Obstacle to Factorization of LPDOs
(Extended Abstract)

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Abstract

We investigate the problem of approximate factorization of linear partial differential operators of arbitrary order and in arbitrarily many variables. Given any such operator L and a specified factorization of its symbol, we define the associated ring of obstacles to the factorization of L extending the specified factorization of the symbol. We derive some facts about obstacles and give an exhaustive enumeration of obstacles for operators of order two and three.

Introduction

We consider the problem of the factorization of Linear Partial Differential Operators (LPDO) over some function space. We start with the approach of Grigoriev and Schwarz [2], who gave an algorithm for factorization of such operators with separable symbol. In each step of this algorithm one has to solve a system of linear equations, which was proved to have at most one solution. So in each step one either finds the next homogeneous component of factors in the factorization or stops and concludes that there is no factorization.

We suggest to use the information obtained by the algorithm even in the case of non-existence of any factorization, that is to describe what actually prevents a factorization. We introduce the notion of a common obstacle to factorization. The idea goes back to Laplace, who found his famous invariants as the common obstacles for second order strictly hyperbolic operators, as described in [4]. Some particular cases of this idea are considered in [3].

A common obstacle is not unique in general. However, we prove that all common obstacles belong to the same class in the ring of obstacles, which is the factor ring of the ring of differential operators modulo the homogeneous ideal generated by the factors of the symbol. We say that this class of common obstacles is the obstacle, which is defined uniquely.

The paper is organized as follows: in Section 1 we fix some notation. In Section 2, we introduce the notion of a common obstacle. Then, in Section 3, we define the ring of obstacles and the uniquely determined obstacle. Uniqueness, order estimation, and behavior of families of factorizations are investigated. In Section 4 and 5 exact formulae for all obstacles of second and third order operators are determined.

1 Notation

Let K be a commutative ring of functions with 1 in the variables $x_1, \dots, x_n$. Consider a differential ring

$$K[X_1, \dots, X_n],$$

where the differential variables $X_1, \dots, X_n$ correspond to the usual partial derivations $\partial_{x_1}, \dots, \partial_{x_n}$ respectively. We use the notation

$$X^{(i_1,\dots,i_n)} := X_1^{i_1} \dots X_n^{i_n}$$

and define the (total) order in the usual fashion:

$$|X^{(i_1,\dots,i_n)}| = ord(X^{(i_1,\dots,i_n)}) := i_1 + \dots + i_n.$$

The differential polynomial ring $K[X_1, \dots, X_n]$ is graded by the total order. The elements of $K[X_1, \dots, X_n]$ are linear partial differential operators, which we abbreviate as LPDO. Consider $L \in K[X_1, \dots, X_n]$ of order d and with the coefficients $a_J \in K,\ J \in \mathbf{N^n}$, that is

$$L = \sum_{|J| \leq d} a_J X^J = \sum_{i=0}^{d} L_i, \tag{1}$$

where $L_i = \{a_J X^J \mid ord(X^J) = i\}$ is called a homogeneous component of L of order i [1]. The homogeneous component L_d is called the *symbol* of L and denoted by Sym_L.

Now, we define operations in $K[X_1, \dots, X_n]$: for two LPDOs L (1) and

$$M = \sum_{J \leq r} b_J X^J$$

we define the common operation of operator composition:

$$L * M := \sum_{|I| \leq d, |J| \leq r} a_I X^I \left(b_J \partial^J \right),$$

the operation of the polynomial multiplication:

$$L \cdot M := \sum_{|I| \leq d, |J| \leq r} \left(a_I b_J X^{I+J} \right).$$

2 Common Obstacle

In this section let L be an LPDO and let its symbol Sym_L be decomposed as $Sym_L = S_1 \cdot S_2$.

Definition 2.1. We say that a factorization $L = L_1 * L_2$ such that

$$Sym_{L_1} = S_1 \quad \text{and} \quad Sym_{L_2} = S_2$$

is of the *type* $(S_1)(S_2)$ (or an extension of the factorization $Sym_L = S_1 \cdot S_2$).

Example 2.2. Consider the second order operator

$$M = (e^x + y)\partial^2_{xx} + (x + (e^x + y)y)\partial^2_{xy} + xy\partial^2_{yy} + \partial_x + (x + y)\partial_y.$$

The decomposition of the symbol

$$Sym_M = (e^x + y)X^2 + (x + (e^x + y)y)XY^2 + xyY^2 = ((e^x + y)X + xY)(X + yY)$$

can be extended to the factorization

$$M = ((e^x + y)\partial_x + x\partial_y + 1) * (\partial_x + y\partial_y).$$

Remark 2.3. In general, not every decomposition of the symbol can be expanded into a factorization of the operator.

Definition 2.4. An LPDO $R \in K[X_1, \ldots, X_n]$ is called a *common obstacle* to factorization of the type $(S_1)(S_2)$ if there exists a factorization of the type $(S_1)(S_2)$ for the operator $L - R$ and R has minimal possible order.

Obviously, a common obstacle is not uniquely defined.

