# Reduced Gröbner Bases in Polynomial Rings over a Polynomial Ring

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**Abstract.** We define reduced Gröbner bases in polynomial rings over a polynomial ring and introduce an algorithm for computing them. There exist some algorithms for computing Gröbner bases in polynomial rings over a polynomial ring. However, we cannot obtain the reduced Gröbner bases by these algorithms. In this paper we propose a new notion of reduced Gröbner bases in polynomial rings over a polynomial ring and we show that every ideal has a unique reduced gröbner basis.

**Keywords.** Gröbner bases.

## 1. Introduction

Many researchers have studied Gröbner bases in several domains (polynomial rings over a Euclidean domain [KRK88], over the integers [NG94], over commutative regular rings [Wei87], over Noetherian rings [AL94] etc ...). In this paper we introduce reduced Gröbner bases in polynomial rings over a polynomial and give an algorithm for computing them. In [IP98] and [AL94], they described the special computation method of Gröbner bases in polynomial rings over a polynomial ring. ( Insa and Pauer described how to compute Gröbner bases in rings of differential operators with coefficients in a polynomial ring. They worked in non-commutative rings, however we can easily apply this method to the commutative case for computing Gröbner bases in polynomial rings over a polynomial ring. This method is the same as [AL94].) This is one of the methods for computing Gröbner bases in polynomial rings over a polynomial ring.

Let $K$ be a field and $\bar{A}, \bar{X}$ variables with $\bar{A} \cap \bar{X} = \emptyset$. It is known that by computing Gröbner bases in polynomial rings over a field with respect to a block order with $\bar{X} \gg \bar{A}$, we can obtain Gröbner bases in $K[\bar{A}][\bar{X}]$ (polynomial rings over a polynomial ring). However, we are not able to obtain reduced (or minimal) Gröbner bases by these methods.

For example, let $a, b, x, y$ be variables and $f_1 = (a-1)x + by^2, f_2 = ay + b$ in $\mathbb{Q}[a,b][x,y]$. If we use the method of computing Gröbner bases with respect to a block order $x \succ_{lex} y \gg a \succ_{lex} b$ where $\succ_{lex}$ is the lexicographic order, to compute a Gröbner basis in $\mathbb{Q}[a,b][x,y]$, then we obtain the following reduced Gröbner basis for $\langle f_1, f_2\rangle$ with respect to the block order

$$g_1 = ay + b, g_2 = (a-1)x + by^2, g_3 = -xy - bx + by^3.$$

We know that $\{g_1, g_2, g_3\}$ is a Gröbner basis for $\langle f_1, f_2\rangle$ with respect to $x \succ_{lex} y$ in $\mathbb{Q}[a,b][x,y]$ (see Lemma 4.3). However there exists a smaller Gröbner basis, because we have $\mathrm{lm}(g_3) = -xy \in \langle \mathrm{lm}(g_1), \mathrm{lm}(g_2)\rangle = \langle ay, (a-1)x\rangle$. (When the coefficient domain is a polynomial ring, we often see this phenomenon.) That is, $g_3$ can be written as $g_3 = -xg_1 + yg_2$. Thus, we do not need $g_3$ for a Gröbner basis $\{g_1, g_2, g_3\}$. However by this method we cannot delete $g_3$. This is a problem of the method of computing Gröbner bases with respect to a block order $\bar{X} \gg \bar{A}$ in $K[\bar{A}, \bar{X}]$. The first method for computing Gröbner bases has problems too, we will see the problems in section 5.
In this paper, we describe the problems and give the answers. Moreover, we propose a new notion of reduced Gröbner bases and we show that every ideal has a unique reduced Gröbner basis.

Our plan is the following: first we introduce two methods of computing Gröbner bases in $K[\bar{A}][\bar{X}]$ in section 3 and 4. In section 5, we explain the problems, and in section 6 we define reduced Gröbner bases and construct algorithms for computing reduced Gröbner bases in $K[\bar{A}][\bar{X}]$. In section 7, we see some examples. Finally, in section 8 we conclude this paper.

## 2. Notations for $K[\bar{A}, \bar{X}]$ and $K[\bar{A}][\bar{X}]$

Let $K$ be a field and $\bar{A} := \{A_1, \dots, A_m\}$ and $\bar{X} := \{X_1, \dots, X_n\}$ finite sets of variables such that $\bar{A} \cap \bar{X} = \emptyset$. $\mathrm{pp}(\bar{X})$, $\mathrm{pp}(\bar{A})$ and $\mathrm{pp}(\bar{A}, \bar{X})$ denote the sets of power products of $\bar{X}$, $\bar{A}$ and $\bar{A} \cup \bar{X}$, respectively. $\mathbb{Q}$ and $\mathbb{N}$ define as the field of rational numbers and the set of natural numbers, respectively. Note that in this paper, the set of natural number $\mathbb{N}$ includes zero 0. In this paper, we define $K[\bar{A}, \bar{X}]$ as a polynomial ring over a field $K$ and $K[\bar{A}][\bar{X}] := (K[\bar{A}])[\bar{X}]$ as a polynomial ring over a polynomial ring $K[\bar{A}]$. Let $f$ and $g$ be non-zero polynomials in $K[\bar{A}, \bar{X}]$ (or $K[\bar{A}][\bar{X}]$ ) and $\succ$ be an arbitrary monomial order on the set of power products in $\mathrm{pp}(\bar{A}, \bar{X})$ (or $\mathrm{pp}(\bar{X})$). If polynomials $f$ and $g$ are in $K[\bar{A}][\bar{X}]$, then we use the subscript $\bar{A}$ as follows:

- The **support of** $f$ (written : $\mathrm{supp}(f)$ (or $\mathrm{supp}_{\bar{A}}(f)$)) is the set of power products of $f$ that appear with a non-zero coefficient.
- The biggest power product of $\mathrm{supp}(f)$ (or $\mathrm{supp}_{\bar{A}}(f)$) with respect to $\succ$ is denoted by $\mathrm{lpp}(f)$ (or $\mathrm{lpp}_{\bar{A}}(f)$) and is called the **leading power product of** $g$ **with respect to** $\succ$.
- The coefficient corresponding to $\mathrm{lpp}(f)$ (or $\mathrm{lpp}_{\bar{A}}(f)$) is called the **leading coefficient of** $f$ **with respect to** $\succ$ which is defined by $\mathrm{lc}(f)$ (or $\mathrm{lc}_{\bar{A}}(f)$).

- The product $\mathrm{lc}(f)\,\mathrm{lpp}(f)$ is called the **leading monomial of $f$ with respect to** $\succ$ which is defined by $\mathrm{lm}(f)$ (or $\mathrm{lm}_{\bar{A}}(f)$).
- The **least common multiple** of $\mathrm{lpp}(f)$ and $\mathrm{lpp}(g)$ (or $\mathrm{lpp}_{\bar{A}}(f)$ and $\mathrm{lpp}_{\bar{A}}(g)$) is defined by $\mathrm{lcm}(\mathrm{lpp}(f), \mathrm{lpp}(g))$ (or $\mathrm{lcm}(\mathrm{lpp}_{\bar{A}}(f), \mathrm{lpp}_{\bar{A}}(g))$.
- The **set of monomials** of $f$ is denoted by $\mathrm{Mono}(f)$ (or $\mathrm{Mono}_{\bar{A}}(f)$).
- If $\mathrm{lpp}(f) = A_1^{\alpha_1} \cdots A_m^{\alpha_m} X_1^{\beta_1} \cdots X_n^{\beta_n} \in \mathrm{pp}(\bar{A}, \bar{X})$, then $\deg_{\{\bar{A},\bar{X}\}}(f) := (\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n) \in \mathbb{N}^{m+n}$.
  If $\mathrm{lpp}_{\bar{A}}(f) = X_1^{\beta_1} \cdots X_n^{\beta_n} \in \mathrm{pp}(\bar{X})$, then $\deg_{\bar{X}}(f) := (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$. **Note that the subscripts are $\{\bar{A}, \bar{X}\}$ and $\bar{X}$.**

**Definition 2.1 (block orders).** *Let $\succ_1$ and $\succ_2$ be admissible orders on $\mathrm{pp}(\bar{A})$ and $\mathrm{pp}(\bar{X})$, respectively, and $t_1, s_1 \in \mathrm{pp}(\bar{A})$, $t_2, s_2 \in \mathrm{pp}(\bar{X})$, $t_1t_2 \succ_{\bar{X},\bar{A}} s_1s_2 \iff t_2 \succ_2 s_2$ or ($t_2 = s_2$, and $t_1 \succ_1 s_1$). This type of order $\succ_{\bar{X},\bar{A}}$ is called a* **block order** *on $\mathrm{pp}(\bar{A}, \bar{X})$. This order is written as $\succ_{\bar{X},\bar{A}} := (\succ_2, \succ_1)$.*

