 0 & \text{if } n \neq i \end{cases} \qquad \text{for } 0 \le n < r.$$

form a basis of this vector space. Indeed, any solution $a\colon \mathbb{Z} \to k$ of $P \cdot a = 0$ can be written as

$$a(n) = \sum_{0 \le i < r} a(i) e_{P,i}(n) \qquad (n \in \mathbb{Z}). \tag{10}$$

(Equation (10) is true by induction on n. For the induction step, note that both sides of it satisfy the same order r recurrence given by P; for the induction base, note that both sides agree for $n = 0, 1, \dots, r-1$.)

Lemma 12. *Let $a\colon \mathbb{Z}^d \to k$ be a C-finite multisequence satisfying the system of recurrences $P_1 \cdot a = 0, \dots, P_d \cdot a = 0$ with $P_i \in k[E_i] \setminus \{0\}$ for $i = 1, \dots, d$. Then*

$$a(n_1,\dots,n_d) := \sum_{0 \le i_1 < r_1} \cdots \sum_{0 \le i_d < r_d} a(i_1,\dots,i_d)\, e_{P_1,i_1}(n_1) \cdots e_{P_d,i_d}(n_d). \tag{11}$$

where $r_i = \deg P_i$ for $i = 1, \dots, d$.

Proof. By induction on d. The induction base $d = 1$ is Equation (10). Let $(n_1,\dots,n_{d-1}) \in \mathbb{Z}^{d-1}$ be arbitrary but fixed and consider $a(n_1,\dots,n_{d-1},n_d)$

as a univariate sequence in n_d. According to Equation (10), it has the representation

$$a(n_1, \ldots, n_{d-1}, n_d) = \sum_{0 \le i_d < r_d} a(n_1, \ldots, n_{d-1}, i_d) e_{P_d, i_d}(n_d). \tag{12}$$

As $(n_1, \ldots, n_{d-1})$ was arbitrary, (12) holds for all $(n_1, \ldots, n_d) \in \mathbb{Z}^d$. Consider the term $a(n_1, \ldots, n_{d-1}, i_d)$ appearing under the sum as a C-finite multisequence of $d-1$ arguments. By the induction hypothesis, it can be written as a $(d-1)$-fold sum of the shape (11). $\square$

Theorem 13. *Any C-finite multisequence can be separated: For any C-finite multisequence $a\colon \mathbb{Z}^d \to k$ there exists an $m \in \mathbb{N}$, C-finite sequences $b_1, \ldots, b_m : \mathbb{Z} \to k$, and a polynomial $f \in k[x_{11}, \ldots, x_{dm}]$ such that*

$$\begin{aligned} a(n_1, \ldots, n_d) = f(\ & b_1(n_1), \ \ldots, \ b_m(n_1), \\ & \ \ \vdots \qquad\qquad\ \ \vdots \\ & b_1(n_d), \ \ldots, \ b_m(n_d) \ ) \end{aligned}$$

for all $(n_1, \ldots, n_d) \in \mathbb{Z}^d$.

Proof. Equation (11) in Lemma 12 gives a suitable representation. $\square$

Theorem 13 states that the set of quasiunivariate multisequences generates the ring of all C-finite multisequences. Note that Equation (11) shows how to compute quasiunivariate representations effectively.

6 Separation and Algebraic Relations

Separation leaves us with the problem of computing the ideal I_* of relations among quasiunivariate multisequences

$$\begin{matrix} b_1(n_1), & \ldots, & b_m(n_1), \\ \vdots & & \vdots \\ b_1(n_d), & \ldots, & b_m(n_d) \end{matrix} \tag{13}$$

where $b_1, \ldots, b_m$ are C-finite. Computing the algebraic relations among the entries of a fixed row in this table is, essentially, a univariate problem; Algorithm 2 applies. Is I_* already generated by the union (taken over all the rows) of the relations among the entries in one row? Indeed, for $R = k[y_{11}, \ldots, y_{dm}]$ and $I_i = I(b_1, \ldots, b_m; k) \subseteq k[y_{i1}, \ldots, y_{im}]$, we have

$$\begin{aligned} R/I_* &\cong k[b_1(n_1), \ldots, b_m(n_1), \ldots\ldots, b_1(n_d), \ldots, b_m(n_d)] \\ &\cong \bigotimes_{i=1}^{d} k[b_1, \ldots, b_m] \cong \bigotimes_{i=1}^{d} k[y_{i1}, \ldots, y_{im}]/I_i \cong R/(RI_1 + \cdots + RI_d), \end{aligned}$$

so it is to be expected that $I_* = RI_1 + \cdots + RI_d$. For the sake of completeness, we shall give a detailed proof of this ideal identity in the remainder of this section. First we consider the special case $d = 2$.

