LINEAR RECURRENCES AND POWER SERIES DIVISION

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Abstract. Bousquet-Mélou and Petkovšek investigated the generating functions of multivariate linear recurrences with constant coefficients. We will give a reinterpretation of their theory by means of division theorems for formal power series, which clarifies the structural background and provides short, conceptual proofs. In addition, extending the division to the context of differential operators, the case of recurrences with polynomial coefficients can be treated in an analogous way.

Throughout this paper we will use the following notation: Let $\mathbb{K}$ be a field and d be the number of variables. Bold letters indicate tuples $\boldsymbol{x} = (x_1, \dots, x_d)$, monomials are written as $\boldsymbol{x}^{\boldsymbol{n}} = x_1^{n_1} \dots x_d^{n_d}$, and the scalar product is denoted by $\boldsymbol{u} \cdot \boldsymbol{w} = u_1 v_1 + \cdots + u_d v_d$. The support $\operatorname{supp}(F(\boldsymbol{x}))$ of a formal power series $F(\boldsymbol{x}) = \sum_{\boldsymbol{n} \in \mathbb{N}^d} f_{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}} \in \mathbb{K}[\![\boldsymbol{x}]\!]$ is the set of all monomials $\boldsymbol{x}^{\boldsymbol{n}}$ whose coefficients $f_{\boldsymbol{n}}$ are nonzero. Let $\mathbb{K}[\![\boldsymbol{x}]\!]^{\not\geq \boldsymbol{p}}$ denote the set of all power series with support in $\mathbb{N}^d \setminus (\boldsymbol{p} + \mathbb{N}^d)$. When we speak of a weight vector, we mean a vector in $\mathbb{R}^d$ with positive, $\mathbb{Q}$-linearly independent components. A weight vector $\boldsymbol{w}$ induces a total order $\prec_{\boldsymbol{w}}$ on $\mathbb{Z}^d$ as well as on the monomials $\boldsymbol{x}^{\boldsymbol{n}}$ in $\mathbb{K}[\![\boldsymbol{x}]\!]$: $\boldsymbol{a} \prec_{\boldsymbol{w}} \boldsymbol{b}$ and $\boldsymbol{x}^{\boldsymbol{a}} \prec_{\boldsymbol{w}} \boldsymbol{x}^{\boldsymbol{b}}$ if $\boldsymbol{w} \cdot \boldsymbol{a} < \boldsymbol{w} \cdot \boldsymbol{b}$. The initial monomial $\operatorname{in}_{\boldsymbol{w}}(F)$ of a power series F w.r.t. to a weight vector $\boldsymbol{w}$ is defined to be the $\prec_{\boldsymbol{w}}$-minimal element of $\operatorname{supp}(F)$.

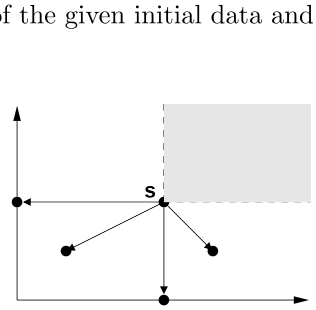
1. Recurrences with constant coefficients

Let $(f_{\boldsymbol{n}})_{\boldsymbol{n} \in \mathbb{N}^d}$ be a sequence in $\mathbb{K}$ given by the recurrence

$$f_{\boldsymbol{n}} = \begin{cases} \varphi(\boldsymbol{n}), & \boldsymbol{n} \in \mathbb{N}^d \setminus (\boldsymbol{s} + \mathbb{N}^d) \\ \displaystyle\sum_{\boldsymbol{t} \in H} c_{\boldsymbol{t}} f_{\boldsymbol{n}+\boldsymbol{t}}, & \boldsymbol{n} \in \boldsymbol{s} + \mathbb{N}^d \end{cases} \tag{1}$$

where $\boldsymbol{s} \in \mathbb{N}^d$ is the starting point of the recurrence and $H \subseteq \mathbb{Z}^d$ is the finite set of shifts such that $\boldsymbol{s} + H \subseteq \mathbb{N}^d$. The function $\varphi : \mathbb{N}^d \setminus (\boldsymbol{s} + \mathbb{N}^d) \to \mathbb{K}$ specifies the initial conditions. The coefficients $c_{\boldsymbol{t}}$ are constants in $\mathbb{K}$. Let H_0 denote the set $H \cup \{\boldsymbol{0}\}$. We define the apex $\boldsymbol{p}$ of the recurrence (1) as the vector $\boldsymbol{p} = (p_1, \dots, p_d)$ with $p_i = \max\{t_i : \boldsymbol{t} \in H \cup \{\boldsymbol{0}\}\}$. The objective then is to determine properties of the generating function $F(\boldsymbol{x}) = \sum_{\boldsymbol{n} \in \mathbb{N}^d} f_{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}}$ in terms of the given initial data and recurrence.

The picture on the right illustrates the situation for $H = \{(-3, 0), (-2, -1), (0, -2), (1, -1)\}$ with starting point $\boldsymbol{s} = (3, 2)$ and apex $\boldsymbol{p} = (1, 0)$. The area $\boldsymbol{s} + \mathbb{N}^d$ is shaded; outside of it the recurrence is given by the initial values.