**Example 2.2.** Let $a, b, x, y$ be variables and $f = 2ax^2y + bx^2y + 3x + by + 1, g = abx^2 + 2xy + by + 2$ be polynomials.
If we consider polynomials $f$ and $g$ as members of $\mathbb{Q}[a, b, x, y]$ with a block order $\succ_{\{x,y\},\{a,b\}} := (x \succ_{lex} y, a \succ_{lex} b)$ where $\succ_{lex}$ is the lexicographic order, then $\mathrm{supp}(f) = \{ax^2y, bx^2y, x, y, 1\}$, $\mathrm{lpp}(f) = ax^2y$, $\mathrm{lc}(f) = 2$, $\mathrm{lm}(f) = 2ax^2y$, $\mathrm{lcm}(\mathrm{lpp}(f), \mathrm{lpp}(g)) = abx^2y$, $\mathrm{Mono}(f) = \{2ax^2y, bx^2y, 3x, by, 1\}$, $\deg_{\{a,b,x,y\}}(f) = (1, 0, 2, 1) \in \mathbb{N}^4$.
If we consider polynomials $f, g$ as members of $\mathbb{Q}[a, b][x, y]$ with the lexicographic order $x \succ_{lex} y$, then $\mathrm{supp}_{\{a,b\}}(f) = \{x^2y, x, y, 1\}$, $\mathrm{lpp}_{\{a,b\}}(f) = x^2y$, $\mathrm{lc}_{\{a,b\}}(f) = 2a+b$, $\mathrm{lm}_{\{a,b\}}(f) = (2a+b)x^2y$, $\mathrm{lcm}(\mathrm{lpp}_{\{a,b\}}(f), \mathrm{lpp}_{\{a,b\}}(g)) = x^2y$, $\mathrm{Mono}_{\{a,b\}}(f) = \{(2a + b)x^2y, 3x, by, 1\}$, $\deg_{\{x,y\}}(f) = (2, 1) \in \mathbb{N}^2$.

## 3. The first computation method

In this section we introduce the special S-polynomial and the special reduction which are from [IP98, AL94], and we give a computation method of Gröbner bases in $K[\bar{A}][\bar{X}]$.

**Proposition 3.1** ([IP98]). *Let $F$ be a finite set of polynomials in $K[\bar{A}][\bar{X}]$, $g \in K[\bar{A}][\bar{X}]$ and $\succ$ a monomial order on $\mathrm{pp}(\bar{X})$. Then there is a polynomial $r \in K[\bar{A}][\bar{X}]$ and there is a family $(h_f)_{f\in F}$ such that*

- $g = \sum_{f\in F} h_f f + r$ *( $r$ is a remainder of $g$ after division by $F$),*
- *for all $f \in F, h_f = 0$ or $g \succ h_f$,*
- *$r = 0$ or $\mathrm{lc}_{\bar{A}}(r) \notin \langle \mathrm{lc}_{\bar{A}}(f) |\ \mathrm{lpp}_{\bar{A}}(f)$ divides $\mathrm{lpp}_{\bar{A}}(r)\rangle$.*

*The polynomials $r, h_f$ $(f \in F)$ can be computed as follows:*
*First set: $r := g$ and $h_f = 0$ $(f \in F)$.*
*While $r \neq 0$ and $\mathrm{lc}_{\bar{A}}(r) \in \langle \mathrm{lc}_{\bar{A}}(f) |\ \mathrm{lpp}_{\bar{A}}(f)$ divides $\mathrm{lpp}_{\bar{A}}(r)\rangle$ do the following:*

*let* $F' := \{f \in F \mid \mathrm{lpp}_{\bar{A}}(f) \text{ divides } \mathrm{lpp}_{\bar{A}}(r)\}$*, compute a family* $(c_f)_{f \in F'}$ *in* $K[\bar{A}][\bar{X}]$ *such that*

$$\sum_{f \in F'} c_f \,\mathrm{lc}_{\bar{A}}(f) = \mathrm{lc}_{\bar{A}}(r).$$

*Replace*

$$r \quad by \quad r - \sum_{f \in F'} c_f X^{\deg_{\bar{X}}(r) - \deg_{\bar{X}}(f)} f$$

*and*

$$h_f \quad by \quad h_f + c_f X^{\deg_{\bar{X}}(r) - \deg_{\bar{X}}(f)}, \quad f \in F',$$

*where* $X^i := X_1^{i_1} \cdots X_n^{i_n}$, $i \in \mathbb{N}^n$.

We can consider this division algorithm as "extended Gröbner bases algorithm [BW93]" (or transformation of Gröbner bases). We can simplify this Proposition to the following definition.

**Definition 3.2 (Reduction).** *Let* $F$ *be a set of polynomials in* $K[\bar{A}][\bar{X}]$ *and* $g = a\beta + g' \in K[\bar{A}][\bar{X}]$ *where* $a \in K[\bar{A}]$, $\beta \in \mathrm{pp}(\bar{X})$ *and* $g' \in K[\bar{A}][\bar{X}]$*. Moreover, let* $F' := \{f \in F \mid \mathrm{lpp}_{\bar{A}}(f) \text{ divides } \beta\}$*. If* $a \in \langle \mathrm{lc}_{\bar{A}}(F') \rangle \subseteq K[\bar{A}]$*, the element* $a$ *can be written as* $a = \sum_{f_i \in F'} h_i \,\mathrm{lc}_{\bar{A}}(f_i)$ *where* $h_i \in K[\bar{A}]$*. Then a reduction* $\xrightarrow{r1}_F$ *is defined as follows:* $g \xrightarrow{r1}_F g - \sum_{f_i \in F'} h_i \frac{\beta}{\mathrm{lpp}_{\bar{A}}(f_i)} f_i$*. In this paper, we define this reduction as* `Reduce1` *(written:* $\xrightarrow{r1}$*). Actually, reducing* $g$ *by* $F$ *and reducing* $g$ *by* $F'$ *is the same. In this case, we can write the reduction* $g \xrightarrow{r1}_{F'}$ *instead of* $g \xrightarrow{r1}_F$*. In the Algorithm 1* Insa-Pauer*, we will write* $g \xrightarrow{r1}_F$ *as* $\texttt{Reduce1}(g, F)$*.*

**Example 3.3.** Let $F = \{f_1 = (a + b + 1)y, f_2 = ax + 1\}$ in $\mathbb{Q}[a, b][x, y]$, $\succ$ the lexicographic order such that $x \succ y$ and $g = (b + 1)xy - y \in \mathbb{Q}[a, b][x, y]$. Then, $\mathrm{lc}_{\bar{A}}(g) = b + 1 \in \langle \mathrm{lc}_{\bar{A}}(f_1), \mathrm{lc}_{\bar{A}}(f_2) \rangle = \langle a + b + 1, a \rangle$, hence, $\mathrm{lc}_{\bar{A}}(g)$ can be written as $\mathrm{lc}_{\bar{A}}(g) = \mathrm{lc}_{\bar{A}}(f_1) - \mathrm{lc}_{\bar{A}}(f_2)$. Clearly, $\mathrm{lpp}_{\bar{A}}(f_1) | \mathrm{lpp}_{\bar{A}}(g), \mathrm{lpp}_{\bar{A}}(f_2) | \mathrm{lpp}_{\bar{A}}(g)$. Therefore, $g$ can be reduced by $F$ as follows: $g \xrightarrow{r1}_F g - (xf_1 - yf_2) = 0$.

Insa and Pauer also introduced the following special S-polynomial.

**Definition 3.4 (S-polynomial** [IP98]**).** *Let* $G$ *be a finite set of polynomials in* $K[\bar{A}][\bar{X}]$ *and let* $I$ *be an ideal in* $K[\bar{A}][\bar{X}]$ *generated by* $G$*. For* $E \subseteq G$*, let*

$$S_E := \left\{ (c_e)_{e \in E} \;\middle|\; \sum_{e \in E} c_e \,\mathrm{lc}_{\bar{A}}(e) = 0 \right\}.$$

(*We can consider* $S_E$ *as a set of syzygies for* $\mathrm{lc}_{\bar{A}}(E)$*.) Then for* $s = (c_e)_{e \in E} \in S_E$*,*

$$\mathsf{Spoly1}(E, s) = \sum_{e \in E} c_e X^{\max(E) - \deg_{\bar{A}}(e)} e$$

*is called S-polynomial with respect to* $s$ *,where*

$$\max(E) := (\max_{e \in E} \deg_{\bar{X}}(e)_1, \ldots, \max_{e \in E} \deg_{\bar{X}}(e)_n) \in \mathbb{N}^n .$$

*In this paper, we call this special S-polynomial "* Spoly1 *".*

**Example 3.5.** Let $E = \{e_1 = (ab+b)x^2 + y, e_2 = (a+b)x + 1, e_3 = axy + by + 2\}$ in $\mathbb{Q}[a,b][x,y]$ and $\succ$ the lexicographic order such that $x \succ y$. Then, a basis of a module of syzygies for $\mathrm{lc}_{\bar{A}}(E)$ is $\{[0,-a,a+b],[-1,1,b-1]\}$. If we take $[0,-a,a+b]$, then $\mathsf{Spoly1}(E,[0,-a,a+b]) = 0e_1 + (-a)ye_2 + (a+b)xe_3 = (a+b)bxy + 2(a+b)x - ay$. If we take $[-1,1,b-1]$, then $\mathsf{Spoly1}(E,[-1,1,b-1]) = (-1)ye_1 + (1)xye_2 + (b-1)xe_3 = (b^2-b-1)xy - y^2 + 2(b-1)x$.

The definition of Gröbner bases in $K[\bar{A}][\bar{X}]$ is the following.

