Conference on Methods of Proof Theory in Mathematics

Max Planck Institute for Mathematics, Bonn

Multi-Summation Tools and Special Functions

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Warmup example (bonus problem 6.69 in "Concrete Mathematics")

FIND a closed form for

$$\sum_{k=1}^{n} k^2 \mathsf{H}_{n+k} = \boxed{?},$$

where $\mathsf{H}_n := \sum_{k=1}^{n} \frac{1}{k}$.

"It would be nice to automate the derivation of formulas such as this."

Telescoping

GIVEN $f(k) = k^2 H_{n+k}$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

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Sigma computes

$$g(k)=\frac{1}{36}(k(-4k^2+(6n+3)k-12n(n+1)+1)$$
$$+6(2k^3-3k^2+k+n(n+1)(2n+1))H_{k+n}).$$

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FIND $g(k)$:

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for all $1\leq k\leq n$ and $n\geq 0$.

Summing this equation over k from 1 to n gives

$$\boxed{\sum_{k=1}^{n}k^2\mathrm{H}_{n+k}}=\boxed{g(n+1)-g(1)}$$

$$=-\frac{1}{36}n(n+1)\left(10n+6(2n+1)H_n-12(2n+1)H_{2n}-1\right).$$

Telescoping

FIND a closed form for

$$\sum_{k=1}^{n} H_k.$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{Q}, & & \\ \sigma(k) &= k+1, & \mathcal{S}\, k &= k+1, \\ \sigma(h) &= h + \frac{1}{k+1}, & \mathcal{S}\, H_k &= H_k + \frac{1}{k+1}. \end{aligned}$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

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with

$$g(k) = (H_k - 1)k.$$

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Hence,

$$(H_{n+1} - 1)(n+1) = \sum_{k=1}^{n} H_k.$$

Example 1: Padé approximation

Example 2

Quadratic Padé approximation to $\log(x)$ at $x = 1$

FIND $r_m(x) = \sum_{k=0}^{m} a_k x^k, \quad s_m(x) = \sum_{k=0}^{m} b_k x^k, \quad t_m(x) = \sum_{k=0}^{m} c_k x^k$:

$$r_m(x)\,(\log x)^2 + s_m(x)\,\log(x) + t_m(x) = O((x-1)^{3m+2}).$$

K. Mahler. *On the approximation of logarithms of algebraic numbers* (1953)

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A. Weideman finds

$$\begin{aligned} r_m(x) &= c_3 A(m,x) \\ s_m(x) &= c_2 A(m,x) + 2c_3 B(m,x) \\ t_m(x) &= c_1 A(m,x) + c_2 B(m,x) + c_3 C(m,x) \end{aligned}$$

where

$$A(m,x) = \sum_{k=0}^{m} \binom{m}{k}^3 (-x)^k \qquad B(m,x) = \sum_{k=0}^{m} \Big[\frac{d}{dk}\binom{m}{k}^3\Big](-x)^k$$

$$C(m,x) = \sum_{k=0}^{m} \Big[\frac{d^2}{dk^2}\binom{m}{k}^3\Big](-x)^k.$$

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$$C(m,x) = \sum_{k=0}^{m} \Big[\frac{d^2}{dk^2}\binom{m}{k}^3\Big](-x)^k.$$

Tests at $x = 1$, $m = 0, 1, \dots$:
$c_1 = \pi^2, \quad c_2 = 0, \quad c_3 = 1$

For all $m \geq 0$*,*

$$\sum_{k=0}^{m}(-1)^k\left[\pi^2\binom{m}{k}^3+\frac{d^2}{dk^2}\binom{m}{k}^3\right]=0$$

For all $m \geq 0$*,*

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}^3\left[3(H_{m-k}-H_k)^2+H_{m-k}^{(2)}+H_k^{(2)}\right]=0$$

where

$$H_k=\sum_{i=1}^{k}\frac{1}{i},\quad H_k^{(2)}=\sum_{i=1}^{k}\frac{1}{i^2}.$$

Theorem (Sigma; 2002)

For all $m \geq 0$*,*

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}^3\left[3(H_{m-k}-H_k)^2+H^{(2)}_{m-k}+H^{(2)}_k\right]=0$$

where

$$H_k=\sum_{i=1}^{k}\frac{1}{i},\quad H^{(2)}_k=\sum_{i=1}^{k}\frac{1}{i^2}.$$

Proof.