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Theorem 1. *If there exists a weight vector $\boldsymbol{w} \in \mathbb{R}^d$ with positive components such that $\boldsymbol{w} \cdot \boldsymbol{t} < 0$ for all $\boldsymbol{t} \in H$, then the recurrence* (1) *has a unique solution.*

The proof can be found in [2].

In order to compute the generating function of $(f_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{N}^d}$, a functional equation for $F_{\boldsymbol{s}}(\boldsymbol{x}) = \sum_{\boldsymbol{n}\in\boldsymbol{s}+\mathbb{N}^d} f_{\boldsymbol{n}}\boldsymbol{x}^{\boldsymbol{n}-\boldsymbol{s}}$ can be deduced in a rather straight-forward manner (for details see [2]):

$$Q(\boldsymbol{x}) \cdot F_{\boldsymbol{s}}(\boldsymbol{x}) = K(\boldsymbol{x}) - U(\boldsymbol{x}), \tag{2}$$

where

$$\begin{aligned}
Q(\boldsymbol{x}) &= \boldsymbol{x}^{\boldsymbol{p}} - \sum_{\boldsymbol{t}\in H} c_{\boldsymbol{t}}\boldsymbol{x}^{\boldsymbol{p}-\boldsymbol{t}}, \\
K(\boldsymbol{x}) &= \sum_{\boldsymbol{t}\in H} \sum_{\boldsymbol{n}\in(\boldsymbol{s}+\boldsymbol{t}+\mathbb{N}^d)\setminus(\boldsymbol{s}+\mathbb{N}^d)} c_{\boldsymbol{t}}\varphi(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}-\boldsymbol{s}+\boldsymbol{p}-\boldsymbol{t}}, \\
U(\boldsymbol{x}) &= \sum_{\boldsymbol{t}\in H} \sum_{\boldsymbol{n}\in(\boldsymbol{s}+\mathbb{N}^d)\setminus(\boldsymbol{s}+\boldsymbol{t}+\mathbb{N}^d)} c_{\boldsymbol{t}}f_{\boldsymbol{n}}\boldsymbol{x}^{\boldsymbol{n}-\boldsymbol{s}+\boldsymbol{p}-\boldsymbol{t}}.
\end{aligned}$$

Here:

- $Q(\boldsymbol{x})$ is a polynomial that is given by the recurrence relation (the characteristic polynomial of the recurrence).
- $K(\boldsymbol{x})$ is known since it contains only coefficients which are given by the initial value function $\varphi(\boldsymbol{n})$. Note that $K(\boldsymbol{x})$ is a formal power series, i.e., no negative exponents occur: The exponents of $K(\boldsymbol{x})$ have the form $\boldsymbol{n}-\boldsymbol{s}+\boldsymbol{p}-\boldsymbol{t}$ with $\boldsymbol{n} \in (\boldsymbol{s}+\boldsymbol{t}+\mathbb{N}^d) \setminus (\boldsymbol{s}+\mathbb{N}^d)$, hence $\boldsymbol{n}-\boldsymbol{t}-\boldsymbol{s} \in \mathbb{N}^d$.
- $U(\boldsymbol{x})$ is a formal power series and is unknown. The exponents of $U(\boldsymbol{x})$ have the form $(\boldsymbol{n}-\boldsymbol{s}-\boldsymbol{t})+\boldsymbol{p}$ with $\boldsymbol{n} \in (\boldsymbol{s}+\mathbb{N}^d)\setminus(\boldsymbol{s}+\boldsymbol{t}+\mathbb{N}^d)$, hence $\boldsymbol{n}-\boldsymbol{t}-\boldsymbol{s} \notin \mathbb{N}^d$. Thus $\operatorname{supp}(U(\boldsymbol{x})) \subseteq \mathbb{N}^d \setminus (\boldsymbol{p}+\mathbb{N}^d)$.

The equation (2) involves two unknown series, namely $F_{\boldsymbol{s}}(\boldsymbol{x})$ and $U(\boldsymbol{x})$, and two given series, $Q(\boldsymbol{x})$ and $K(\boldsymbol{x})$. It is now immediate to write (2) in a slightly different way:

$$K(\boldsymbol{x}) = Q(\boldsymbol{x}) \cdot F_{\boldsymbol{s}}(\boldsymbol{x}) + U(\boldsymbol{x}). \tag{3}$$

This is nothing else but a Euclidean division of power series with remainder: The formal power series $K(\boldsymbol{x})$ is divided by the polynomial $Q(\boldsymbol{x})$ yielding the quotient $F_{\boldsymbol{s}}(\boldsymbol{x})$ and the remainder $U(\boldsymbol{x})$.