Remark 2.5. A common obstacle always exists, although it may be equal to 0.

Theorem 2.6. *Let L be an LPDO in two variables, $ord(L) = d$, and $Sym_L = S_1 \cdot S_2$, where S_1 and S_2 are coprime. Then the order of a common obstacle to a factorization of the type $(S_1)(S_2)$ is less or equal to $d - 2$.*

Remark 2.7. Let $deg(S_1) = deg(S_2) = 1$ and the number n of variables be 2. Then a common obstacle to factorization of the type $(S_1)(S_2)$ has order less or equal to 0. That is, any common obstacle is a zero order operator.

Remark 2.8. Let $deg(S_1) = 1$, $deg(S_2) = 2$ and the number n of variables be 2. Then a common obstacle to factorizations has order less or equal to 1.

3 Ring of obstacles

In this section let $L \in K[X_1, \ldots, X_n]$ and $Sym_L = S_1 \cdot S_2$, where S_1 and S_2 are coprime. Denote the orders of S_1 and S_2 by k and l, respectively.

Definition 3.1. We define the *ring of obstacles* as the factor ring

$$K(S_1, S_2) := K[X_1, \ldots, X_n]/\langle S_1, S_2\rangle,$$

where $\langle S_1, S_2\rangle$ is the homogeneous ideal generated by S_1 and S_2.

When L has no factorization of the type $(S_1)(S_2)$, one may, nevertheless, apply the algorithm of Grigoriev and Schwarz [2] to L, looking for a factorization of such a type. In this way, at every step one has to solve an equation in order to find the next homogeneous components of the factors of L. So, either there is a solution and we may proceed one more step, or, otherwise, we stop and have a common obstacle, which is necessarily unique by construction.

Definition 3.2. We call the common obstacle obtained by the above algorithm the *main obstacle.*

Definition 3.3. We define the *obstacle* to factorizations of the type $(S_1)(S_2)$ as the whole coset of the main obstacle in $K(S_1, S_2)$.

Theorem 3.4. *Any common obstacle belongs to the same coset in $K(S_1, S_2)$, that is the obstacle is uniquely defined as an element of the ring $K(S1, S2)$.*

Remark 3.5. So, the common obstacle is not unique, but there are the main obstacle and the obstacle, which are uniquely defined.

Remark 3.6. The factorization of L of the type $(S_1)(S_2)$ exists if and only if the obstacle equals zero. The factorization of L of the type $(S_1)(S_2)$ exists if and only if the main obstacle is zero.

Theorem 3.7. *The dimension of the ring of obstacles $K(S1, S2)$ in order $d < k + l$ is*

$$\binom{n+d-1}{n-1} - \chi(d-k)\binom{n+d-k-1}{n-1} - \chi(d-l)\binom{n+d-l-1}{n-1},$$

where

$$\chi(c) := \begin{cases} 1 & \text{if } c \geq 0 \ , \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.8. Let $k = l = 1$. Then by the theorem 3.7 the dimension of the ring of obstacles

- in order 0 is 1,
- in order 1 is $n - 2$.

Example 3.9. Let $k = 1$ and $l = 2$. Then by the theorem 3.7 the dimension of the ring of obstacles

- in order 0 is 1,
- in order 1 is $n - 1$,
- in order 2 is $\frac{n^2-n-2}{2}$.

Theorem 3.10. *Let f be an arbitrary function in K. Then the obstacle for the type $(S_1 \cdot \frac{1}{f})(S_2 \cdot f)$ agrees with the obstacle for the type $(S_1)(S_2)$.*

4 Obstacle for second order LPDO

In this section we assume that $n = 2$, i.e. we are in the case of two variables, and $L \in K[X_1, X_2]$ is an LPDO of second order. Suppose its symbol is decomposed as $Sym_L = S_1 \cdot S_2$, where S_1 and S_2 are coprime. Because of theorem 3.10, it is enough to find the obstacle only in the case where the coefficients in X_1 in S_1 and S_2 are 1. So we may consider

$$S_1 = X_1 + \alpha X_2, \quad S_2 = X_1 + \beta X_2,$$

where $\alpha, \beta \in K$, as the general forms of S_1 and S_2. Then the LPDO L may be written as

$$L = S_1 \cdot S_2 + a_{10}X_1 + a_{01}X_2 + a_{00},$$

for some $a_{10}, a_{01}, a_{00} \in K$. In this situation we can give an exact formula for the main obstacle of this type.

Theorem 4.1. *The main obstacle of the type* $(S_1)(S_2)$ *is*

$$c^2 - a_{10}c + a_{00},$$

where

$$c = \frac{1}{\alpha - \beta} \left(\partial_x(\beta) + \alpha \partial_y(\beta) + a_{10}\alpha - a_{01} \right).$$

Remark 4.2. The main obstacle to a factorization of the type is $(S_2)(S_1)$ is

$$c^2 - a_{10}c + a_{00},$$

where

$$c = \frac{1}{\beta - \alpha} \left(\partial_x(\alpha) + \beta \partial_y(\alpha) + a_{10}\beta - a_{01} \right).$$

Remark 4.3. So, now, from the formulae it is clear, that the obstacle to factorization of the type $(S_1)(S_2)$ is not the same as that of the type $(S_2)(S_1)$.