Lemma 14. *Assume that the functions $a_1, \ldots, a_r \colon U \times V \to k$ depend only on their first argument, i.e., the one in U, while the functions $b_1, \ldots, b_s \colon U \times V \to k$ depend only on their second argument, i.e., the one in V. Let us write their algebraic relations in the ring $R = k[x_1, \ldots, x_r, y_1, \ldots, y_s]$ where x_i corresponds to a_i and y_j to b_j, for $i = 1, \ldots, r$ and $j = 1, \ldots, s$.*

(1) Let F be a Gröbner basis for $I(a_1, \ldots, a_r; k)$ and let G be a Gröbner basis for $I(b_1, \ldots, b_s; k)$ with respect to some fixed term order. Then $F \cup G$ is a Gröbner basis for $I(a_1, \ldots, a_r, b_1, \ldots, b_s; k)$.
(2) The relations among $a_1, \ldots, a_r, b_1, \ldots, b_s$ are generated by the relations among $a_1, \ldots, a_r$ together with the relations among $b_1, \ldots, b_s$:

$$I(a_1, \ldots, a_r, b_1, \ldots, b_s; k) = RI(a_1, \ldots, a_r; k) + RI(b_1, \ldots, b_s; k).$$

Proof. Part 2 immediately follows from Part 1; we prove Part 1.

Let $I_* = I(a_1, \ldots, a_r, b_1, \ldots, b_s; k)$. To show that $F \cup G$ is a Gröbner basis for $I_* := I(a_1, \ldots, a_r, b_1, \ldots, b_s; k)$, it suffices to show (a) that $F \cup G \subseteq I_*$ and (b) that any element of I_* reduces to 0 by $F \cup G$.

(a) $F \cup G \subseteq I_*$ since $F \subseteq I(a_1, \ldots, a_r; k) \subseteq I_*$ and $G \subseteq I(b_1, \ldots, b_s; k) \subseteq I_*$.

(b) Let $f \in I_*$ be fully reduced with respect to $F \cup G$. We have to show that $f = 0$. Fix an arbitrary $u \in U$. Define a ring map

$$\phi_u \colon k[x_1, \ldots, x_r, y_1, \ldots, y_s] \to k[y_1, \ldots, y_s]$$

fixing k by $\phi_u(x_i) = a_i(u)$ for $i = 1, \ldots, r$ and $\phi_u(y_i) = y_i$ for $i = 1, \ldots, s$. Note that $f \in I_*$ implies $\phi_u(f) \in I(b_1, \ldots, b_s; k)$. By assumption, f is fully reduced with respect to G. Since the head terms of elements of G involve only $y_1, \ldots, y_s$ while they are free of $x_1, \ldots, x_r$, this implies that also $\phi_u(f)$ is fully reduced with respect to G. As $\phi_u(f) \in I(b_1, \ldots, b_s; k)$ is fully reduced by a Gröbner basis of $I(b_1, \ldots, b_s; k)$, we know that, in fact, $\phi_u(f) = 0$.

Let us write the polynomial $f \in k[x_1, \dots, x_r, y_1, \dots, y_s]$ as a finite sum

$$f = \sum_{m \in \mathbb{N}^s} f_m \, y_1^{m_1} \dots y_s^{m_s} \tag{14}$$

with coefficient polynomials $f_m \in k[x_1, \dots, x_r]$. Since $\phi_u(f) = 0$, we have $\phi_u(f_m) = 0$ for all $m \in \mathbb{N}^s$. To show that $f = 0$, it remains to show that all coefficient polynomials f_m vanish. Fix an arbitrary m. As we have shown $\phi_u(f_m) = 0$ for an arbitrary $u \in U$, we know that $f_m \in I(a_1, \dots, a_r; k)$. Since, by assumption, f is fully reduced with respect to F, and since $F \subseteq k[x_1, \dots, x_r]$, we know by (14) that also f_m is fully reduced with respect to F. We have shown that $f_m \in I(a_1, \dots, a_r; k)$ is fully reduced with respect to a Gröbner basis of $I(a_1, \dots, a_r; k)$. Therefore, $f_m = 0$. $\square$

Generalizing Lemma 14 from functions of 2 to functions of d arguments is a simple matter of induction. The result is:

Theorem 15. *Consider an array*

$$\begin{matrix} b_{11}(n_1), & \dots, & b_{1m}(n_1) \\ \vdots & & \vdots \\ b_{d1}(n_d), & \dots, & b_{dm}(n_d) \end{matrix}$$

of $d \times m$ quasiunivariate multisequences $b_{ij} \colon \mathbb{Z}^d \to k$, in which multisequences in the i-th row depend only on their i-th argument n_i. Let $I_i = I(b_{i1}, \dots, b_{im}; k) \subseteq k[y_{i1}, \dots, y_{im}]$ be the ideal of relations of the entries in the i-th row, and let $I_* = I(b_{11}, \dots, b_{dm}; k) \subseteq k[y_{11}, \dots, y_{dm}]$ be the ideal of relations of all the entries in the array. Then I_* is generated by $I_1, \dots, I_d$:

$$I_* = \sum_{i=1}^{d} k[y_{11}, \dots, y_{dm}] I_i.$$

Proof. By induction on d. For $d = 1$, there is nothing to prove. In the induction step from d to $d+1$, use Lemma 14 Part 2 with $U = \mathbb{Z}^d$, $V = \mathbb{Z}$, $(a_1, \dots, a_r) = (b_{1,1}, \dots, b_{d,m})$, and $(b_1, \dots, b_s) = (b_{d+1,1}, \dots, b_{d+1,m})$. $\square$

Example 16. *Determine the ideal*

$$I_* := I(F_m, F_{m+1}, (-1)^m, F_n, F_{n+1}, (-1)^n; \mathbb{Q}) \subseteq R := \mathbb{Q}[x_1, x_2, x_3, y_1, y_2, y_3].$$

(Notation: F_m stands for the multisequence $(m, n) \mapsto F_m$, etc.)