Sigma

□

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{SUM}(m) := \sum_{k=0}^{m} \underbrace{(-1)^k \binom{m}{k}^3 \left[3(H_{m-k} - H_k)^2 + H_{m-k}^{(2)} + H_k^{(2)}\right]}_{=:\, f(m,k)}.$$

FIND $c_0(m)$, $c_1(m)$, $c_2(m)$, and $g(m,k)$:

$$\boxed{g(m,k+1) - g(m,k)} = \boxed{c_0(m)\, f(m,k) + c_1(m)\, f(m+1,k) + c_2(m)\, f(m+2,k)}$$

for all $0 \leq k \leq m$ and all $m \geq 0$.

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for all $0 \leq k \leq m$ and all $m \geq 0$.

Sigma:

$$c_0(m) := 3(3m+2)(3m+4)(3m+8), \quad c_1(m) := 0,$$
$$c_2(m) := (m+2)^2(3m+8),$$

$$g(m,k) := (-1)^k \binom{m}{k}^3 \frac{p_1(k, m, H_k, H_k^{(2)}, H_{m-k}, H_{m-k}^{(2)})}{(m-k+1)^5(m-k+2)^5},$$

$$g(m,k+1) := (-1)^k \binom{m}{k}^3 \frac{p_2(k, m, H_k, H_k^{(2)}, H_{m-k}, H_{m-k}^{(2)})}{(m-k+1)^5}.$$

Zeilberger's creative telescoping paradigm

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for all $0 \le k \le m$ and all $m \ge 0$.

Summing this equation over k from 0 to m gives:

$$\boxed{g(m,m+1) - g(m,0)}$$

$$= \boxed{\begin{array}{l} c_0(m)\,\mathsf{SUM}(m) + \\ c_1(m)\,\left[\mathsf{SUM}(m+1) - f(m+1,m+1)\right] \\ c_2(m)\,\left[\mathsf{SUM}(m+2) - f(m+2,m+1) - f(m+2,m+2)\right]. \end{array}}$$

Example 2:
Apéry's proof of the irrationality of $\zeta(3)$

Example 3

Apéry's proof (1979) of the irrationality of $\zeta(3)$ relies on the following fact:

$$a(n) = \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2$$

and

$$b(n) = \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2 \left(H_n^{(3)} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n+m}{m} \binom{n}{m}} \right)$$

satisfy both the recurrence relation

$$(n+1)^3 A(n) - (2n+3)\left(17n^2 + 51n + 39\right) A(n+1) + (n+2)^3 A(n+2) = 0.$$

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Van der Poorten (1979) points out that Henri Cohen and Don Zagier showed this fact by

> *"some rather complicated but ingenious explanations"*

based on the creative telescoping method.

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$a(n)$-case: trivial exercise by Zeilberger's algorithm (1991)

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$b(n)$-case: skilful application of computer algebra

1. Generalization of the Cohen/Zagier method in the WZ-setting (Zeilberger, 1993)
2. Multi-summation + holonomic closure properties (Chyzak, 1998)

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satisfy both the recurrence relation

$$(n+1)^3A(n)-(2n+3)\left(17n^2+51n+39\right)A(n+1)+(n+2)^3A(n+2)=0.$$

$b(n)$-case: plain sailing (and not plane sailing) by Sigma

Example 3: Evaluation of a quadruple sum

Example 4

A challenging email (2004)

```
From: Doron Zeilberger
To: Robin Pemantle, Herbert Wilf
CC:Carsten Schneider


Robin and Herb,

I am willing to bet that Carsten Schneider's SIGMA package
for handling sums with harmonic numbers (among others)
can do it in a jiffy. I am Cc-ing this to Carsten.

Carsten: please do it, and Cc- the  answer to me.
             -Doron
```

The problem

From: Robin Pemantle [University of Pennsylvania]
To: herb wilf; doron zeilberger

Herb, Doron,

I have a sum that, when I evaluate numerically, looks suspiciously like it comes out to exactly 1.
Is there a way I can automatically decide this?
The sum may be written in many ways, but one is:

$$\sum_{j,k=1}^{\infty} \frac{H_j(H_{k+1}-1)}{jk(k+1)(j+k)}; \qquad H_j := \sum_{i=1}^{j} \frac{1}{i}.$$

Of course you can expand out the H's and get a quadruple sum.
There are zillions of ways to play with it, summing by parts, but I have never managed to get rid of all the summations.