To justify this, we have to show that $U(\boldsymbol{x})$ satisfies the appropriate support condition. We choose the monomial order that is induced by the weight vector $\boldsymbol{w}$ from Theorem 1 in order to make $\boldsymbol{x}^{\boldsymbol{p}}$ the initial monomial of $Q(\boldsymbol{x})$. Since $\boldsymbol{w} \cdot \boldsymbol{t} < 0$ we have that $\boldsymbol{x}^{\boldsymbol{p}} \prec_w \boldsymbol{x}^{\boldsymbol{p}-\boldsymbol{t}}$ for all $\boldsymbol{t} \in H$, and therefore $x^{\boldsymbol{p}}$ is $\prec_{\boldsymbol{w}}$-minimal in $\operatorname{supp}(Q)$. To make the order total, $\boldsymbol{w}$ additionally has to have $\mathbb{Q}$-linearly independent components. We have seen that $U(\boldsymbol{x})$ is in $\mathbb{K}[\![\boldsymbol{x}]\!]^{\not\geq \boldsymbol{p}}$, and therefore contains only monomials that are smaller (w.r.t. $\prec_{\boldsymbol{w}}$) than the initial monomial $\boldsymbol{x}^{\boldsymbol{p}}$ of $Q(\boldsymbol{x})$.

2. Division of formal power series

The division (3) can be carried out explicitly by generalizing the usual Euclidean division in $\mathbb{K}[x]$ to the multivariate power series ring $\mathbb{K}[\![\boldsymbol{x}]\!]$ (Weierstraß division). We interpret the division by a power series as a perturbation of the division by its initial monomial. Let's have a short look on a special case:

Example 1. The division of a formal power series $P(\boldsymbol{x})$ by a monomial $\boldsymbol{x}^{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^d$, is equivalent to the direct sum decomposition $\mathbb{K}[\![\boldsymbol{x}]\!] = \boldsymbol{x}^{\boldsymbol{n}}\mathbb{K}[\![\boldsymbol{x}]\!] \oplus \mathbb{K}[\![\boldsymbol{x}]\!]^{\not\geq \boldsymbol{n}}$ (when viewed as vector spaces). For the division we get $P(\boldsymbol{x}) = x^{\boldsymbol{n}} \cdot F(\boldsymbol{x}) + R(\boldsymbol{x})$ where the remainder $R(\boldsymbol{x})$ has to fulfill the support condition $\operatorname{supp}(R(\boldsymbol{x})) \subseteq \mathbb{K}[\![\boldsymbol{x}]\!]^{\not\geq \boldsymbol{n}}$. Note that $\mathbb{K}[\![\boldsymbol{x}]\!]^{\not\geq \boldsymbol{n}}$ is isomorphic to $\mathbb{K}[\![\boldsymbol{x}]\!]/\langle \boldsymbol{x}^{\boldsymbol{n}}\rangle$, again when viewed as vector spaces.

In a straightforward manner this example can be extended to the division by a power series $A(\boldsymbol{x}) \in \mathbb{K}[\![\boldsymbol{x}]\!]$ with initial monomial $\boldsymbol{x}^{\boldsymbol{n}}$ (w.r.t. some monomial order), and one gets $\mathbb{K}[\![\boldsymbol{x}]\!] = A(\boldsymbol{x})\mathbb{K}[\![\boldsymbol{x}]\!] \oplus \mathbb{K}[\![\boldsymbol{x}]\!]^{\not\geq \boldsymbol{n}}$.

In our setting where the division (3) arises from a recurrence, we are in fact not interested in performing the division explicitly, because we can obtain the result of the division (a power series representation of the generating function) by just applying the recurrence relation. We are more interested in deducing properties of the generating function.

Assume that the apex $\boldsymbol{p}$ is $\boldsymbol{0}$; from the support condition on $U(\boldsymbol{x})$ follows that $U(\boldsymbol{x}) = 0$. Equation (3) simplifies to $F_{\boldsymbol{s}}(\boldsymbol{x}) = K(\boldsymbol{x})/Q(\boldsymbol{x})$. Hence, if the apex is $\boldsymbol{0}$ and $K(\boldsymbol{x})$ is a rational function, then the generating function $F_{\boldsymbol{s}}(\boldsymbol{x})$ is a rational function.

In the case of convergent power series, we invoke the Grauert-Hironaka-Galligo division theorem (cf. [12, 9, 10, 3, 4]:

Theorem 2. *Let $\mathbb{K}$ be $\mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{Q}_p$ or any complete valued field, and let $A(\boldsymbol{x}) \in \mathbb{K}\{\boldsymbol{x}\}$ be a convergent power series. Let again $\boldsymbol{x}^{\boldsymbol{n}}$ be the initial monomial of $A(\boldsymbol{x})$ with respect to some monomial order on $\mathbb{N}^d$. Then*

$$\mathbb{K}\{\boldsymbol{x}\} = A(\boldsymbol{x})\mathbb{K}\{\boldsymbol{x}\} \oplus \mathbb{K}\{\boldsymbol{x}\}^{\not\geq \boldsymbol{n}}.$$

We conclude that the solution $F_{\boldsymbol{s}}(\boldsymbol{x})$ of (3) is a convergent power series provided that the initial conditions constitute a convergent series $K(\boldsymbol{x}) \in \mathbb{K}\{\boldsymbol{x}\}$. This has been proven in Theorem 7 of [2].