**Definition 3.6 (Gröbner bases).** *Fix a monomial order. A finite subset $G = \{g_1, \ldots, g_s\}$ of an ideal $I$ in $K[\bar{A}][\bar{X}]$ is said to be a* **Gröbner basis** *if*

$$\langle \mathrm{lm}_{\bar{A}}(g_1), \ldots, \mathrm{lm}_{\bar{A}}(g_s)\rangle = \langle \mathrm{lm}_{\bar{A}}(I)\rangle.$$

**Remark:** This definition is equivalent to the following. We are able to understand the definition as follows:
Let $I$ be an ideal in $K[\bar{A}][\bar{X}]$ and let $G$ be a finite subset of $I$. For $i \in \mathbb{N}^n$ let $\mathrm{lc}(i,I) := \langle \mathrm{lc}_{\bar{A}}(f) | f \in I, \deg(f) = i\rangle$. Then $G$ is a **Gröbner basis** of $I$ (with respect to $\succ$ ) if and only if $\forall i \in \mathbb{N}^n$ the ideal $\mathrm{lc}(i,I) \subseteq K[\bar{A}]$ is generated by $\{\mathrm{lc}_{\bar{A}}(g) \mid g \in G, i \in \deg(g) + \mathbb{N}^n\}$.

**Example 3.7.** Consider the ring $\mathbb{Q}[a,b][x,y]$ with the lexicographic order $x \succ_{lex} y$, and let $I = \langle f_1, f_2\rangle = \langle ax+bx+y, bxy\rangle$. Since $\mathrm{lm}_{\{a,b\}}(\texttt{Spoly1}(\{f_1,f_2\},[b,a+b])) = by^2 \notin \langle \mathrm{lm}_{\{a,b\}}(f_1), \mathrm{lm}_{\{a,b\}}(f_2)\rangle = \langle (a+b)x, bxy\rangle$, $\{f_1,f_2\}$ is not a Gröbner basis for $I$. Actually, a Gröbner basis for $I$ is $\{f_1, f_2, by^2\}$.

There are a lot of applications of Gröbner bases in $K[\bar{A}][\bar{X}]$ which are well-known in polynomial rings over a field. For instance, if $G$ is a Gröbner basis for an ideal $I$ in $K[\bar{A}][\bar{X}]$, then $\forall g \in I$, $g \xrightarrow{r1}_G 0$. In this paper, we do not describe the detail of properties of Gröbner bases in $K[\bar{A}][\bar{X}]$ (see [IP98] Proposition 3 and [AL94]). The following algorithm is for computing Gröbner bases in $K[\bar{A}][\bar{X}]$.

**Algorithm 1** FirstGB
**Input**: $F$: a finite set of polynomials in $K[\bar{A}][\bar{X}]$,
**Output**: $G$: a Gröbner basis of $F$ in $K[\bar{A}][\bar{X}]$.
**begin**
  $G \leftarrow F$; $B \leftarrow \{(f_{i_1}, f_{i2} \ldots f_{ip}) \mid 1 \le i_1 < i_2 \cdots < i_p \le s, 2 \le p \le s\}$
   **while** $B \ne \emptyset$ **do**
    Take any element $E$ from $B$;  $B \leftarrow B \backslash \{E\}$
    $S_E \leftarrow$ Compute a basis of a module of syzygies for $\mathrm{lc}_{\bar{A}}(E)$
     **while** $S_E \ne \emptyset$ **do**
      Take any element $\alpha$ from $S_E$; $S_E \leftarrow S_E \backslash \{\alpha\}$
      $h \leftarrow$Spoly1$(E,\alpha)$;  $r \leftarrow$`Reduce1`$(h,G)$
       **if** $r \ne 0$ **then**
      $B \leftarrow B \cup \big\{(r, g_{j_1}, .., g_{j_q}) \,\big|$ distinct elements $g_{j_1}, \ldots, g_{j_p} \in G, 1 \le p \le |G|\big\}$
        $G \leftarrow G \cup \{r\}$

**end-if**
**end-while**
**end-while**
`return`$(G)$
**end**

**Remark:** As we said earlier, we need the special S-polynomial `Spoly1` and the special reduction `Reduce1` in order to compute Gröbner bases in $K[\bar{A}][\bar{X}]$. In this point, this algorithm is more complicated than the Buchberger algorithm. There exist some criteria for computing Gröbner bases in $K[\bar{A}][\bar{X}]$. We can apply Buchberger's criteria [Buc79] and Zhou and Winkler's work [ZW06] for computing Gröbner bases.

## 4. The second computation method (Approach via block orders)

In this section, we introduce another computation method of Gröbner bases in $K[\bar{A}][\bar{X}]$. Actually, we are able to compute a Gröbner bases in $K[\bar{A}][\bar{X}]$ by computing Gröbner bases in $K[\bar{A}, \bar{X}]$ with respect to a block order $\succ_{\bar{X},\bar{A}}$. First, we define a normal S-polynomial and reduction in $K[\bar{A}, \bar{X}]$ as `Spoly2` and `Reduce2` to distinguish them from `Spoly1` and `Reduce1`.

**Definition 4.1.** *Fix a monomial order. Let $f, g \in K[\bar{A}, \bar{X}]$ be nonzero polynomials. The S-polynomial of $f$ and $g$ is the following*

$$\texttt{Spoly2}(f, g) = \frac{\mathrm{lcm}(\mathrm{lpp}(f), \mathrm{lpp}(g))}{\mathrm{lm}(f)} f - \frac{\mathrm{lcm}(\mathrm{lpp}(f), \mathrm{lpp}(g))}{\mathrm{lm}(g)} g.$$

*In this paper, we define this S-polynomial as "* `Spoly2` *".*

**Definition 4.2.** *Fix a monomial order. Let $f = a\alpha + f_1, g = b\alpha\beta + g_1$ with $\mathrm{lm}(f) = a\alpha$ in $K[\bar{A}, \bar{X}]$ where $a, b \in K$, $\alpha, \beta \in \mathrm{pp}(\bar{A}, \bar{X})$ and $f_1, g_1 \in K[\bar{A}, \bar{X}]$. Then a reduction $\xrightarrow{r2}_f$ is defined as follows: $g \xrightarrow{r2}_f b\alpha\beta + g_1 - ba^{-1}\beta(a\alpha + f_1)$, where $b\alpha\beta$ need not be the leading monomial of $g$. In this paper we call this reduction "* `Reduce2` *". A reduction $\xrightarrow{r2}_F$ by a set $F$ of polynomials is also natural defined* [BW93, Win96].

Before we describe an algorithm for computing Gröbner bases in $K[\bar{A}][\bar{X}]$, we need the following theorem.

**Theorem 4.3.** *Let $F$ be a finite set of polynomials in $K[\bar{A}][\bar{X}]$. $F$ can be seen as a finite subset of polynomials in $K[\bar{A}, \bar{X}]$ and we write the set as $F$ again. Let $G = \{g_1, \dots, g_s\}$ be a Gröbner basis for $\langle F \rangle$ in $K[\bar{A}, \bar{X}]$ with respect to a block order $\succ_{\bar{X},\bar{A}} := (\succ_1, \succ_2)$(i.e., $\bar{X} \gg \bar{A}$). $G$ can be seen as a set of $K[\bar{A}][\bar{X}]$ and we write the set as $G$ again. Then, $G$ is also a Gröbner basis for $\langle F \rangle$ with respect to $\succ_1$ in $K[\bar{A}][\bar{X}]$.*

*Proof.* For all $h \in \langle F \rangle \subseteq K[\bar{A}][\bar{X}]$, we prove that $\mathrm{lm}_{\bar{A}}(h)$ is generated by $\{\mathrm{lm}_{\bar{A}}(g) | g \in G\}$. Since $h$ can be seen as an element of $K[\bar{A}, \bar{X}]$ and $G$ is a Gröbner basis for $\langle F \rangle$ in $K[\bar{A}, \bar{X}]$, $h$ can be written as $h = h_1 g_1 + \cdots + h_s g_s$ such that $\mathrm{lm}(h) \succ_{\bar{X},\bar{A}}$

$\mathrm{lm}(h_1g_1) \succ_{\bar{X},\bar{A}} \cdots \succ_{\bar{X},\bar{A}} \mathrm{lm}(h_sg_s)$ where $h_1,\ldots,h_s \in K[\bar{A},\bar{X}]$. As $\succ_{\{\bar{X},\bar{A}\}}$ is a block order on $K[\bar{A},\bar{X}]$, we have $\mathrm{lm}_{\bar{A}}(h) \succ_1 \mathrm{lm}_{\bar{A}}(h_1g_1) \succ_1 \cdots \succ_1 \mathrm{lm}_{\bar{A}}(h_sg_s)$ in $K[\bar{A}][\bar{X}]$. W.l.o.g., $h_1g_1,\ldots,h_kg_k$ have the same leading power product "$\mathrm{lpp}_{\bar{A}}(h)$" where $k \le s$. That is, $\mathrm{lm}_{\bar{A}}(h) = \mathrm{lm}_{\bar{A}}(h_1g_1)+\cdots+\mathrm{lm}_{\bar{A}}(h_kg_k)$. We have $\mathrm{lm}_{\bar{A}}(h_ig_i) = \mathrm{lm}_{\bar{A}}(h_i)\,\mathrm{lm}_{\bar{A}}(g_i)$, hence $\mathrm{lm}_{\bar{A}}(h) = \mathrm{lm}_{\bar{A}}(h_1)\,\mathrm{lm}_{\bar{A}}(g_1)+\cdots+\mathrm{lm}_{\bar{A}}(h_k)\,\mathrm{lm}_{\bar{A}}(g_k)$. Therefore, $\mathrm{lm}_{\bar{A}}(h) \in \langle \mathrm{lm}_{\bar{A}}(g_1),\ldots,\mathrm{lm}_{\bar{A}}(g_s)\rangle$. $G$ is a Gröbner basis for $\langle F\rangle$ with respect to $\succ_1$ in $K[\bar{A}][\bar{X}]$. □

By Theorem 4.3, we are able to compute Gröbner bases in $K[\bar{A}][\bar{X}]$ by computing Gröbner bases with respect to a block order $\succ_{\bar{X},\bar{A}}$ in $K[\bar{A},\bar{X}]$.