By Example 7 (twice), both $I_1 := I(F_m, F_{m+1}, (-1)^m; \mathbb{Q}) \subseteq \mathbb{Q}[x_1, x_2, x_3]$ and $I_2 := I(F_n, F_{n+1}, (-1)^n; \mathbb{Q}) \subseteq \mathbb{Q}[y_1, y_2, y_3]$ are known. Clearly, I_* contains $RI_1 + RI_2$. The question is whether or not I_* contains anything beyond that.

As F_m, F_{m+1} and $(-1)^m$ depend only on m while F_n, F_{n+1} and $(-1)^n$ depend only on n, this is not the case, by Lemma 14. Therefore,

$$I_* = \langle x_1^2 + x_1x_2 - x_2^2 + x_3,\, x_3^2 - 1,\, y_1^2 + y_1y_2 - y_2^2 + y_3,\, y_3^2 - 1\rangle.$$

□

7 Relations among C-finite Multisequences

Now we have all the tools for solving Problem MCRels. All we need to do is to combine separation (Section 5, Theorem 13) with Theorem 15 and Algorithm 2; the result is Algorithm 3 below. This algorithm, like Algorithm 2, exploits (Adams and Loustaunau, 1994, Theorem 2.4.2).

Algorithm 3 (solving Problem MCrels)**.**
Input: C-finite multisequences $a_1, \ldots, a_r \colon \mathbb{Z}^d \to \mathbb{Q}$, where each sequence is given by d recurrences (one for each argument) and sufficiently many initial values.
Output: A finite set $G \subseteq \mathbb{Q}[x_1, \ldots, x_r]$ generating $I(a_1, \ldots, a_r; \mathbb{Q})$.

<u>function</u> $\textsc{MCRels}(a_1, \ldots, a_r)$

Compute a separated representation for $a_1, \ldots, a_r$. It consists of polynomials $p_1, \ldots, p_r \in \mathbb{Q}[y_{11}, \ldots, y_{dm}]$and univariate C-finite sequences $b_1, \ldots, b_m : \mathbb{Z} \to \mathbb{Q}$ such that

$$a_k(n_1, \ldots, n_d) = p_k\Big(\begin{array}{ccc} b_1(n_1), & \ldots, & b_m(n_1), \\ \vdots & & \vdots \\ b_1(n_d), & \ldots, & b_m(n_d) \end{array} \Big)$$

for $k = 1, \ldots, r$ and all $(n_1, \ldots, n_d) \in \mathbb{Z}^d$.

$F := \textsc{CRels}(b_1, \ldots, b_m)$ as an ideal of $\mathbb{Q}[z_1, \ldots, z_m]$.

$A := \bigcup_{i=1}^{d} \{f(y_{i1}, \ldots, y_{im}) : f \in F\}$

$B := \{x_k - p_k : k = 1, \ldots, r\}$

Endow $R := \mathbb{Q}[y_{11}, \ldots, y_{dm}; x_1, \ldots, x_r]$ with a term order $\prec$ for eliminating $y_{11}, \ldots, y_{dm}$.

$G := \textsc{MonicReducedGröbnerBasis}(A \cup B)$ in R with respect to $\prec$

<u>return</u> $G \cap \mathbb{Q}[x_1, \ldots, x_r]$

Theorem 17. *Algorithm 3 is correct: Its output G generates $I(a_1, \ldots, a_r; \mathbb{Q})$.*

Proof. By the correctness of Algorithm 2 and renaming of variables, the set $\{f(y_{i1},\dots,y_{im}) : f \in F\}$ generates the ideal $I_i := I(b_1(n_i),\dots,b_m(n_i);\mathbb{Q}) \subseteq k[y_{i1},\dots,y_{im}]$ for $i=1,\dots,d$. By Theorem 15, this implies that A generates $I_* := I(b_1(n_1),\dots,b_m(n_d);\mathbb{Q})$. From the representation of $a_1,\dots,a_r$ in terms of $b_1(n_1),\dots,b_m(n_d)$ computed in step 2, it follows that $I(a_1,\dots,a_r;\mathbb{Q})$ is the kernel of the ring map $\psi\colon \bar{\mathbb{Q}}[x_1,\dots,x_r] \to \mathbb{Q}[y_{11},\dots,y_{dm}]$ given by $\psi(x_j) := p_k + I_*$ for $j=1,\dots,r$ and $\psi(c) = c + I_*$ for $c \in \mathbb{Q}$. By (Adams and Loustaunau, 1994, Theorem 2.4.2), the set G computed in Step 5 – Step 8 generates the kernel of ψ. □

8 Finding Representations

It is sometimes of interest to know whether a given C-finite sequence can be represented in terms of other given C-finite sequences.

Problem Rep (variants: LinRep, PolyRep, RatRep, AlgRep).
Input: A C-finite (multi-)sequence a and C-finite (multi-)sequences $b_1,\dots,b_r$.
Output: Either a linear combination (resp. a polynomial, resp. a rational function, resp. an algebraic function) f in r variables such that

$$a(n_1,\dots,n_d) = f(b_1(n_1,\dots,n_d),\dots,b_r(n_1,\dots,n_d)) \tag{15}$$

for all $(n_1,\dots,n_d) \in \mathbb{Z}^d$ or the string "no such representation exists."