A power series $A(\boldsymbol{x}) \in \mathbb{K}[\![\boldsymbol{x}]\!]$ is called algebraic, if there exists a polynomial $P(\boldsymbol{x}, t) \in \mathbb{K}[\boldsymbol{x}][t]$ such that $P(\boldsymbol{x}, A(\boldsymbol{x})) = 0$, or, more explicitely, if there are polynomials $p_0, \ldots, p_m \in \mathbb{K}[\boldsymbol{x}], p_m \neq 0$ such that

$$p_m(\boldsymbol{x})A(\boldsymbol{x})^m + \cdots + p_1(\boldsymbol{x})A(\boldsymbol{x}) + p_0(\boldsymbol{x}) = 0.$$

Let $\mathbb{K}[\![\boldsymbol{x}]\!]^{\text{alg}} \subseteq \mathbb{K}[\![\boldsymbol{x}]\!]$ denote the subalgebra of algebraic power series. In this case we invoke the Lafon-Hironaka division theorem (cf. [14, 11]):

Theorem 3. *Let $A(\boldsymbol{x}) \in \mathbb{K}[\![\boldsymbol{x}]\!]^{\text{alg}}$ and let $\boldsymbol{x}^{\boldsymbol{n}}$ be the initial monomial of $A(\boldsymbol{x})$ (w.r.t. some monomial order) with $\boldsymbol{n} = (0, \ldots, 0, n_k, 0, \ldots, 0)$. Then*

$$\mathbb{K}[\![\boldsymbol{x}]\!]^{\text{alg}} = A(\boldsymbol{x})\mathbb{K}[\![\boldsymbol{x}]\!]^{\text{alg}} \oplus \left(\mathbb{K}[\![\boldsymbol{x}]\!]^{\text{alg}}\right)^{\not\geq \boldsymbol{n}}.$$

A constructive version of this theorem using polynomial codes of algebraic power series has been developed in [1].

In particular, the theorem implies that in the division (3) the quotient $F_{\boldsymbol{s}}(\boldsymbol{x})$ and the remainder $U(\boldsymbol{x})$ are algebraic, provided that $K(\boldsymbol{x})$ is algebraic and the initial monomial of $Q(\boldsymbol{x})$ involves only one variable. Hence, if the apex $\boldsymbol{p}$ of (1) has exactly one nonzero component and if the initial conditions constitute an algebraic power series $K(\boldsymbol{x})$, then the generating function $F_{\boldsymbol{s}}(\boldsymbol{x})$ is algebraic. This has been proven in Theorem 13 of [2].

3. Recurrences with polynomial coefficients

We are now considering P-finite recurrences, i.e., recurrences with polynomial coefficients $c_{\boldsymbol{t}}(\boldsymbol{n}) \in \mathbb{K}[\boldsymbol{n}]$, written in the following form:

$$\left\{\begin{array}{rll} f_{\boldsymbol{n}} & = \varphi(\boldsymbol{n}), & \boldsymbol{n} \in \mathbb{N}^d \setminus (\boldsymbol{s} + \mathbb{N}^d) \\ c_{\boldsymbol{0}}(\boldsymbol{n}) f_{\boldsymbol{n}} & = -\displaystyle\sum_{\boldsymbol{t} \in H} c_{\boldsymbol{t}}(\boldsymbol{n}) f_{\boldsymbol{n}+\boldsymbol{t}}, & \boldsymbol{n} \in \boldsymbol{s} + \mathbb{N}^d \end{array}\right. \tag{4}$$

The existence of a unique solution for P-finite recurrences can be stated in a similar way as in Theorem 1 for C-finite recurrences:

Corollary 4. *If there exists a weight vector $\boldsymbol{w} \in \mathbb{R}^d$ with positive components such that $\boldsymbol{w} \cdot \boldsymbol{t} < 0$ for all $\boldsymbol{t} \in H$, and if additionally the polynomial $c_{\mathbf{0}}(\boldsymbol{n})$ has no integer root in $\boldsymbol{s} + \mathbb{N}^d$, then the recurrence (4) has a unique solution.*

In contrast to Theorem 1, we additionally require that the polynomial $c_{\mathbf{0}}(\boldsymbol{n})$ does not have integer roots in the region $\boldsymbol{s} + \mathbb{N}^d$ where the recurrence relation is applied; this condition is trivially fulfilled for constant coefficients. If it happens that $c_{\mathbf{0}}$ does have an integer root there, the whole recursion would break down. This situation can often be avoided by an adequate choice of the starting point $\boldsymbol{s}$. In the case $d = 1$ this is always possible, whereas for $d > 1$ there are instances for which there is no such $\boldsymbol{s}$. Corollary 4 also follows immediately from Theorem 5 in [2]. In the following we will always assume that the recurrence fulfills the conditions of the theorem.

The proof of Theorem 4 via division theorems is slightly more delicate than for Theorem 1. The functional equation (2) has to be replaced by a differential equation. This is done as follows:

Let $x^{\underline{k}}$ denote the falling factorial $x(x-1)\cdots(x-k+1)$ where $x^{\underline{0}}$ is set equal to 1. The falling factorials constitute a basis for the polynomial ring $\mathbb{K}[x]$ via the formula $x^n = \sum S(n,k) x^{\underline{k}}$ where $S(n,k)$ denote the Stirling numbers of the second kind. For several variables the falling factorial is defined as $\boldsymbol{x}^{\underline{\boldsymbol{k}}} = \prod_{i=1}^d x_i^{\underline{k_i}}$, and obviously also any multivariate polynomial can be written in terms of falling factorials $\boldsymbol{x}^{\underline{\boldsymbol{k}}}$. We now transform the polynomials $c_{\boldsymbol{t}}(\boldsymbol{n})$ in this manner, but take for convenience shifted falling factorials:

$$c_{\boldsymbol{t}}(\boldsymbol{n}) = \tilde{c}_{\boldsymbol{t}}(\boldsymbol{n} - \boldsymbol{s} + \boldsymbol{p}) = \sum_{\boldsymbol{k} \in S_{\boldsymbol{t}}} c_{\boldsymbol{t}\boldsymbol{k}} (\boldsymbol{n} - \boldsymbol{s} + \boldsymbol{p})^{\underline{\boldsymbol{k}}}$$

with certain coefficients $c_{\boldsymbol{t}\boldsymbol{k}} \in \mathbb{K}$ and a finite index set $S_{\boldsymbol{t}} \subset \mathbb{N}^d$.

Let $F_{\boldsymbol{s}}(\boldsymbol{x})$ again denote the generating function $\sum_{\boldsymbol{n} \in \boldsymbol{s} + \mathbb{N}^d} f_{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}-\boldsymbol{s}}$. Then our recurrence (4) rewrites as follows:

$$\begin{aligned}
0 &= \sum_{\boldsymbol{t} \in H_0} c_{\boldsymbol{t}}(\boldsymbol{n}) f_{\boldsymbol{n}+\boldsymbol{t}} \\
&= \sum_{\boldsymbol{n} \in \boldsymbol{s}+\mathbb{N}^d} \sum_{\boldsymbol{t} \in H_0} c_{\boldsymbol{t}}(\boldsymbol{n}) f_{\boldsymbol{n}+\boldsymbol{t}} \boldsymbol{x}^{\boldsymbol{n}-\boldsymbol{s}+\boldsymbol{p}} \\
&= \sum_{\boldsymbol{t} \in H_0} \sum_{\boldsymbol{n} \in \boldsymbol{s}+\boldsymbol{t}+\mathbb{N}^d} c_{\boldsymbol{t}}(\boldsymbol{n}-\boldsymbol{t}) f_{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}-\boldsymbol{s}+\boldsymbol{p}-\boldsymbol{t}} \\
&= \sum_{\boldsymbol{t} \in H_0} \sum_{\boldsymbol{n} \in \boldsymbol{s}+\mathbb{N}^d} \sum_{\boldsymbol{k} \in S_{\boldsymbol{t}}} c_{\boldsymbol{t}\boldsymbol{k}} (\boldsymbol{n}-\boldsymbol{s}+\boldsymbol{p}-\boldsymbol{t})^{\underline{\boldsymbol{k}}} f_{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}-\boldsymbol{s}+\boldsymbol{p}-\boldsymbol{t}} - K(\boldsymbol{x}) + U(\boldsymbol{x}) \\
&= \sum_{\boldsymbol{t} \in H_0} \sum_{\boldsymbol{k} \in S_{\boldsymbol{t}}} \sum_{\boldsymbol{n} \in \boldsymbol{s}+\mathbb{N}^d} (c_{\boldsymbol{t}\boldsymbol{k}} \boldsymbol{x}^{\boldsymbol{k}} \partial^{\boldsymbol{k}} \boldsymbol{x}^{\boldsymbol{p}-\boldsymbol{t}})[f_{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}-\boldsymbol{s}}] - K(\boldsymbol{x}) + U(\boldsymbol{x}) \\
&= \sum_{\boldsymbol{t} \in H_0} \sum_{\boldsymbol{k} \in S_{\boldsymbol{t}}} (c_{\boldsymbol{t}\boldsymbol{k}} \boldsymbol{x}^{\boldsymbol{k}} \partial^{\boldsymbol{k}} \boldsymbol{x}^{\boldsymbol{p}-\boldsymbol{t}}) (F_{\boldsymbol{s}}(\boldsymbol{x})) - K(\boldsymbol{x}) + U(\boldsymbol{x}).
\end{aligned}$$

Here, $D = \sum_{\boldsymbol{t} \in H_0} \sum_{\boldsymbol{k} \in S_{\boldsymbol{t}}} c_{\boldsymbol{t}\boldsymbol{k}} \boldsymbol{x}^{\boldsymbol{k}} \partial^{\boldsymbol{k}} \boldsymbol{x}^{\boldsymbol{p}-\boldsymbol{t}}$ is now a differential operator with polynomial coefficients. The power series

$$K(\boldsymbol{x}) = -\sum_{\boldsymbol{t} \in H} \sum_{\boldsymbol{n} \in (\boldsymbol{s}+\boldsymbol{t}+\mathbb{N}^d) \setminus (\boldsymbol{s}+\mathbb{N}^d)} \varphi(\boldsymbol{n}) c_{\boldsymbol{t}}(\boldsymbol{n}-\boldsymbol{t}) \boldsymbol{x}^{\boldsymbol{n}-\boldsymbol{s}+\boldsymbol{p}-\boldsymbol{t}}$$