Remark 4.4. As mentioned at the beginning of this section, the obstacle in the case $S_1 = X_1, S_2 = X_2$ can be obtained from Theorem 4.1 by a change of variables. But for a second order operator it is easy to consider the case $S_1 = X_1, S_2 = X_2$ separately. Namely, we may find that the obstacle P_1 for the type $(X_1)(X_2)$ is

$$P_1 = a_{00} - \partial_{x_1}(a_{10}) - a_{10}a_{01}$$

and the obstacle P_2 for the type $(X_2)(X_1)$ is

$$P_2 = a_{00} - \partial_{x_2}(a_{01}) - a_{10}a_{01}.$$

Remark 4.5. One may note that the obtained obstacles P_1 and P_2 are exactly the Laplace invariants of a strictly hyperbolic second order LPDO [4] .

5 Obstacle for third order LPDO

In this section we assume that $n = 2$, i.e. we are in the case of two variables, and $L \in K[X_1, X_2]$ is an LPDO of third order. Suppose its symbol is decomposed as

$$Sym_L = S_1 \cdot S_2 \cdot S_3.$$

Because of Theorem 3.10, we may assume that the coefficients in X_1 in S_1 and S_2 are 1. So, we may say that the following are the general forms for S_1, S_2, S_3:

$$S_1 = X_1 + s_1 X_2,\ S_2 = X_1 + s_2 X_2,\ S_3 = X_1 + s_3 X_2,$$

where $s_1, s_2, s_3 \in K$. Thus, in this section we may consider the following as the general form of L:

$$L = S_1 \cdot S_2 \cdot S_3 + L_2 + L_1 + L_0,$$

where

$$L_2 = a_{20} X_1^2 + a_{11} X_1 X_2 + a_{02} X_2^2,\ L_1 = a_{10} X_1 + a_{01} X_2,\ L_0 = a_{00}$$

and all $a_{ij} \in K$.

In he following we determine the main obstacle to the factorization of L for every combination of S_1, S_2, S_3 into two factors of the form $(S_i)(S_j S_k)$ or $(S_j S_k) S_i$. It is convenient to introduce the following notation.

Definition 5.1. For $\alpha, \beta, \gamma \in K$ and $s \in \{-1, 1\}$ we define

$$\begin{aligned}
det(\alpha, \beta, \gamma, s) &:= s(\gamma - \alpha)(\beta - \alpha), \\
p_{10}(\alpha, \beta, \gamma, s) &:= \frac{1}{det(\alpha, \beta, \gamma, s)}((-\beta\gamma + \beta\alpha + \gamma\alpha) a_{20} - \alpha a_{11} + a_{02}), \\
p_{01}(\alpha, \beta, \gamma, s) &:= \frac{1}{det(\alpha, \beta, \gamma, s)}(\alpha\beta\gamma a_{20} - \beta\gamma a_{11} + (-\alpha + \beta + \gamma) a_{02}), \\
p_{00}(\alpha, \beta, \gamma, s) &:= \frac{1}{det(\alpha, \beta, \gamma, s)}(\alpha^2 a_{20} - \alpha a_{11} + a_{02}),
\end{aligned}$$

$$\begin{aligned}
P_1(\alpha, \beta, \gamma) &:= (a_{10} - S_1(p_{10}(\alpha, \beta, \gamma, s)) + g_{00} p_{01}(\alpha, \beta, \gamma, s)) \cdot X_1 + \\
&\quad (a_{01} + S_1(p_{01}(\alpha, \beta, \gamma, s)) + g_{00} p_{01}(\alpha, \beta, \gamma, s)) \cdot X_2 + a_{00}, \\
P_2(\alpha, \beta, \gamma) &:= a_{10} X_1 + (a_{01} - (S_1 \cdot S_2 + p_{10}(\gamma, \beta, \alpha, 1) X_1 + \\
&\quad p_{01}(\gamma, \beta, \alpha, 1) X_2)(\gamma)) X_2 + a_{00} - (S_1 \cdot S_2)(p_{00}(\gamma, \beta, \alpha, 1)).
\end{aligned}$$

Remark 5.2. Note that in the $p_I(\alpha, \beta, \gamma, s)$, $I \in \{(10), (01), (00)\}$ and $P_1(\alpha, \beta, \gamma)$ the second and the third variables commute, while in $P_2(\alpha, \beta, \gamma)$ the first and the second variables commute.

Theorem 5.3. *For the types $(S_i)(S_j \cdot S_k)$ and $(S_j \cdot S_k) \cdot (S_i)$, where S_i is coprime with $S_j \cdot S_k$, the main obstacles are $P_1(s_i, s_j, s_k)$ and $P_2(s_j, s_k, s_i)$, respectively.*

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