All four variants of the problem can be easily solved by looking at a Gröbner basis of

$$I(a,b_1,\dots,b_r;k) \subseteq \mathbb{Q}[x_0,x_1,\dots,x_r]$$

with respect to an elimination ordering for the variable x_0 corresponding to a:

(1) A linear combination $f = c_1x_1 + \cdots c_rx_r$ ($c_i \in \mathbb{Q}$) such that (15) holds exists if and only if the reduced Gröbner basis contains a polynomial of the form $c_0x_0 - c_1x_1 - \cdots - c_rx_r$ for some $c_i \in \mathbb{Q}$; in this case $f = c_1x_1 + \cdots + c_rx_r$.
(2) A polynomial $f \in \mathbb{Q}[x_1,\dots,x_r]$ such that (15) holds exists if and only if the reduced Gröbner basis contains a polynomial of the form $x_0 + q$ for some polynomial $q \in \mathbb{Q}[x_1,\dots,x_m]$; in this case, $f = -q$.
(3) A rational function $f \in \mathbb{Q}(x_1,\dots,x_r)$ such that (15) holds exists if and only if the Gröbner basis contains a polynomial of the form $px_0 + q$ for some polynomials $p,q \in \mathbb{Q}[x_1,\dots,x_m]$, $p \neq 0$; in this case, $f = -q/p$.
(4) An algebraic function $f(x_1,\dots,x_r)$ such that (15) holds exists if and only if the Gröbner basis contains a polynomial in which x_0 appears.

From another point of view, Problem Rep is about solving recurrences: We solve the defining recurrence of a in terms of the sequences $b_1,\dots,b_r$.

Example 1 (continued from page 2). *A lexicographic Gröbner basis of* $I(\mathfrak{F}(n), F_n, F_{n+1}; \mathbb{Q})$ *with respect to* $x_0 \succ x_1 \succ x_2$ *is* $\{-1+x_1^4+2x_1^3x_2-x_1^2x_2^2-2x_1x_2^3+x_2^4\}$. *As the generator of this ideal is free of* x_0, *we can conclude that there does not exist any algebraic function* A *with* $\mathfrak{F}_n = A(F_n, F_{n+1})$.

Taking the arithmetic sequence $n \mapsto n$ *into account, we find that a lexicographic Gröbner basis of* $I(\mathfrak{F}(n), F_n, F_{n+1}, n; \mathbb{Q})$ *with respect to* $x_0 \succ x_1 \succ x_2 \succ x_3$ *is* $\{-5x_0+2x_1+2x_1x_3+x_2x_3, -1+x_1^4+2x_1^3x_2-x_1^2x_2^2-2x_1x_2^3+x_2^4, 16-40x_0x_1^3-60x_0x_1^2x_2-8x_1^3x_2+70x_0x_1x_2^2-12x_1^2x_2^2+45x_0x_2^3+14x_1x_2^3-16x_2^4+16x_3-25x_2^4x_3\}$, *the first generator of which implies* $\mathfrak{F}_n = \frac{1}{5}(2(n+1)F_n + nF_{n+1})$. □

9 Minimal Recurrences

A C-finite sequence given by a linear recurrence equation of some order s may already satisfy a linear recurrence of smaller order than s. The well-known Berlekamp-Massey-Algorithm can be used for computing the shortest (least order) linear recurrence that a given C-finite sequence satisfies. More generally, consider recurrences of the form

$$a(n+s) = f\Big(a(n), \dots, a(n+s-1)\Big) \quad (n \in \mathbb{Z}).$$

We call such a recurrence linear, polynomial, rational, or algebraic, if f is a linear combination, a polynomial, a rational function, or an algebraic function of its arguments, respectively. Given a C-finite sequence, it might also be of interest to know the minimal order recurrence of any of these types.

Problem MinRec. (variants: LinMinRec, PolyMinRec, RatMinRec, AlgMinRec)
Input: A C-finite sequence a.
Output: A linear (resp. polynomial, resp. rational, resp. algebraic) recurrence equation of minimal order satisfied by a.

Problem MinRec and its variants can be easily reduced to the respective variant of problem Rep. Suppose that a is a univariate C-finite sequence, defined by a recurrence of order s. To find its minimal recurrence, use the algorithm for problem Rep to check whether $a(n+r)$ can be expressed in terms of $a(n), \dots, a(n+r-1)$, for $r = 0, \dots, s-1$. The first representation found is the smallest recurrence. If no representation is found for any r then the recurrence by which a was defined is already minimal.

Example 18. *For the Fibonacci numbers with even index,* F_{2n}, *we find the first order algebraic recurrence*

$$F_{2(n+1)} = \tfrac{1}{2}\Big(3F_{2n} + \sqrt{4+5F_{2n}^2}\Big).$$

There does not exist a rational first order recurrence for F_{2n}. $\square$

In Algorithm 3, we have assumed that C-finite multisequences $a\colon \mathbb{Z}^d \to \mathbb{Q}$ are defined by d separated recurrence equations, one per argument. Other recurrence equations, which the sequence may satisfy in addition, can be found by an application of Algorithm 3.