is known since it is determined by the initial conditions. The series

$$U(\boldsymbol{x}) = -\sum_{\boldsymbol{t}\in H} \sum_{\boldsymbol{n}\in(\boldsymbol{s}+\mathbb{N}^d)\setminus(\boldsymbol{s}+\boldsymbol{t}+\mathbb{N}^d)} f_{\boldsymbol{n}} c_{\boldsymbol{t}}(\boldsymbol{n}-\boldsymbol{t})\boldsymbol{x}^{\boldsymbol{n}-\boldsymbol{s}+\boldsymbol{p}-\boldsymbol{t}}$$

is unknown, and satisfies the support condition $\operatorname{supp}(U) \subseteq \mathbb{N}^d \setminus (\boldsymbol{p} + \mathbb{N}^d)$. Analogously to equation (2) we get

$$K(\boldsymbol{x}) = D\left(F_{\boldsymbol{s}}(\boldsymbol{x})\right) + U(\boldsymbol{x}). \tag{5}$$

To make this precise, we briefly review the theory of perfect differential operators and their division (cf. [6]).

4. Perfect differential operators

We consider linear partial differential operators with polynomial coefficients of the form $D = \sum_{\boldsymbol{a},\boldsymbol{b}\in\mathbb{N}^d} c_{\boldsymbol{a}\boldsymbol{b}}\boldsymbol{x}^{\boldsymbol{a}}\partial^{\boldsymbol{b}}$. They define $\mathbb{K}$-linear maps $D : \mathbb{K}[\![\boldsymbol{x}]\!] \to \mathbb{K}[\![\boldsymbol{x}]\!]$, $A \mapsto D(A)$. The differences $\boldsymbol{r} = \boldsymbol{a} - \boldsymbol{b} \in \mathbb{Z}^d$ with $c_{\boldsymbol{a}\boldsymbol{b}} \neq 0$ are called the shifts of D. A differential operator is called a monomial operator if all its summands have the same shift $\boldsymbol{r}$; this is equivalent to saying that the operator sends monomials to monomials. A monomial operator can be represented as $\kappa_{\boldsymbol{r}}\boldsymbol{x}^{\boldsymbol{r}}$ where $\kappa_{\boldsymbol{r}} : \mathbb{N}^d \to \mathbb{K}$ is called the coefficient function: The monomial operator $\boldsymbol{x}^{\boldsymbol{a}}\partial^{\boldsymbol{b}}$ with $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{r}$ has the coefficient function $\kappa_{\boldsymbol{r}}(\boldsymbol{n}) = \boldsymbol{n}^{\underline{\boldsymbol{b}}}$ and the shift $\boldsymbol{r}$:

$$(\boldsymbol{x}^{\boldsymbol{a}}\partial^{\boldsymbol{b}})\boldsymbol{x}^{\boldsymbol{n}} = \boldsymbol{n}^{\underline{\boldsymbol{b}}}\boldsymbol{x}^{\boldsymbol{n}+\boldsymbol{r}} = \kappa_{\boldsymbol{r}}(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}+\boldsymbol{r}}.$$

A monomial subspace M is a vector subspace of $\mathbb{K}[\![\boldsymbol{x}]\!]$ for which there is a set $\Sigma \subseteq \mathbb{N}^d$ such that M is formed by all power series with support in Σ. The canonical monomial direct complement of M is the vector subspace N of power series with support in the complement $\mathbb{N}^d \setminus \Sigma$.

The initial form of D with respect to a weight vector $\boldsymbol{w}$, denoted by D°, is defined by $D^\circ = \sum_{\boldsymbol{a}-\boldsymbol{b}=\boldsymbol{r}} c_{\boldsymbol{a}\boldsymbol{b}}\boldsymbol{x}^{\boldsymbol{a}}\partial^{\boldsymbol{b}}$, where $\boldsymbol{r}$ is the minimal shift of D (i.e., $\boldsymbol{w}\cdot\boldsymbol{r}$ is minimal). Clearly D° is a monomial operator; we denote its coefficient function with $\kappa^\circ(\boldsymbol{n})$. Let $\overline{D}$ denote the tail of the operator, i.e., $D = D^\circ + \overline{D}$. We say that the initial form D° dominates D if there is a constant $C > 0$ such that for all $\boldsymbol{b} \in \mathbb{N}^d$ with $c_{\boldsymbol{a}\boldsymbol{b}} \neq 0$ for some $\boldsymbol{a}$, and all $\boldsymbol{n} \in \mathbb{N}^d$ with $\kappa^\circ(\boldsymbol{n}) \neq 0$, we have $\boldsymbol{n}^{\underline{\boldsymbol{b}}} \leq C \cdot |\kappa^\circ(\boldsymbol{n})|$.

A differential operator D is called perfect if for any $A \in \mathbb{K}[\![\boldsymbol{x}]\!]$ there exists an $\boldsymbol{n} \in \mathbb{N}^d$ such that $\operatorname{in}_{\boldsymbol{w}}(D(A)) = (D^\circ\boldsymbol{x}^{\boldsymbol{n}})/\kappa^\circ(\boldsymbol{n})$. In other words, if for all power series A the initial monomial of $D(A)$ lies in the image $\operatorname{Im}(D^\circ)$ of D°. The image $\operatorname{Im}(D^\circ)$ is spanned by the monomials $\{\boldsymbol{x}^{\boldsymbol{n}+\boldsymbol{r}} \mid \boldsymbol{n} \in \mathbb{N}^d \text{ and } \kappa^\circ(\boldsymbol{n}) \neq 0\}$ where $\boldsymbol{r}$ is the shift of D°.