**Algorithm 2** GröbnerBasis-Block
**Input** $F$: a finite set of polynomials in $K[\bar{A}][\bar{X}]$,
**Output** $G$: a Gröbner basis of $\langle F\rangle$ in $K[\bar{A}][\bar{X}]$.

1. Consider $F$ as a set of polynomials in $K[\bar{A},\bar{X}]$.
2. Compute a Gröbner basis $G$ for $\langle F\rangle$ with respect to a block order $\succ_{\bar{X},\bar{A}}$ in $K[\bar{A},\bar{X}]$.
3. Consider $G$ as a set of polynomials in $K[\bar{A}][\bar{X}]$. Then, by Theorem 4.3, $G$ is a Gröbner basis for $\langle F\rangle$ with respect to $\succ_{\bar{X}}$ in $K[\bar{A}][\bar{X}]$.

Since we do not need the special S-polynomial `Spoly1` and the special reduction `Reduce1` in this algorithm, this algorithm is much more efficient than the algorithm Insa-Pauer.

## 5. Problems

### 5.1. The first method

In this subsection, we consider a problem of the approach of the first method by the following example.

**Example 5.1.** Let $f_1 = a^2x - a$ and $f_2 = (a^3 - a)x - a^2 + 1$ be polynomials in $\mathbb{Q}[a][x]$. Then, a Gröbner basis of $\langle f_1, f_2\rangle$ is $\{f_1, f_2\}$, because $\mathtt{Spoly1}(f_1, f_2) = 0$, $f_1 \xrightarrow{r1}_{f_2} f_1$ and $f_2 \xrightarrow{r1}_{f_1} f_2$ in $\mathbb{Q}[a][x]$. However, we have $f_3 = a\cdot f_1 - f_2 = ax - 1$. The polynomial $f_3$ is an element of $\langle f_1, f_2\rangle$, and $f_3$ divides $f_1$ and $f_2$. This means $\langle f_3\rangle = \langle f_1, f_2\rangle$. That is, $\{f_3\}$ is a Gröbner basis for $\langle f_1, f_2\rangle$, too. $\{f_3\}$ is simpler than $\{f_1, f_2\}$. However, $\{ax-1\}$ cannot be computed by the method of Insa-Pauer.

### 5.2. The second method (Approach via block orders)

In this subsection, we give a problem of the approach of the second method by the following example.

**Example 5.2.** Let $F = \{f_1 = ax + 1, f_2 = (b + 1)y, f_3 = az + bz + z\}$ be a set of polynomials in $\mathbb{Q}[a, b][x, y, z]$. $F$ can be seen as a set of $\mathbb{Q}[a, b, x, y, z]$. We have a block order $\succ_{\{x,y,z\},\{a,b\}} = (\succ_{lex}, \succ_{grlex})$ with $x \succ_{lex} y \succ_{lex} z$ and $a \succ_{grlex} b$ where $\succ_{lex}$ is the lexicographic order and $\succ_{grlex}$ is the graded reverse lexicographic order. Then the reduced Gröbner bases $G$ for $\langle F\rangle$ with respect to a block order $\succ_{\{x,y,z\},\{a,b\}}$ is the following.

$$G = \{g_1 = (a+b+1)z, g_2 = (b+1)y, g_3 = yz, g_4 = ax+1, g_5 = (b+1)xz - z\}.$$

Since $G$ is the reduced Gröbner basis of $\langle F\rangle$, $g \in G$ cannot be reduced by $G\backslash\{g\}$ with respect to the block order in $\mathbb{Q}[a,b,x,y,z]$. However, look at $g_5$. Then, we have $\mathrm{lm}_{\bar{A}}(g_5) \in \langle \mathrm{lm}_{\bar{A}}(G\backslash\{g_5\})\rangle$ in $\mathbb{Q}[a,b][x,y,z]$. That is, $g_5$ can be written $g_5 = x\cdot g_1 - z\cdot g_4$. This means that $g_5$ can be still reduced to 0 by $g_1$ and $g_4$ in $\mathbb{Q}[a,b][x,y,z]$. The polynomial $g_5$ is a redundant polynomial in $\mathbb{Q}[a,b][x,y,z]$. By the method, $G\backslash\{g_5\}$ cannot be computed.

> **Problem**: Sometimes there exits a Gröbner basis which has less elements than Gröbner bases computed by either of the two methods above.
> **This Gröbner basis cannot be computed by the two methods.**

## 6. Algorithms for Computing Reduced Gröbner Bases

In this section, we define a reduced Gröbner basis for an ideal in $K[\bar{A}][\bar{X}]$ and give an algorithm for computing a reduced Gröbner basis. First we define weak reduced Gröbner bases in $K[\bar{A}][\bar{X}]$.

**Definition 6.1 (Weak reduced Gröbner bases).** *Let $\succ_{\bar{X},\bar{A}} := (\succ_1, \succ_2)$ be a block order and $I$ an ideal in $K[\bar{A}][\bar{X}]$. Then, a weak reduced Gröbner basis $G$ for $I$ with respect to $\succ_1$, and $\succ_{\bar{X},\bar{A}}$ is a Gröbner basis for $I$ in $K[\bar{A}][\bar{X}]$ such that*

1. *for all $p \in G$, $\mathrm{lc}(p) = 1$ with respect to $\succ_{\bar{X},\bar{A}}$,*
2. *for all $p \in G$, no monomial in $\mathrm{Mono}(p)$ lies in $\langle \mathrm{lm}(G\backslash\{p\})\rangle$ in $K[\bar{A},\bar{X}]$ with respect to $\succ_{\bar{X},\bar{A}}$,*
3. *for all $p \in G$, no monomial in $\mathrm{Mono}_{\bar{A}}(p)$ lies in $\langle \mathrm{lm}_{\bar{A}}(G\backslash\{p\})\rangle$ in $K[\bar{A}][\bar{X}]$ with respect to $\succ_1$.*

As we said earlier, the approaches of the first and second method cannot always compute reduced Gröbner bases in $K[\bar{A}][\bar{X}]$.

**How do we compute (weak) reduced Gröbner bases in $K[\bar{A}][\bar{X}]$?**

A polynomial ring $K[\bar{A}][\bar{X}]$ can be seen as a polynomials ring $K[\bar{A},\bar{X}]$. This means that the polynomial ring $K[\bar{A}][\bar{X}]$ has properties of $K[\bar{A},\bar{X}]$. In this sense, we have two reduction systems `Reduce1`, `Reduce2` and two S-polynomial systems `Spoly1`, `Spoly2` for computing weak reduced Gröbner bases in $K[\bar{A}][\bar{X}]$.

For instance, in Example 5.1. we have a Gröbner basis $\{f_1 = ax^2 - a, f_2 = (a^3 - a)x - a^2 + 1\}$. If we use `Reduce2` or `Spoly2` to the Gröbner basis $\{f_1, f_2\}$, then we can obtain $ax-1$ by the computation $f_2 \xrightarrow{r2}_{\{f_1\}} ax-1$, or `Spoly2`$(f_1,f_2) = ax-1$. Since $f_1 \xrightarrow{r2}_{\{ax-1\}} 0$ and $f_2 \xrightarrow{r2}_{\{ax-1\}} 0$, $\{ax-1\}$ is a weak reduced Gröbner basis for $\langle f_1, f_2\rangle$. In Example 5.2, we obtained a Gröbner basis $G =$

$\{g_1, g_2, g_3, g_4, g_5\}$ by the algorithm GröbnerBasis-Block . Let's apply `Reduce1` to $G$. Then, since $\mathrm{lpp}_{\{a,b\}}(g_1) | \mathrm{lpp}_{\{a,b\}}(g_5)$, $\mathrm{lpp}_{\{a,b\}}(g_4) | \mathrm{lpp}_{\{a,b\}}(g_5)$ and $\mathrm{lc}_{\{a,b\}}(g_5) = -\mathrm{lc}_{\{a,b\}}(g_1) + \mathrm{lc}_{\{a,b\}}(g_4) = -(a+b+1)+a = -b-1$, we have $g_5 \xrightarrow{r1}_{\{g_1,g_2\}} 0$. Thus, $g_5$ is a redundant polynomial which is found by `Reduce2`. By Definition 6.1, a weak reduced Gröbner basis for $G$ is $\{g_1, g_2, g_3, g_4\}$ with respect to the lexicographic order with $x \succ y \succ z$.