10 C-finite Sequences over $\mathbb{Q}(z_1, \ldots, z_n)$

So far, our algorithms deal with C-finite sequences over the field $\mathbb{Q}$ of rational numbers. In fact, they work also for C-finite sequences over the algebraic numbers $\bar{\mathbb{Q}}$ without any modification. In this section, we briefly sketch how to extend them to C-finite sequences over a field of rational functions $\mathbb{Q}(z_1, \ldots, z_n)$.

It turns out that the only problem with generalizing the algorithms from $\mathbb{Q}$ to $\mathbb{Q}(z_1, \ldots, z_n)$ is that Ge's algorithm ExponentLattice works for algebraic numbers $\zeta_1, \ldots, \zeta_r \in \mathbb{Q}[\alpha]^\times$ with $\alpha \in \bar{\mathbb{Q}}$, while for our present generalization we would need it for algebraic functions $\zeta_1, \ldots, \zeta_r \in \mathbb{Q}(z_1, \ldots, z_n)[\alpha]^\times$ with $\alpha \in \overline{\mathbb{Q}(z_1, \ldots, z_n)}$. There is a pragmatic approach for extending Ge's algorithm to the latter case: To get rid of the indeterminates $z_1, \ldots, z_n$, substitute randomly chosen rational numbers $z_1^{(1)}, \ldots, z_n^{(1)}$ for them in the defining relations of $\zeta_1, \ldots, \zeta_r$ and α. That way we obtain images $\zeta_1^{(1)}, \ldots, \zeta_r^{(1)} \in \mathbb{Q}[\alpha^{(1)}]^\times$, with $\alpha^{(1)} \in \bar{\mathbb{Q}}$, of $\zeta_1, \ldots, \zeta_r$, except possibly when some $\zeta_i^{(1)}$ has a singularity at $(z_1^{(1)}, \ldots, z_n^{(1)})$, which we can always avoid. Note that any multiplicative relation $\zeta_1^{m_1} \cdots \zeta_r^{m_r} = 1$ among $\zeta_1, \ldots, \zeta_r$ implies a corresponding relation $(\zeta_1^{(1)})^{m_1} \cdots (\zeta_r^{(1)})^{m_r} = 1$ among their images $\zeta_1^{(1)}, \ldots, \zeta_r^{(1)}$. Therefore, the lattice $L = L(\zeta_1, \ldots, \zeta_r)$ is contained in the lattice $L^{(1)} = L(\zeta_1^{(1)}, \ldots, \zeta_r^{(1)})$. Generators for $L^{(1)}$ can be computed by Ge's algorithm. In unlucky cases, the images $\zeta_1^{(1)}, \ldots, \zeta_r^{(1)}$ may satisfy additional multiplicative relations, and so we cannot conclude at this point that $L = L^{(1)}$. To make sure that we did not run into an unlucky case, all we have to do is to check membership in L for each generator $m \in \mathbb{Z}^r$ of $L^{(1)}$, i.e., to check that indeed $\zeta_1^{m_1} \ldots \zeta_r^{m_r} = 1$. This can be done, for instance, by an ideal membership test using Gröbner basis methods. If this check succeeds, ExponentLattice$(\zeta_1, \ldots, \zeta_r)$ finishes by returning the generators of $L = L^{(1)}$. Otherwise, in the unlucky case, the algorithm repeats the same steps with different values for $z_1, \ldots, z_n$, and so on. Unlucky cases can be made unlikely by drawing $z_1, \ldots, z_n$ from a large enough (finite) subset of $\mathbb{Q}^n$ with uniform probability. It would be interesting to find bounds for the probability of running into an unlucky case, or, better, to give a deterministic – but still efficient – algorithm.

In case we use N different images of $\zeta_1, \dots, \zeta_r$, leading to N superlattices $L^{(1)}, \dots, L^{(N)}$ of L, an optimization is possible: As a candidate for L, use their intersection $L^{(1)} \cap \cdots \cap L^{(N)}$, as it is, in general, smaller than each of them; Cohen (1993) describes how to intersect integer lattices.

Example 19. *The Chebyshev polynomials of the first kind $T_n(z)$ are C-finite over $\mathbb{Q}(z)$:*

$$T_{n+2}(z) - 2zT_{n+1}(z) + T_n(z) = 0 \qquad (n \in \mathbb{Z}).$$

With Algorithm 3 we can compute

$$I(T_{n-m}(z), T_n(z), T_{m+n}(z), T_m(z); \mathbb{Q}(z)) =$$
$$\langle -x_1 - x_3 + 2x_2x_4, x_2^2 + x_4^2 - x_1x_3 - 1, -2x_4^3 + 2x_1x_3x_4 + 2x_4 - x_1x_2 - x_2x_3 \rangle.$$

The second generator gives the identity

$$T_m(z)^2 + T_n(z)^2 - T_{n-m}(z)T_{m+n}(z) - 1 = 0$$

which is a well-known analog of Catalan's identity (1) *for the Chebyshev polynomials.* □

11 Examples and Applications

If the ideal of algebraic relations of some C-finite sequences is explicitly known, then a lot of information about these sequences can be computed algorithmically.