Robin

Simplify:

$$\sum_{k=1}^{\infty}\frac{H_{k+1}-1}{k(k+1)}\sum_{j=1}^{\infty}\frac{H_j}{j(j+k)}.$$

$$\sum_{j=1}^{a}\frac{H_j}{j(j+k)} = \text{FIND}$$

Simplify:

$$\sum_{k=1}^{\infty}\frac{H_{k+1}-1}{k(k+1)}\sum_{j=1}^{\infty}\frac{H_j}{j(j+k)}.$$

$$\sum_{j=1}^{a}\frac{H_j}{j(j+k)}=\frac{kH_k^2-2H_k+kH_k^{(2)}+2kH_a^{(2)}}{2k^2}$$

$$\underbrace{-\frac{(kH_a-1)}{k^2}\sum_{i=1}^{k}\frac{1}{a+i}-\frac{1}{k}\sum_{i=1}^{k}\frac{1}{i}\sum_{j=1}^{i}\frac{1}{a+j}}_{\text{Limits}}$$

Simplify:

$$\sum_{k=1}^{\infty}\frac{H_{k+1}-1}{k(k+1)}\sum_{j=1}^{\infty}\frac{H_j}{j(j+k)}.$$

$$\sum_{j=1}^{\infty}\frac{H_j}{j(j+k)}= \qquad \frac{kH_k^2-2H_k+kH_k^{(2)}+2k\zeta(2)}{2k^2}$$

Simplify:

$$\sum_{k=1}^{\infty}\frac{H_{k+1}-1}{k(k+1)}\sum_{j=1}^{\infty}\frac{H_j}{j(j+k)}.$$

$$\sum_{k=1}^{b}\frac{H_{k+1}-1}{k(k+1)}\sum_{j=1}^{\infty}\frac{H_j}{j(j+k)}=\sum_{k=1}^{b}\frac{H_{k+1}-1}{k(k+1)}\frac{kH_k^2-2H_k+kH_k^{(2)}+2k\zeta(2)}{2k^2}$$

$$=$$ Sigma

Simplify:

$$\sum_{k=1}^{\infty}\frac{H_{k+1}-1}{k(k+1)}\sum_{j=1}^{\infty}\frac{H_j}{j(j+k)}.$$

$$\begin{aligned}
\sum_{k=1}^{b}\frac{H_{k+1}-1}{k(k+1)}\sum_{j=1}^{\infty}\frac{H_j}{j(j+k)} &= \sum_{k=1}^{b}\frac{H_{k+1}-1}{k(k+1)}\frac{kH_k^2-2H_k+kH_k^{(2)}+2k\zeta(2)}{2k^2}\\
&= \frac{1}{2(b+1)^2}\Big(6H_b+4bH_b+4H_b^2+3bH_b^2+H_b^3+bH_b^3-6b\zeta(2)\\
&+2H_b\zeta(2)+2bH_b\zeta(2)-2H_b^{(2)}-7bH_b^{(2)}+H_bH_b^{(2)}+bH_bH_b^{(2)}\Big)\\
&-\frac{2b^2}{(b+1)^2}\Big(\zeta(2)+H_b^{(2)}\Big)\\
&+(\zeta(2)-1)\sum_{i=1}^{b}\frac{H_i}{i^2}-\sum_{i=1}^{b}\frac{H_i^2}{i^3}+\frac{1}{2}\sum_{i=1}^{b}\frac{H_i^3}{i^2}+\frac{1}{2}\sum_{i=1}^{b}\frac{H_iH_i^{(2)}}{i^2}
\end{aligned}$$

ζ-relations

This gives

$$S=\sum_{j,k=1}^{\infty}\frac{H_j(H_{k+1}-1)}{jk(k+1)(j+k)}$$

$$=-4\zeta(2)+(\zeta(2)-1)\sum_{i=1}^{\infty}\frac{H_i}{i^2}-\sum_{i=1}^{\infty}\frac{H_i^2}{i^3}+\frac{1}{2}\sum_{i=1}^{\infty}\frac{H_i^3}{i^2}+\frac{1}{2}\sum_{i=1}^{\infty}\frac{H_iH_i^{(2)}}{i^2}.$$

ζ-relations

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$$= -4\zeta(2) + (\zeta(2)-1)\sum_{i=1}^{\infty}\frac{H_i}{i^2} - \sum_{i=1}^{\infty}\frac{H_i^2}{i^3} + \frac{1}{2}\sum_{i=1}^{\infty}\frac{H_i^3}{i^2} + \frac{1}{2}\sum_{i=1}^{\infty}\frac{H_i H_i^{(2)}}{i^2}.$$

E.g., in J.M. Borwein, Girgensohn. Evaluation of triple Euler sums, 1996.

Flajolet, Salvy. Euler sums and contour integral representations, 1998.

Granville. A decomposition of Riemann's zeta-function. 1997.