By the observation above, we need `Reduce1`, `Reduce2`, `Spoly1` and `Spoly2` for computing weak reduced Gröbner bases in $K[\bar{A}][\bar{X}]$. We can easily construct an algorithm for computing weak reduced Gröbner bases. Now we know that by the algorithms FirstGB or GröbnerBasis-Block, we can compute a Gröbner basis $G_1$ in $K[\bar{A}][\bar{X}]$. The Gröbner basis $G_1$ is not always a weak reduced Gröbner basis, hence we need `Reduce1` and `Reduce2` to reduce $G_1$ to a weak Gröbner basis. Actually, we need two reduction systems `Reduce1`, `Reduce2` and one of two S-polynomial systems `Spoly1` and `Spoly2`. We introduce an algorithm which returns a weak reduced Gröbner basis. In the first step of this algorithm, we apply FitstGB or GröbnerBasis-Block.

**Algorithm 3** WRGB (Weak reduced Gröbner bases)
**Input** $F$: a finite set of polynomials in $K[\bar{A}][\bar{X}]$, $\succ_1$ : a monomial order on $\mathrm{pp}(\bar{X})$,
$\succ_2$ : a monomial order on $\mathrm{pp}(\bar{A})$, $\succ_{\bar{X},\bar{A}} := (\succ_1, \succ_2)$ : a block order,
**Output** $G$: a weak reduced Gröbner basis of $\langle F \rangle$ w.r.t. $\succ_1$ and $\succ_{\bar{X},\bar{A}}$ in $K[\bar{A}][\bar{X}]$.
**begin**
 $G \leftarrow$ Compute a Gröbner basis $G$ for $\langle F \rangle$ by FirstGB or GröbnerBasis-Block
 $E1 \leftarrow 0$
 **while** $E1 \neq 1$ **do**
  **if** there exists $p \in G$ such that
  $\left(p \xrightarrow{r1}_{\{G\backslash\{p\}\}} p_1\right)$ or $\left(p \xrightarrow{r2}_{\{G\backslash\{p\}\}} p_1 \text{ and w.r.t } \succ_{\bar{X},\bar{A}}\right)$
  **then**
   **if** $p_1 \neq 0$ **then** $G \leftarrow \{G\backslash\{p\}\} \cup \{p_1\}$
   **else if** $G \leftarrow G\backslash\{p\}$
   **end-if**
  **else-if** $E1 \leftarrow 1$
  **end-if**
 **end-while**
`return`$(G)$
**end**

**Theorem 6.2.** *The algorithm* WRGB *terminates. The output forms a weak reduced Gröbner basis for* $\langle F \rangle$.

*Proof.* In the first line of Algorithm 3 WRGB, if we apply FirstGB for computing a Gröbner basis for $\langle F \rangle$ in $K[\bar{A}][\bar{X}]$, then FirstGB terminates. (Since $K[\bar{A}]$ is a Noetherian ring, $K[\bar{A}][\bar{X}]$ is a Noetherian ring too. Thus, the termination of FirstGB is guaranteed because we have a finite ascending chain condition of properly contained ideals over a Noetherian ring.) In the first step of Algorithm 3 WRGB, if

we apply GröbnerBasis-Block for computing a Gröbner basis for $\langle F\rangle$, obviously GröbnerBasis-Block terminates. (see [Buc65]).
Let $G$ be a finite set of polynomials in $K[\bar{A}][\bar{X}]$. In the **while-loop** step, if there exists an element $p$ of $\mathrm{Mono}(g)$ or $\mathrm{Mono}_{\bar{A}}(g)$ which can be reduced to $p_1$ by some polynomials of $G\backslash\{g\}$ in `Reduce1` or `Reduce2`, then we always have $\mathrm{lm}(p) \succ_{\bar{X},\bar{A}} \mathrm{lm}(p_1)$ ($\mathrm{lm}(p_1)$ is smaller or equal than $\mathrm{lm}(p)$ with respect to the monomial order $\succ_{\bar{X},\bar{A}}$). That is, the result of applying `Reduce1` or `Reduce2` to any monomial $m \in \mathrm{Mono}(g) \cup \mathrm{Mono}_{\bar{A}}(g)$ has a leading monomial which cannot be greater than $m$ with respect to $\succ_{\bar{X},\bar{A}}$. Therefore, iterated application of `Reduce1` and `Reduce2` to $G$ will eventually terminate. This algorithm terminates and the outputs satisfy the properties of Definition 6.1. □

**Corollary 6.3.** *Let $I$ be an ideal in $K[\bar{A}][\bar{X}]$ and $\succ_{\bar{X},\bar{A}}:= (\succ_1, \succ_2)$ a block order. Since the algorithms terminates, there exists a weak reduced Gröbner basis for $I$ in $K[\bar{A}][\bar{X}]$. By Example* 5.2*, the reduced Gröbner basis for I with respect to $\succ_{\bar{X},\bar{A}}$ in $K[\bar{A},\bar{X}]$ is always not a weak reduced Gröbner basis for $I$ with respect to $\succ_1$, $\succ_1$ and $\succ_{\bar{X},\bar{A}}$ in $K[\bar{A}][\bar{X}]$.*

In the Algorithm 3 WRGB, if we apply the algorithm FirstGB for computing a Gröbner basis, then we need syzygy computations `Spoly1` and "extended Gröbner bases algorithm [BW93] `Reduce1` (transformation of Gröbner bases)". In general, syzygy computations and "extended Gröbner bases algorithm [BW93]" are expensive. However, in Algorithm 3 WRGB, if we apply the algorithm GröbnerBasis-Block, then we do not need any syzygy computation. Actually, the algorithm GröbnerBasis-Block is a normal Gröbner bases computation in polynomial rings over a field with respect to a block order. At present, we have very powerful programs for computing Gröbner basis in $K[\bar{A},\bar{X}]$ in computer algebra systems `Singular`[1], `Risa/Asir`[2][NT92] and `Magma`[3]. We can apply these powerful programs for computing weak reduced Gröbner bases in $K[\bar{A}][\bar{X}]$. Thus, in implementation and computation speed points of view, WRGB with GröbnerBasis-Block is better than WRGB with FirstGB.

Before concluding this section, we consider a property of reduced Gröbner bases. Now we have a question. **"Is a weak reduced Gröbner basis uniquely determined by an ideal $I \subseteq K[\bar{A}][\bar{X}]$ and monomial orders?"** In fact, this answer is **"NO"**. A weak Reduced Gröbner basis is not unique. We have the following easy example for this question.

**Example 6.4.** Let $F = \{ab+1, ac+1\}$ be a subset of $\mathbb{Q}[a,b,c][x,y]$ and $\succ_{\{x,y\},\{a,b,c\}} = (\succ_{lex}, \succ_{lex})$ a block order with $x \succ_{lex} y$ and $a \succ_{lex} b \succ_{lex} c$ where $\succ_{lex}$ is the lexicographic order. Then, $F$ is a weak Gröbner basis for $\langle F\rangle$ in $\mathbb{Q}[a,b,c][x,y]$. Actually, $F$ satisfies the property 1,2 of Definition 6.1. However, we can say $\langle F\rangle =$

[1] `http://www.singular.uni-kl.de/`
[2] `http://www.math.kobe-u.ac.jp/Asir/`
[3] `http://magma.maths.usyd.edu.au/magma/`

$\langle ac+1, -b+c\rangle$. The set $\{ac+1, -b+c\}$ is a weak reduced Gröbner basis for $\langle F\rangle$, too. Therefore, a weak reduced Gröbner basis is not uniquely determined.

We give one more example facilitate the understanding of the next definition. Let $F = \{(ac+b)x^2, (ac-c+bd^2)x^2, (-cd-bc+bd^3)x\} \subset \mathbb{Q}[a,b,c,d][x]$. We have the lexicographic order $\succ$ such that $a \succ b \succ c \succ d$. In fact, $F$ is a weak Gröbner basis for $\langle F\rangle$. For $e \in \mathrm{lpp}_{\{a,b,c,d\}}(F)$, let $F_e = \{f \,|\, \mathrm{lpp}_{\{a,b,c,d\}}(f) = e\}$. We have $\mathrm{lpp}_{\{a,b,c,d\}}(F) = \{x, x^2\}$, so $F_x = \{(-cd-bc+bd^3)x\}$ and $F_{x^2} = \{(ac+b)x^2, (ac-c+bd^2)x^2\}$. Let's consider all coefficients of $F_x$ and $F_{x^2}$. Since $\mathrm{lc}_{\{a,b,c,d\}}(F_x) = \{-cd-bc+bd^3\}$ has only one element, $\{-cd-bc+bd^3\}$ is the reduced Gröbner basis for an ideal generated by itself. Next we consider $\mathrm{lc}_{\{a,b,c,d\}}(F_{x^2}) = \{ac+b, ac-c+bd^2\}$. Actually, $\{ac+b, ac-c+bd^2\}$ is **NOT** the reduced Gröbner basis for the ideal generated by itself $\{ac+b, ac-c+bd^2\}$ with respect to $\succ$ in $\mathbb{Q}[a,b,c,d]$. However, since (the main variable) $x$ divides $x^2$, by definition of Reduce1, $\mathrm{lc}_{\{a,b,c,d\}}(F_{x^2})$ is constrained by $\langle \mathrm{lc}_{\{a,b,c,d\}}(F_x)\rangle$. Therefore, we have to consider $\mathrm{lc}_{\{a,b,c,d\}}(F_{x^2})$ in $\mathbb{Q}[a,b,c,d]/\langle \mathrm{lc}_{\{a,b,c,d\}}(F_x)\rangle$.