Proving and Finding Identities

In order to decide whether a conjectured algebraic relation of some given C-finite multisequences holds, it suffices to compute the ideal of the algebraic relations of these sequences by Algorithm 3 and to check whether the polynomial corresponding to the conjectured identity belongs to that ideal. For instance, Catalan's identity (1) can be proved in that way. Textbooks on Fibonacci numbers (Hoggatt, 1979, e.g.) list dozens of such identities. More interesting might be that such identities can also be *found* in an automated way, provided that it is specified where to search. In order to find, for instance, an identity that relates F_n, F_m, F_{n+m}, F_{n-m}, $(-1)^n$ and $(-1)^m$, it is sufficient to compute

$$I(F_n, F_m, F_{n+m}, F_{n-m}, (-1)^n, (-1)^m; \mathbb{Q}).$$

The ideal basis returned by Algorithm 3 contains a polynomial corresponding to (1).

We are by no means restricted to the Fibonacci numbers. Many other combinatorial sequences also obey C-finite recurrences, and Algorithm 2 can be used to study their algebraic relations.

Example 20. *The sequence f defined via*

$$f(n+3) = 5f(n+2) - 7f(n+1) + 4f(n), \quad f(0) = \tfrac{5}{16}, f(1) = \tfrac{3}{4}, f(2) = 2$$

describes the number of HC-polyominoes for $n \geq 2$ (Stanley, 1997, Example 4.7.18). With Algorithm 2, we find that $f(n), f(n+1), f(n+2)$ are algebraically dependent with 2^n via

$$\begin{aligned} 2^{2n} = 256f(n)^3 &- 896f(n)^2 f(n+1) + 1104f(n)f(n+1)^2 - 496f(n+1)^3 \\ &+ 320f(n)^2 f(n+2) - 752f(n)f(n+1)f(n+2) \\ &+ 512f(n+1)^2 f(n+2) + 112f(n)f(n+2)^2 \\ &- 160f(n+1)f(n+2)^2 + 16f(n+2)^3 \qquad (n \geq 0). \end{aligned}$$

This identity might not have been known before, and it seems hard to prove it in a combinatorial way.

With the algorithm for Problem AlgRep, we prove that $f(n)$ cannot be represented as an algebraic function in terms of $F_n, F_{n+1}, (-1)^n$ and n. We do not know of any other method – combinatorially or not – for proving the absence of such representations. □

Example 21. *The "Tribonacci" numbers T_n, defined via*

$$T_{n+3} = T_n + T_{n+1} + T_{n+2}, \quad T_0 = 0, T_1 = T_2 = 1$$

(Sloane and Plouffe, 1995, A000073), satisfy the identity

$$T_{2n}^3 + T_n^2 T_{4n} + 2T_{3n}T_{4n}T_{5n} + T_{2n}T_{4n}T_{6n} = 2T_nT_{2n}T_{3n} + T_{4n}^3 + T_{2n}T_{5n}^2 + T_{3n}^2 T_{6n}.$$

This identity was discovered by Algorithm 2. It appeared, together with some further polynomials, as basis element of $I(T_n, T_{2n}, \ldots, T_{6n}; \mathbb{Q})$. □

Example 22. *For the Perrin numbers P_n (Sloane and Plouffe, 1995, A001608), defined via*

$$P_{n+3} = P_n + P_{n+1}, \quad P_0 = 3, P_1 = 0, P_2 = 2,$$

we find

$$I(P_n, P_{2n}, P_{3n}; \mathbb{Q}) = \langle x_1^3 - 3x_1x_2 + 2x_3 - 6 \rangle,$$

and hence the identity $P_n^3 - 3P_nP_{2n} + 2P_{3n} = 6$. □

Solving Recurrences

Example 23. *It is easy to see that the sum*

$$a(n) = \sum_{k=0}^{n} \binom{n}{k} F_k$$

satisfies

$$a(n+2) = 3a(n+1) - a(n), \quad a(0) = 0, \quad a(1) = 1.$$

Using the algorithm for Problem PolyRep, we can solve this recurrence in terms of Fibonacci numbers, i.e., $b_1(n) = F_n$ *and* $b_2(n) = F_{n+1}$*, getting*

$$a(n) = F_n(2F_{n+1} - F_n)$$

which is well-known. □

Example 24. *The sum*

$$a(n) = \sum_{k=0}^{n} \binom{n}{k} F_{n+k}$$

satisfies the recurrence

$$a(n+2) = 4a(n+1) + a(n) \quad a(0) = 0, \quad a(1) = 2.$$

Using the algorithm for Problem PolyRep, we find the representation

$$a(n) = F_n(2F_n^2 - 3F_nF_{n+1} + 3F_{n+1}^2).$$

□

Example 25. *The sum*

$$a(n) = \sum_{k=0}^{n} \binom{n}{k} F_{2k}$$

satisfies the recurrence

$$a(n+2) = 5a(n+1) - 5a(n) \quad a(0) = 0, \quad a(1) = 1.$$

The algorithm for Problem AlgRep proves that $a(n)$ *cannot be written as an algebraic function in* n*,* F_n*, and* F_{n+1}*.* □