Example 2. Let $D = 4y - xy^2\partial_x\partial_y + x^2$. The involved shifts are $(0,1)$ and $(2,0)$. We choose a weight vector such that $(0,1) \prec_{\boldsymbol{w}} (2,0)$ and get the initial form $D^\circ = 4y - xy^2\partial_x\partial_y$ with coefficient function $\kappa^\circ(n_1, n_2) = 4 - n_1n_2$. We see that $\kappa^\circ(\boldsymbol{n}) = 0$ for $\boldsymbol{n} \in Z = \{(1,4),(2,2),(4,1)\}$, hence the image $\operatorname{Im}(D^\circ)$ is spanned (as a vector space) by the monomials $\{x^{n_1}y^{n_2+1} : (n_1,n_2) \notin Z\}$. This operator is not perfect since, e.g., D applied to x^2y^2 gives $x^4y^2 \notin \operatorname{Im}(D^\circ)$.

This example also illustrates that in general it might be impossible to decide whether an operator is perfect or not: The computation of $\operatorname{Im}(D^\circ)$ needs a diophantine equation of arbitrary form to be solved.

Example 3. Consider now the operator $D = 4y - xy^2\partial_x\partial_y + x^2y^4$ with D° being the same as in the example above, but now $\overline{D} = x^2y^4$. Clearly we have $\operatorname{Im}(\overline{D}) = x^2y^4\mathbb{K}[\![x,y]\!] \subset \operatorname{Im}(D^\circ)$ which implies that in this case D is perfect.

Note that the concept of perfect operators is more subtle than these two examples suggest. For more details we refer to [5, 6] from where we cite a division theorem for differential operators (in fact a specialized version that is sufficient for our setting):

Theorem 5. *Let $\mathcal{K}$ be either $\mathbb{K}[\![\boldsymbol{x}]\!]$ or $\mathbb{K}\{\boldsymbol{x}\}$. Let $D \in \mathbb{K}[\boldsymbol{x}][\partial]$ be a perfect differential operator and let D° be its initial form with respect to some weight vector $\boldsymbol{w}$. Choose the canonical direct monomial complements L° of $\operatorname{Ker}(D^\circ)$ and J° of $\operatorname{Im}(D^\circ)$ in $\mathcal{K}$. In the case of convergent power series, assume in addition that D is dominated by D°. Then we have the direct sum decompositions*

$$\operatorname{Im} D \oplus J^\circ = \mathcal{K} \text{ and } \operatorname{Ker} D \oplus L^\circ = \mathcal{K}.$$

In other words, if D is perfect, then the division $K = D(F_{\boldsymbol{s}}) + U$ exists and is unique. The support condition on U is given by D°.

5. Back to P-finite recurrences

Proposition 6. *A differential operator*

$$D = \sum_{\boldsymbol{t}\in H_0} \sum_{\boldsymbol{k}\in S_{\boldsymbol{t}}} c_{\boldsymbol{t}\boldsymbol{k}} \boldsymbol{x}^{\boldsymbol{k}} \partial^{\boldsymbol{k}} \boldsymbol{x}^{\boldsymbol{p}-\boldsymbol{t}} = \sum_{\boldsymbol{t}\in H_0} D_{\boldsymbol{t}}$$

that evolves from a recurrence which is of type (4) *and satisfies the conditions of Theorem 4, is perfect.*

Proof. The shift of all summands in any of the operators $D_{\boldsymbol{t}}$ is $\boldsymbol{p} - \boldsymbol{t}$. It does not change when $D_{\boldsymbol{t}}$ is converted to the standard form $\sum c_{\boldsymbol{a}\boldsymbol{b}} \boldsymbol{x}^{\boldsymbol{a}} \partial^{\boldsymbol{b}}$ by means of the commutation rule $\partial x = x\partial + 1$. Thus all the $D_{\boldsymbol{t}}$'s are monomial operators. Let $\boldsymbol{w}$ be the weight vector from Theorem 4 with $\boldsymbol{w} \cdot \boldsymbol{t} < 0$ for all $\boldsymbol{t} \in H$. Then $D_{\boldsymbol{0}}$ is the monomial operator with the minimal shift, hence we have $D^\circ = D_{\boldsymbol{0}}$ and $\overline{D} = \sum_{\boldsymbol{t}\in H} D_{\boldsymbol{t}}$. The coefficient function of the initial form turns out to be $\kappa^\circ(\boldsymbol{n}) = \sum_{\boldsymbol{k}\in S_0} c_{\boldsymbol{0}\boldsymbol{k}} (\boldsymbol{n} + \boldsymbol{p})^{\underline{\boldsymbol{k}}} = c_{\boldsymbol{0}}(\boldsymbol{n} + \boldsymbol{s})$. Since the polynomial $c_{\boldsymbol{0}}(\boldsymbol{n})$ does not have any zeros in $\boldsymbol{s} + \mathbb{N}^d$, we see that $\kappa^\circ(\boldsymbol{n}) \neq 0$ for all $\boldsymbol{n} \in \mathbb{N}^d$. Consequently $\operatorname{Im}(D^\circ) = \boldsymbol{x}^{\boldsymbol{p}} \mathbb{K}[\![\boldsymbol{x}]\!]$ which matches the support condition on $U(\boldsymbol{x})$. The kernel of D° is 0, hence

$$\operatorname{in}_{\boldsymbol{w}}(D(A)) = D^\circ(\operatorname{in}_{\boldsymbol{w}}(A)) \text{ for all } A \in \mathbb{K}[\![\boldsymbol{x}]\!],$$

and this proves that D is perfect. □

We conclude that the division (5) has always a unique solution. This corresponds exactly to the statement of Theorem 4 which asserts that the recurrence has a unique solution.