We did not take care of conditions of all coefficients in $K[\bar{A}]$ and thus a weak reduced Gröbner basis was not uniquely determined. Since the coefficient domain is a polynomial ring, we need some conditions to obtain a unique reduced Gröbner basis.

**Definition 6.5.** *The* **normal form** *of a subset $F$ in $K[\bar{A}]$ with respect to an ideal $I$ in $K[\bar{A}]$ and a monomial order $\succ$ is the set of all non-zero remainders of elements of $F$ after division by a Gröbner basis of $I$ with respect to $\succ$.*

**Definition 6.6.** *Let $F$ be a subset of $K[\bar{A}]$, $I \subset K[\bar{A}]$ an ideal and $\succ$ a monomial order on $\mathrm{pp}(\bar{A})$. $G \subset K[\bar{A}]$ is called a* **reduced Gröbner basis** *for $F$ with respect to $\succ$ in a quotient ring $K[\bar{A}]/I$ if $G$ is the normal form of the reduced Gröbner basis for $\langle F\rangle$ with respect to an ideal $I$ and $\succ$.*

**Remark 6.7.** We know algorithm for computing Gröbner bases and division. It is possible to compute a reduced Gröbner basis $G$ in $K[\bar{A}]/I$. **A reduced Gröbner basis $G$ in $K[\bar{A}]/I$ is uniquely determined.** It is an easy exercise.

Now we define more strict reduced Gröbner bases than weak reduced Gröbner bases. We call this reduced Gröbner basis **"strong reduced Gröbner basis"**.

**Definition 6.8 (Strong reduced Gröbner bases).** *Let $\succ_{\bar{X},\bar{A}} := (\succ_1, \succ_2)$ be a block order and $I$ an ideal in $K[\bar{A}][\bar{X}]$. For $e \in \mathrm{lpp}_{\bar{A}}(G)$, let $G_e = \{f \,|\, \mathrm{lpp}_{\bar{A}}(f) = e\}$. Then, a strong reduced Gröbner basis $G$ for $I$ with respect to $\succ_1$, $\succ_2$ and $\succ_{\bar{X},\bar{A}}$ is a Gröbner basis for $I$ in $K[\bar{A}][\bar{X}]$ such that*

1. *for all $p \in G$, no monomial in $\mathrm{Mono}(p)$ lies in $\langle \mathrm{lm}(G \setminus \{p\})\rangle$ in $K[\bar{A}, \bar{X}]$ with respect to $\succ_{\bar{X},\bar{A}}$,*
2. *for all $p \in G$, no monomial in $\mathrm{Mono}_{\bar{A}}(p)$ lies in $\langle \mathrm{lm}_{\bar{A}}(G \setminus \{p\})\rangle$ in $K[\bar{A}][\bar{X}]$ with respect to $\succ_1$,*

3. *for* $e \in \mathrm{lpp}_{\bar{A}}(G)$, $\mathrm{lc}_{\bar{A}}(G_e)$ *is the reduced Gröbner basis for an ideal generated by itself with respect to* $\succ_2$ *in the quotient ring* $K[\bar{A}]/J_e$ *where* $J_e$ *is an ideal generated by* $F = \{\mathrm{lc}_{\bar{A}}(g) \in K[\bar{A}] | g \in G\backslash G_e$ *such that* $\mathrm{lpp}_{\bar{A}}(g)|e\}$.
   (*If* $F = \emptyset$, $K[\bar{A}]/J_e = K[\bar{A}]$.)

In order to consider the strong reduced Gröbner basis, let's see Example 5.1. In the example, we obtained a Gröbner basis $G = \{f_1 = a^2x - a, f_2 = (a^3 - a)x - a^2 + 1\}$ by FirstGB. $G$ does not satisfy the property 3 of Definition 6.8. Since the set of all power products is $\mathrm{lpp}_{\bar{A}}(G) = \{x\}$, we have $G_x := \{f_1, f_2\}$ and $\mathrm{lc}_{\bar{A}}(G_x) := \{a^2, a^3 - a\}$. Since the reduced Gröbner basis for $\langle \mathrm{lc}_{\bar{A}}(G_x)\rangle$ is $\{a\}$ in $K[\bar{A}]$, $G$ is not a strong reduced Gröbner basis. However, we can construct the strong reduced Gröbner basis. Since $\langle a \rangle = \langle \mathrm{lc}_{\bar{A}}(G_x)\rangle$, $a$ can be written as $a = c_1 \mathrm{lc}_{\bar{A}}(f_1) + c_2 \mathrm{lc}_{\bar{A}}(f_2)$, where $c_1, c_2 \in \mathbb{Q}[a]$. In this case, $c_1 = a, c_2 = -1$. Now we can compute a new polynomial $g$ such that $\langle g \rangle = \langle G \rangle$, $\langle \mathrm{lm}_{\bar{A}}(g)\rangle = \langle \mathrm{lm}_{\bar{A}}(G)\rangle$ and $\{\mathrm{lc}_{\bar{A}}(g)\}$ is the reduced Gröbner basis for $\mathrm{lc}_{\bar{A}}(G_x)$. $g = c_1f_1 + c_2f_2 = af_1 - f_2 = ax - 1$. Therefore, $\{g\}$ is a strong reduced Gröbner basis.

We introduce an algorithm which returns a strong reduced Gröbner basis.

**Algorithm 4** SRGB (Strong reduced Gröbner bases)
**Input** $F$: a finite set of polynomials in $K[\bar{A}][\bar{X}]$, $\succ_1$ : a monomial order on $\mathrm{pp}(\bar{X})$,
$\succ_2$ : a monomial order on $\mathrm{pp}(\bar{A})$, $\succ_{\bar{X},\bar{A}} := (\succ_1, \succ_2)$ : a block order,
**Output** $L$: a strong reduced Gröbner basis of $\langle F \rangle$ with respect to $\succ_1, \succ_2$ and $\succ_{\bar{X},\bar{A}}$ in $K[\bar{A}][\bar{X}]$.
**begin**
$G \leftarrow$ Compute a Gröbner basis for $\langle F \rangle$; $B \leftarrow \mathrm{lpp}_{\bar{A}}(G)$; $L \leftarrow \emptyset$
  **while** $B \neq \emptyset$ **do**
    Select **the lowest power product** $p$ w.r.t. $\succ_1$ from $B$; $B \leftarrow B\backslash\{p\}$
   $G_p \leftarrow \{f \in F | \mathrm{lpp}_{\bar{A}}(f) = p\}$; $G \leftarrow G\backslash G_p$; $J_p \leftarrow \{\mathrm{lc}_{\bar{A}}(f) | f \in G$ s.t $\mathrm{lpp}_{\bar{A}}(f)|p\}$
    **if** $\mathrm{lc}_{\bar{A}}(G_p)$ is **NOT** the reduced Gröbner basis with respect to $\succ_2$ in $K[\bar{A}]/\langle J_p\rangle$
**then**
      $Q \leftarrow$ Compute $Q$ such that $\langle Q \rangle = \langle G_p \rangle$, $\langle \mathrm{lm}_{\bar{A}}(Q)\rangle = \langle \mathrm{lm}_{\bar{A}}(G_p)\rangle$ and
       $\mathrm{lc}_{\bar{A}}(Q)$ is the reduced Gröbner basis for $\langle \mathrm{lc}_{\bar{A}}(G_p)\rangle$ w.r.t. $\succ_2$ in $K[\bar{A}]/\langle J_p\rangle$
      ($\mathrm{lm}_{\bar{A}}(Q)$ is irreducible by $J_p$)
      $L \leftarrow L \cup \{Q \downarrow_L\}$
      **else-if** $L \leftarrow L \cup \{G_p \downarrow_L\}$
      **end-if**
  **end-while**
   `return`($L$)
**end**

In the algorithm, we used the notations $Q \downarrow_L$ and $G_p \downarrow_L$ where $Q, G_p, L \subset K[\bar{A}][\bar{X}]$. This meaning is the following.

$Q \downarrow_L :=$ **begin**
    $S \leftarrow \emptyset$
    **while** $Q \neq \emptyset$ **do**

Select $q$ from $Q$; $Q \leftarrow Q \backslash \{q\}$; $q_1 \leftarrow q\downarrow_L$ (by Reduce1 and Reduce2)
  **if** $q_1 \neq 0$ **then**
    $S \leftarrow S \cup \{q_1\}$
  **end-if**
**end-while**
`return`$(S)$
**end**

**Theorem 6.9.** *The algorithm* SRGB *terminates. The output forms a strong reduced Gröbner basis for* $\langle F\rangle$.

*Proof.* We know that how to compute a weak reduced Gröbner basis $G$, and this step terminates. Since we have a Gröbner basis $G$, we have to check $\mathrm{lc}_{\bar{A}}(G_p)$ where $p \in \mathrm{lpp}_{\bar{A}}(G)$. If $\mathrm{lc}_{\bar{A}}(G_p)$ is not a reduced Gröbner basis with respect to $\succ_2$ in $K[\bar{A}]/\langle J_p\rangle$, then, we have to compute the following;

---

$Q \leftarrow$ Compute $Q$ such that $\langle Q\rangle = \langle G_p\rangle$, $\langle \mathrm{lm}_{\bar{A}}(Q)\rangle = \langle \mathrm{lm}_{\bar{A}}(G_p)\rangle$ and $\mathrm{lc}_{\bar{A}}(Q)$ is the reduced Gröbner basis for $\langle \mathrm{lc}_{\bar{A}}(G_p)\rangle$ with respect to $\succ_2$ in $K[\bar{A}]/\langle J_p\rangle$.