Proving Divisibility Relations

Example 26. *In order to prove the divisibility property*

$$L_n \mid L_{n+2m}^4 - (L_{2m}^2 - 4)^2 \qquad (n, m \geq 0) \tag{16}$$

for the Lucas numbers L_n defined by $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$, it suffices to find an identity of the form

$$L_{n+2m}^4 - (L_{2m}^2 - 4)^2 = q(n,m)L_n \qquad (n, m \geq 0)$$

for some integer sequence $q(n,m)$. If $q(n,m)$ can itself be expressed in terms of L_n, L_{2m}, and L_{n+2m}, then it can be computed. For, if

$$\mathfrak{a} := I(L_n, L_{2m}, L_{n+2m}; \mathbb{Q}) = \langle g_1, \ldots, g_\ell \rangle,$$

then, by an extended Gröbner basis computation (Becker et al., 1993, Section 5.6) we can find polynomials $c_0, \ldots, c_\ell$ such that

$$x_3^4 - (x_2^2 - 4)^2 = c_0 x_1 + c_1 g_1 + \cdots + c_\ell g_\ell.$$

In this way, we have found that

$$q(n,m) = (L_n - 2L_{n+2m}L_{2m})(L_n^2 + 2L_{n+2m}^2 - L_n L_{2m+n} L_{2m})$$

does the job. (Observe that $q(n,m) \neq 0$ for all $n, m \geq 0$.)

In fact, the present example is even simpler: (16) *follows by inspection from*

$$\mathfrak{a} = \langle -16 + x_1^4 + 8x_2^2 - x_2^4 - 2x_1^3 x_2 x_3 + 2x_1^2 x_3^2 + x_1^2 x_2^2 x_3^2 - 2x_1 x_2 x_3^3 + x_3^4 \rangle.$$

□

Example 27. *The problem proposed by Furdui (2002) can be treated in a similar way: Prove that* $\gcd(L_n, F_{n+1}) = 1$ *for all $n \geq 1$.*

Using Algorithm 2, we find that

$$I(L_n, F_{n+1}; \mathbb{Q}) = \langle x_1^4 - 10x_1^3 x_2 + 35x_1^2 x_2^2 - 50x_1 x_2^3 + 25x_2^4 - 1 \rangle.$$

Let us denote the generator of this ideal by g. An extended Gröbner basis computation shows that

$$1 = (x_1)^3 \cdot (x_1) + (-10x_1^3 + 35x_1^2 x_2 - 50x_1 x_2^2 + 25x_2^3) \cdot (x_2) + (-1) \cdot g.$$

Hence there are integer sequences $p(n), q(n)$ such that

$$1 = p(n)L_n + q(n)F_{n+1} + 0 \qquad (n \geq 1).$$

The claim follows. □

Example 28. *For the sequence $a(n)$ defined via*

$$a(n+2) = 5a(n+1) - a(n) \quad (n \geq 0), \qquad a(0) = a(1) = 1$$

we have

$$I(a(n), a(n+1); \mathbb{Q}) = \langle x_2^2 + x_1^2 + 3 - 5x_1x_2 \rangle.$$

An immediate consequence is that $a(n)a(n+1) \mid a(n+1)^2 + a(n)^2 + 3$ *for all* $n \in \mathbb{N}$. *Friendman (1995) has asked for a proof of this divisibility property. Such problems can easily be generated using our algorithm.* □

12 An Implementation

A package for the computer algebra system Mathematica 5 implementing Algorithm 3 is available for download at

http://www.risc.uni-linz.ac.at/research/combinat/software/

It provides a function "Dependencies" which computes the ideal of algebraic relations among a given list of C-finite multisequences over $\mathbb{Q}$. We illustrate the usage of this package by a short example, and refer to the user manual (Kauers and Zimmermann, 2007) for further information.

Example 1 (continued). *In order to compute the algebraic relations among* $\mathfrak{F}(n)$, F_n, F_{n+1} *and* n, *we type*

In[1]:= $\mathrm{Dependencies}[\{\mathfrak{F}[n], \mathrm{Fibonacci}[n], \mathrm{Fibonacci}[n+1], n\}, x,$
$\mathrm{Where} \to \{\mathfrak{F}[n+2] == \mathfrak{F}[n+1] + \mathfrak{F}[n] + \mathrm{Fibonacci}[n+2],$
$\mathfrak{F}[0] == 0, \mathfrak{F}[1] == 1\}]$

and obtain in less than a second the following basis:

Out[1]= $\{-5x_1 + 2x_2 + 2x_2x_4 + x_3x_4, -1 + x_2^4 + 2x_2^3x_3 - x_2^2x_3^2 - 2x_2x_3^3 + x_3^4, 16 - 40x_1x_2^3 - 60x_1x_2^2x_3 - 8x_2^3x_3 + 70x_1x_2x_3^2 - 12x_2^2x_3^2 + 45x_1x_3^3 + 14x_2x_3^3 - 16x_3^4 + 16x_4 - 25x_3^4x_4\}$

□

13 Concluding Remarks

Our algorithm depends heavily on the fact that linear recurrence equations (or differential equations) with constant coefficients admit closed form solutions in terms of exponentials and polynomials. In general, this is no longer true if the coefficients $c_i(n)$ in a recurrence equation

$$c_0(n)a(n) + c_1(n)a(n+1) + \cdots + c_r(n)a(n+r) = 0 \qquad (n \in \mathbb{Z})$$

can be polynomials in n. Solutions $a(n)$ of such recurrence equations are called *P-finite*. It would be very interesting to have an algorithm for computing the algebraic relations among given P-finite sequences. Such an algorithm would be extremely useful in the field of symbolic summation and integration of special functions.