Let's turn to the case of convergent power series; here we get a sufficient but not necessary condition for the convergence of the generating function. Theorem 5 states that the generating function $F_{\boldsymbol{s}}(\boldsymbol{x})$ is a convergent power series if the operator D corresponding to its recurrence relation is dominated by its initial form. This is exactly the case when the polynomial $c_{\boldsymbol{0}}(\boldsymbol{n})$ dominates all the polynomials $c_{\boldsymbol{t}}(\boldsymbol{n}), \boldsymbol{t} \in H$, i.e., there is a constant $C > 0$ such that for all $\boldsymbol{n} \in \mathbb{N}^d$ we have

$$|c_{\boldsymbol{t}}(\boldsymbol{n})| \leq C \cdot |c_{\boldsymbol{0}}(\boldsymbol{n} + \boldsymbol{s})|.$$

6. Examples and outlook

In order to illustrate the applicability of our theory we choose the Eulerian numbers (see e.g. [8, chap. 6.2]).

Example 4. The recurrence

$$a_{n,k} = (k+1)a_{n-1,k} + (n-k)a_{n-1,k-1} \tag{6}$$

defines the Eulerian numbers, together with the initial conditions $a_{n,0} = 1$ and $a_{n,n-1} = 1$. Furthermore $a_{n,k} = 0$ for $n < 0$ or $k < 0$ or $k \geq n$. The initial conditions border an area of triangular shape which prevents us from applying our method. For this reason we do a transform, namely we define $b_{n,k} := a_{n+k+1,k}$; the initial conditions for $b_{n,k}$ are now $b_{n,0} = 1$ and $b_{0,k} = 1$ for all $n, k \in \mathbb{N}$ (and $b_{n,k} = 0$ for $n < 0$ or $k < 0$). By substituting $n \to n + k + 1$ in (6) we get the recurrence

$$b_{n,k} = (n+1)b_{n,k-1} + (k+1)b_{n-1,k}. \tag{7}$$

Since we have $H = \{(0,-1), (-1,0)\}$ it is natural to choose the starting point $\boldsymbol{s} = (1,1)$; the apex $\boldsymbol{p}$ is obviously $(0,0)$. Hence the generating function in question is $F_{\boldsymbol{s}}(x,y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} b_{n+1,k+1} x^n y^k$. The known part in this case is

$$K(x,y) = -\frac{x^2 - 2x}{(x-1)^2 x} - \frac{y^2 - 2y}{(y-1)^2 y}.$$

From $\boldsymbol{p} = (0,0)$ it follows that $U(x,y) = 0$. The differential operator corresponding to recurrence (7) is

$$D = 1 - 2x - 2y - xy(\partial_x + \partial_y).$$

If we plug in a truncated power series expansion of $F_{\boldsymbol{s}}(x,y)$, we see that indeed $K(x,y) = D(F_{\boldsymbol{s}}(x,y))$ holds.

Gnedin and Olshanski [7] studied nonnegative solutions of the dual recurrence. The dual (or backwards) recurrence is obtained by changing the signs of all shifts. The problem is now to describe initial conditions for the dual recurrence such that its solution does not involve negative values.

Example 4 (continued)**.** The backwards recurrence of (7) is $d_{n,k} = (n+1)d_{n,k+1} + (k+1)d_{n+1,k}$ which we write as

$$(n+1)d_{n,k} = d_{n,k-1} - k d_{n+1,k-1}. \tag{8}$$

It is claimed that $d_{0,0} = 1$, the other initial conditions $d_{n,0}, n \geq 1$ and $d_{0,k}, k \geq 1$ have to be determined such that $d_{n,k} \geq 0$ for all $n, k \in \mathbb{N}$. We choose $\boldsymbol{s} = (0,1)$, and we see that in this case the apex $\boldsymbol{p}$ is $(1,0)$. We can compute a differential operator for (8):

$$D = x + 2y - xy + x^2 \partial_x + y^2 \partial_y.$$

Hence we have to determine all power series $K(x,y)$ for which the division (5) yields a power series solution $F_{\boldsymbol{s}}(x,y)$ with nonnegative coefficients. This seems to be an interesting research problem.

It would be nice if we could also state some results about the algebraicity of the generating function. But here even the univariate case is still open: The famous p-curvature conjecture of Grothendieck [13, 15] asserts that a linear differential equation with coefficients in $\mathbb{Q}(x)$ admits a complete system of solutions if and only if the differential equation reduced modulo p has a complete system of rational solutions for almost all p.

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