---

As we said in **Remark** 6.7, it is possible to compute $Q$. This step clearly terminates. Since $B$ is a finite set, the first **while-loop** terminates. Therefore, this algorithm terminates. In the **if**-part of the algorithm, if $\mathrm{lc}_{\bar{A}}(G_p)$ is not the reduced Gröbner basis with respect to $\succ_2$ in $K[\bar{A}]/\langle J_p\rangle$, then the algorithm computes $Q$. Next the algorithm computes $Q\downarrow_L$. In fact, by reduce1, reduce2 and the weak reduced Gröbner basis, we have $\mathrm{lm}_{\bar{A}}(Q) = \mathrm{lm}_{\bar{A}}(Q\downarrow_L)$. That is, in this step, the algorithm does not reduce any leading monomials of $Q$ for the properties 1,2 of Definition 6.8. By the same reasons, if $\mathrm{lc}_{\bar{A}}(G_p)$ is the reduced Gröbner basis with respect to $\succ_2$ in $K[\bar{A}]/\langle J_p\rangle$, then we have $\mathrm{lm}_{\bar{A}}(G_p) = \mathrm{lm}_{\bar{A}}(G_p\downarrow_L)$, and $G_p\downarrow_L$ satisfies the properties 1,2 of Definition 6.8. Therefore, this algorithm outputs a strong reduced Gröbner basis with respect to $\succ_1$, $\succ_2$ and $\succ_{\bar{X},\bar{A}}$. □

A strong reduced Gröbner bases have the following nice property.

**Theorem 6.10.** Let $\succ_{\bar{X},\bar{A}} := (\succ_1, \succ_2)$ be a block order on $\mathrm{pp}(\bar{X},\bar{A})$. Let $I$ be an ideal in $K[\bar{A}][\bar{X}]$. Then, $I$ has a unique strong reduced Gröbner basis.

*Proof.* Since the existence of strong reduced Gröbner bases was proved by Theorem 6.9, we prove the uniqueness. First we prove the following claim.

**Claim 1** Let $\succ_{\bar{X},\bar{A}} := (\succ_1, \succ_2)$ be a block order on $\mathrm{pp}(\bar{X},\bar{A})$ and $I$ an ideal in $K[\bar{A}][\bar{X}]$. Let $G_1$ and $G_2$ be strong reduced Gröbner bases for $I$ with respect to $\succ_1$ and $\succ_{\bar{X},\bar{A}}$. Then, $\mathrm{lm}(G_1) = \mathrm{lm}(G_2)$. Namely, the set of all leading monomials of strong reduced Gröbner bases for $I$ is unique.

(*Proof of Claim 1*) Assume that $G_1$ and $G_2$ are strong reduced Gröbner bases for $I$ in $K[\bar{A}][\bar{X}]$. We set $G_1 = \{g_1, \dots, g_s\}$, $G_2 = \{h_1, \dots, h_p\}$. W.l.o.g., $\mathrm{lpp}_{\bar{A}}(g_1) = \dots = \mathrm{lpp}_{\bar{A}}(g_k)$ is the lowest leading power product of $G_1$ with respect to $\succ_1$ for $1 \le k \le s$, and $\mathrm{lpp}_{\bar{A}}(h_1) = \dots = \mathrm{lpp}_{\bar{A}}(h_l)$ is the lowest leading power product of

$G_2$ with respect to $\succ_1$ for $1 \le l \le p$. If $\mathrm{lpp}_{\bar{A}}(g_1) \succ_1 \mathrm{lpp}_{\bar{A}}(h_1)$ ($\mathrm{lpp}_{\bar{A}}(g_1)$ is bigger than $\mathrm{lpp}_{\bar{A}}(h_1)$), $h_1$ can not be in $\langle G_1 \rangle$, because there is no element such that $\mathrm{lpp}_{\bar{A}}(g_i) | \mathrm{lpp}_{\bar{A}}(h_1)$ where $g_i \in G_1$. However, by $\langle G_1 \rangle = \langle G_2 \rangle$, we have $h_1 \in \langle G_1 \rangle$. Hence, $\mathrm{lpp}_{\bar{A}}(h_1) \succeq \mathrm{lpp}_{\bar{A}}(g_1)$ (by the order $\succ_1$). By the same reason, we have also $\mathrm{lpp}_{\bar{A}}(g_1) \succeq \mathrm{lpp}_{\bar{A}}(h_1)$ (by the order $\succ_1$). Therefore, $\mathrm{lpp}_{\bar{A}}(h_1) = \mathrm{lpp}_{\bar{A}}(g_1)$. We have two sets

$$\{\mathrm{lm}_{\bar{A}}(g_1), \dots, \mathrm{lm}_{\bar{A}}(g_k)\} = \{\mathrm{lc}_{\bar{A}}(g_1)\,\mathrm{lpp}_{\bar{A}}(g_1), \dots, \mathrm{lc}_{\bar{A}}(g_k)\,\mathrm{lpp}_{\bar{A}}(g_1)\},$$
$$\{\mathrm{lm}_{\bar{A}}(h_1), \dots, \mathrm{lm}_{\bar{A}}(h_l)\} = \{\mathrm{lc}_{\bar{A}}(h_1)\,\mathrm{lpp}_{\bar{A}}(g_1), \dots, \mathrm{lc}_{\bar{A}}(h_l)\,\mathrm{lpp}_{\bar{A}}(g_1)\}.$$

Since $G_1, G_2$ are strong reduced Gröbner bases for $I$ with respect to $\succ_1$, $\succ_2$ and $\succ_{\bar{X},\bar{A}}$ in $K[\bar{A}][\bar{X}]$, $\{\mathrm{lc}_{\bar{A}}(g_1), \dots, \mathrm{lc}_{\bar{A}}(g_k)\}$ is the reduced Gröbner basis for an ideal generated by itself in $K[\bar{A}]$, and $\{\mathrm{lc}_{\bar{A}}(h_1), \dots, \mathrm{lc}_{\bar{A}}(h_l)\}$ is also the reduced Gröbner basis for an ideal generated by itself in $K[\bar{A}]$. By the property of Gröbner bases $G_1, G_2$ and $\mathrm{lpp}_{\bar{A}}(h_1) = \mathrm{lpp}_{\bar{A}}(g_1)$, we have the following relations

$$\begin{aligned}
\mathrm{lm}_{\bar{A}}(h_{j_1}) &= \alpha_1 \,\mathrm{lm}_{\bar{A}}(g_1) + \cdots + \alpha_k \,\mathrm{lm}_{\bar{A}}(g_k) \\
&= \alpha_1 \,\mathrm{lc}_{\bar{A}}(g_1)\,\mathrm{lpp}_{\bar{A}}(g_1) + \cdots + \alpha_k \,\mathrm{lc}_{\bar{A}}(g_k)\,\mathrm{lpp}_{\bar{A}}(g_1), \\
\mathrm{lm}_{\bar{A}}(g_{j_2}) &= \beta_1 \,\mathrm{lm}_{\bar{A}}(h_1) + \cdots + \beta_l \,\mathrm{lm}_{\bar{A}}(h_l) \\
&= \beta_1 \,\mathrm{lc}_{\bar{A}}(h_1)\,\mathrm{lpp}_{\bar{A}}(g_1) + \cdots + \beta_l \,\mathrm{lc}_{\bar{A}}(h_l)\,\mathrm{lpp}_{\bar{A}}(g_1),
\end{aligned}$$

where $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l \in K[\bar{A}]$, $1 \le j_1 \le k$ and $1 \le j_2 \le l$. Hence we can say $\langle \mathrm{lc}_{\bar{A}}(g_1), \dots, \mathrm{lc}_{\bar{A}}(g_k)\rangle = \langle \mathrm{lc}_{\bar{A}}(h_1), \dots, \mathrm{lc}_{\bar{A}}(h_l)\rangle$. Since the two sets $\{\mathrm{lc}_{\bar{A}}(g_1), \dots, \mathrm{lc}_{\bar{A}}(g_k)\}$ and $\{\mathrm{lc}_{\bar{A}}(h_1), \dots, \mathrm{lc}_{\bar{A}}(h_p)\}$ are the reduced Gröbner bases with respect to $\succ_2$ in $K[\bar{A}]$, we have $\{\mathrm{lc}_{\bar{A}}(g_1), \dots, \mathrm{lc}_{\bar{A}}(g_k)\} = \{\mathrm{lc}_{\bar{A}}(h_1), \dots, \mathrm{lc}_{\bar{A}}(h_l)\}$. Therefore we have

$$\{\mathrm{lm}_{\bar{A}}(g_1), \dots, \mathrm{lm}_{\bar{A}}(g_k)\} = \{\mathrm{lm}_{\bar{A}}(h_1), \dots, \mathrm{lm}_{\bar{A}}(h_l)\}.$$