Another line of generalization concerns Karr's $\Pi\Sigma$-theory. Recall that Karr's celebrated summation algorithm (Karr, 1981) is able to determine the algebraic relations among terms that are composed of nested indefinite sums and products (subject to some technical restrictions). For instance, Karr's algorithm finds

$$I(\sum_{k=1}^{n} \frac{H_k}{k}, H_n, H_n^{(2)}; \mathbb{Q}) = \langle 2x_1 - x_2^2 - x_3 \rangle,$$

where $H_k := \sum_{k=1}^{n} 1/k$ and $H_k^{(2)} := \sum_{k=1}^{n} 1/k^2$ denote the Harmonic numbers and the Harmonic numbers of second order, respectively. Karr's algorithm requires the constituents of each sum (e.g., k and H_k in the first sum above) to be algebraically independent. Schneider (2001) has extended Karr's algorithm such as to allow the appearence of $(-1)^n$ in summands. We believe that with our algorithms, this restriction could be relaxed further. This would for instance allow to compute a complete list of generators of

$$\begin{aligned} I(\sum_{k=1}^{n} \frac{1}{F_k F_{k+2}}, F_n, F_{n+1}; \mathbb{Q}) \supseteq \langle & x_1 x_3^2 - x_3^2 + x_1 x_2 x_3 - x_2 x_3 + 1, \\ & x_2^4 + 2x_3 x_2^3 - x_3^2 x_2^2 - 2x_3^3 x_2 + x_3^4 - 1 \rangle, \end{aligned}$$

which neither Karr's nor our algorithms can do alone. In particular, such a generalization would immediately lead to a summation algorithm for nested sums and products involving arbitrary C-finite sequences.

We did not analyze the complexity of our algorithms. The computation of a primitive element (Algorithm 2, line 3) is costly and dominates the runtime in many cases. Experiments suggest that it is the runtime bottleneck if the degrees of the minimal polynomials for $\zeta_1, \ldots, \zeta_\ell$ exceeds approximately 15. Less frequently, the runtime bottleneck is the Gröbner basis computation in Algorithm 2.

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References

Adams, W. W., Loustaunau, P., 1994. An Introduction to Gröbner Bases. No. 3 in Graduate Studies in Math. Amer. Math. Soc., New York.

Becker, T., Weispfenning, V., Kredel, H., 1993. Gröbner Bases. Springer.

Buchberger, B., 1965. Ein Algorithmus zum Auffinden der Basiselemente des Restklassenrings nach einem nulldimensionalen Polynomideal. Ph.D. thesis, Universität Innsbruck.

Cohen, H., 1993. A Course in Computational Algebraic Number Theory. Springer.

Everest, G., van der Poorten, A., Shparlinski, I., Ward, T., 2003. Recurrence Sequences. Vol. 104 of Mathematical Surveys and Monographs. American Mathematical Society.

Friendman, J., 1995. Problem B-785. The Fibonacci Quarterly 33 (2).

Furdui, O., 2002. Problem B-931. The Fibonacci Quarterly 40 (1), 85.

Ge, G., 1993. Algorithms related to multiplicative representations of algebraic numbers. Ph.D. thesis, U.C. Berkeley.

Graham, R. L., Knuth, D. E., Patashnik, O., 1994. Concrete Mathematics, 2nd Edition. Addison-Wesley.

Hoggatt, V. E., 1979. Fibonacci and Lucas numbers. The Fibonacci Association.

Karr, M., 1981. Summation in finite terms. Journal of the ACM 28, 305–350.

Kauers, M., Zimmermann, B., 2007. Dependencies — a Mathematica package for computing algebraic dependencies of C-finite sequences. Tech. rep., SFB F013, Johannes Kepler Universität, (in preparation).

Milne-Thomson, L. M., 1933. The Calculus of Finite Differences. Macmillan and Co., ltd.

Nemes, I., Petkovšek, M., 1995. RComp: a mathematica package for computing with recursive sequences. Journal of Symbolic Computation 20 (5–6), 745–753.

Salvy, B., Zimmermann, P., 1994. Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable. ACM Transactions on Mathematical Software 20 (2), 163–177.

Schneider, C., 2001. Symbolic summation in difference fields. Ph.D. thesis, RISC-Linz, Johannes Kepler Universität Linz.

Sloane, N. J. A., Plouffe, S., 1995. The Encyclopedia of Integer Sequences. Academic Press, http://research.att.com/˜njas/sequences/.

Stanley, R. P., 1997. Enumerative Combinatorics, Volume 1. Cambridge Studies in Advanced Mathematics 62. Cambridge University Press.

Sturmfels, B., Weismantel, R., Ziegler, G. M., 1995. Gröbner bases of lattices, corner polyhedra, and integer programming. Contributions to Algebra and Geometry 36 (2), 281–298.

Zeilberger, D., 1990. A holonomic systems approach to special function identities. Journal of Computational and Applied Mathematics 32, 321–368.