Next we consider two sets $G_{11} := G_1 \backslash \{g_1, \dots, g_k\}$ and $G_{21} := G_2 \backslash \{h_1, \dots, h_l\}$. W.l.o.g., $\mathrm{lpp}_{\bar{A}}(g_{k+1}) = \ldots = \mathrm{lpp}_{\bar{A}}(g_{k_1})$ is the lowest leading power product of $G_{11}$ with respect to $\succ_1$ for $2 \le k_1 \le s$. That is, $g_{k+1}, \dots, g_{k_1} \in G_{11} \subseteq G_1$. Since $G_1$ is a strong reduced Gröbner basis, $\mathrm{lm}_{\bar{A}}(g_{k+1}), \dots, \mathrm{lm}_{\bar{A}}(g_{k_1})$ can not be reduced by $\mathrm{lm}_{\bar{A}}(g_1), \dots, \mathrm{lm}_{\bar{A}}(g_k)$. W.l.o.g., $\mathrm{lpp}_{\bar{A}}(h_{l+1}) = \ldots = \mathrm{lpp}_{\bar{A}}(h_{l_1})$ is the lowest leading power product of $G_{21}$ with respect to $\succ_1$ for $2 \le l_1 \le p$. That is, $h_{l+1}, \dots, h_{l_1} \in G_{21} \subseteq G_2$. By the same reason above, we have $\mathrm{lpp}_{\bar{A}}(g_{k+1}) = \mathrm{lpp}_{\bar{A}}(h_{l+1})$ and $\langle \mathrm{lc}_{\bar{A}}(g_{k+1}), \dots, \mathrm{lc}_{\bar{A}}(g_{k_1})\rangle = \langle \mathrm{lc}_{\bar{A}}(h_{l+1}), \dots, \mathrm{lc}_{\bar{A}}(h_{l_1})\rangle$. We know that $\mathrm{lpp}_{\bar{A}}(g_1)$ is the lowest leading power product of $G_1$ and $G_2$. If $\mathrm{lpp}_{\bar{A}}(g_1)$ does not divide $\mathrm{lpp}_{\bar{A}}(g_{k+1})$, then by the same reason above,

$$\{\mathrm{lm}_{\bar{A}}(g_{k+1}), \dots, \mathrm{lm}_{\bar{A}}(g_{k_1})\} = \{\mathrm{lm}_{\bar{A}}(h_{l+1}), \dots, \mathrm{lm}_{\bar{A}}(h_{l_1})\}.$$

If $\mathrm{lpp}_{\bar{A}}(g_1)$ divide $\mathrm{lpp}_{\bar{A}}(g_{k+1})$, then the reduced Gröbner basis for $\langle \mathrm{lc}_{\bar{A}}(g_{k+1}), \dots, \mathrm{lc}_{\bar{A}}(g_{k_1})\rangle$ is unique in $K[\bar{A}]/J$. Since $G_1$ and $G_2$ are strong reduced Gröbner bases, we have

$$\{\mathrm{lm}_{\bar{A}}(g_{k+1}), \dots, \mathrm{lm}_{\bar{A}}(g_{k_1})\} = \{\mathrm{lm}_{\bar{A}}(h_{l+1}), \dots, \mathrm{lm}_{\bar{A}}(h_{l_1})\}.$$

By the Hilbert basis theorem, $G_1$ and $G_2$ have finite many elements. Therefore, repeat the same procedure, then we have $\mathrm{lm}_{\bar{A}}(G_1) = \mathrm{lm}_{\bar{A}}(G_2)$. □

Suppose that $G_1$ and $G_2$ are strong reduced Gröbner bases for $I$. Then, by the claim 1, we have $\mathrm{lm}_{\bar{A}}(G_1) = \mathrm{lm}_{\bar{A}}(G_2)$. Thus, given $g_1 \in G_1$, there is $g_2 \in G_2$ such that $\mathrm{lm}_{\bar{A}}(g_1) = \mathrm{lm}_{\bar{A}}(g_2)$. If we can show that $g_1 = g_2$, it will follow that $G_1 = G_2$, and uniqueness will be proved.
To show $g_1 = g_2$, consider $g_1 - g_2$. This is in $I$, and since $G_1$ is a Gröbner basis, it follows that $g_1 - g_2 \xrightarrow{r1}_{G_1} \circ \xrightarrow{r2}_{G_1} \circ \cdots \circ \xrightarrow{r1}_{G_1} 0$ (by Reduce1 and Reduce2). However, we also know $\mathrm{lm}_{\bar{A}}(g_1) = \mathrm{lm}_{\bar{A}}(g_2)$. Hence, these monomials cancel in $g_1 - g_2$, and the remaining monomials are divisible by none of $\mathrm{lm}_{\bar{A}}(G_1) = \mathrm{lm}_{\bar{A}}(G_2)$ since $G_1$ and $G_2$ are reduced. This shows that $g_1 - g_2 \xrightarrow{r1}_{G_1} \circ \xrightarrow{r2}_{G_1} \circ \cdots \circ \xrightarrow{r1}_{G_1} g_1 - g_2$, and then $g_1 - g_2 = 0$ follows. This completes the proof. □

## 7. Computation Examples

The algorithms WRGB (with GröbnerBasis-Block) have been implemented for the case $K = \mathbb{Q}$ in the computer algebra system Risa/Asir by the author. In this section, we give three easy examples of reduced Gröbner bases .

**Example 7.1.** Let $a, x, y$ be variables and $f_1 = (a-1)x+y^2$, $f_2 = ay+a$ polynomials in $\mathbb{Q}[a][x, y]$. We compute a reduced Gröbner basis for $\langle f_1, f_2\rangle$ with respect to the lexicographic order with $x \succ y$ in $\mathbb{Q}[a][x, y]$.
By the procedure of WRGB, first we compute the reduced Gröbner basis $G$ for $\langle f_1, f_2\rangle$ with respect to a block order $\succ_{\{x,y\},\{a\}}$ in $\mathbb{Q}[a, x, y]$. The reduced Gröbner basis $G$ in $\mathbb{Q}[a, x, y]$ is the following

$$G = \left\{g_1 = ay + a, g_2 = ax - x + y^2, g_3 = -xy - x + y^3 + y^2\right\}.$$

Second, we need to check whether there exists a polynomial $p \in G$ which can be reduced by $G\backslash\{p\}$ or not.
We have $\mathrm{lpp}_{\{a\}}(g_1)|\,\mathrm{lpp}_{\{a\}}(g_3)$, $\mathrm{lpp}_{\{a\}}(g_2)|\,\mathrm{lpp}_{\{a\}}(g_3)$ and $\mathrm{lc}_{\{a\}}(g_3) = \mathrm{lc}_{\{a\}}(g_1) - \mathrm{lc}_{\{a\}}(g_2) = -a + (a - 1) = -1$, therefore $g_3$ can be reduced as follows

$$g_3 \xrightarrow{r1}_{\{g_1,g_2\}} g_3 - (-xg_1 + yg_2) = ax - x + y^2 = g_2.$$

The set $\{g_1, g_2\}$ is a weak reduced Gröbner basis for $\langle f_1, f_2\rangle$ in $\mathbb{Q}[a][x, y]$. Actually, $\{g_1, g_2\}$ is the strong reduced Gröbner basis, too.

In the following example, we compare three algorithms GröbnerBasis-Block, FirstGB and WRGB. We used a PC with [CPU: Pentium M 1.73 GHZ, OS: Windows XP].

**Example 7.2.** Let $a, b, x, y, z$ be variables and $F = \{ax^2z + ay + a, axz + b, (a + 1)xz + ab\}$ in $\mathbb{Q}[a, b][x, y, z]$. We have the lexicographic order with $x \succ y \succ z$. We compute a Gröbner basis for $\langle F\rangle$ in $\mathbb{Q}[a, b][x, y, z]$ by three algorithms FirstGB, GröbnerBasisB, and WRGB (with GröbnerBasis-Block).

1. By FirstGB, we have the following Gröbner basis

```
[b*a^2-b*a-b,-b*x+a*y+a,(a+1)*z*x+b*a,a*z*x+b,
 a*z*y+a*z+b^2*a-b^2,(-a^3+a^2+a)*y-a^3+a^2+a].
(cputime: 0.04688sec)
```

This list has six polynomials.

2. By GröbnerBasis-Block, we have the following Gröbner basis

```
[-b*a^2+b*a+b,(a^3-a^2-a)*y+a^3-a^2-a,b*z*y+b*z-b^3*a+2*b^3,
 a*z*y+a*z+b^2*a-b^2,-b*x+a*y+a,-z*x-b*a+b].
(cputime: 0sec)
```

This list has six polynomials.

3. By WRGB, we have the following reduced Gröbner basis

```
[-b*a^2+b*a+b,(a^3-a^2-a)*y+a^3-a^2-a,a*z*y+a*z+b^2*a-b^2,
 -b*x+a*y+a,-z*x-b*a+b].
(cputime: 0.01563sec)
```

This list has five polynomials.

## 8. Conclusions

The existing algorithms cannot compute reduced Gröbner bases. In this paper, we defined reduced Gröbner bases in polynomial rings over a polynomial ring and gave algorithms for computing them. By the algorithm SRGB, we can uniquely obtain the strong reduced Gröbner basis in polynomial rings over a polynomial ring. We can apply the technique of reduced Gröbner bases in $K[\bar{A}][\bar{X}]$ for computing comprehensive Gröbner bases [Wei92, Mon02, SS06] as one of applications. If we apply the technique for computing comprehensive Gröbner bases, then we can obtain comprehensive Gröbner bases which are more optimal.

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