Zagier. Values of zeta functions and their applications. 1994.

we find

$$\sum_{i=1}^{\infty}\frac{H_i}{i^2} = 2\zeta(3), \qquad \sum_{i=1}^{\infty}\frac{H_i^2}{i^3} = -\zeta(2)\zeta(3) + \frac{7}{2}\zeta(5),$$

$$\sum_{i=1}^{\infty}\frac{H_i^3}{i^2} = \zeta(2)\zeta(3) + 10\zeta(5), \qquad \sum_{i=1}^{\infty}\frac{H_i H_i^{(2)}}{i^2} = \zeta(2)\zeta(3) + \zeta(5).$$

ζ-relations

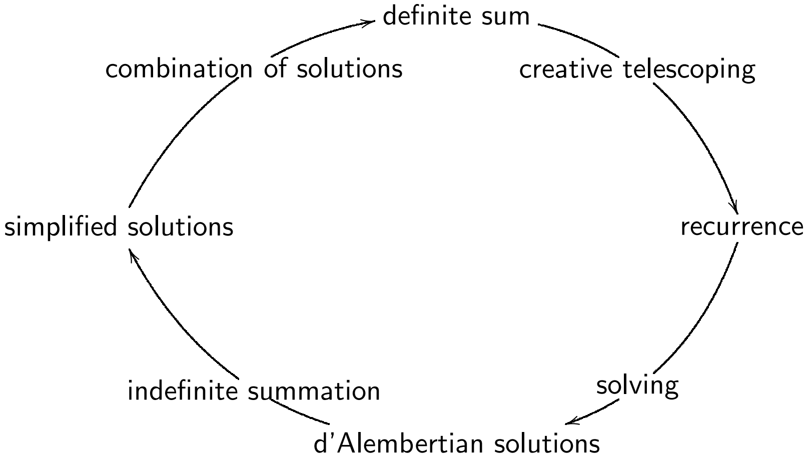
Theorem.

$$S = \sum_{j,k=1}^{\infty} \frac{H_j(H_{k+1}-1)}{jk(k+1)(j+k)}$$
$$= -4\zeta(2) - 2\zeta(3) + 4\zeta(2)\zeta(3) + 2\zeta(5) = 0.999222...$$

Pemantle, CS. When is 0.999... equal to 1? 2007.

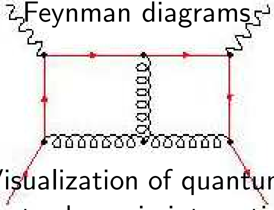
Panholzer, Prodinger. Computer-free evaluation of an infinite double sum. 2005.

The Sigma-summation spiral:



Example 4: Particle physics

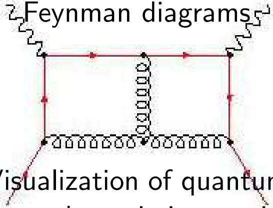
Feynman diagrams



Visualization of quantum electrodynamic interaction

Example 4: Particle physics

Feynman diagrams



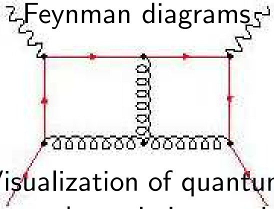
Visualization of quantum electrodynamic interaction

$$\longrightarrow \quad \int d^D p_1 \dots d^D p_n \frac{1}{(p_1^2)^{v_1} \dots (p_n^2)^{v_n}}$$

Task: Evaluation of Feynman integrals

Example 4: Particle physics

Feynman diagrams



Visualization of quantum electrodynamic interaction

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Task: Evaluation of Feynman integrals

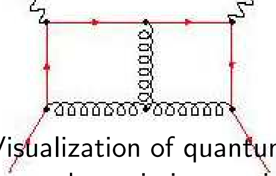
Reduction
(J. Blümlein; DESY)

↓

Feynman sums

Example 4: Particle physics

Feynman diagrams



Visualization of quantum electrodynamic interaction

$$\longrightarrow \int d^D p_1 \dots d^D p_n \frac{1}{(p_1^2)^{v_1} \dots (p_n^2)^{v_n}}$$

Task: Evaluation of Feynman integrals

Reduction (J. Blümlein; DESY) ↓

Feynman sums

Simplification ←

New insight (CERN, SLC)

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006.

GIVEN

$$F(N)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\left(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(\varepsilon)\Gamma(1-\varepsilon)\frac{\Gamma(j+1-2\varepsilon)\Gamma(j+1+\varepsilon)\Gamma(k+j+1+N}{\Gamma(j+1-\varepsilon)\Gamma(j+2+N)\Gamma(k+j+2)} + \frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(-\varepsilon)\Gamma(1+\varepsilon)\frac{\Gamma(j+1+2\varepsilon)\Gamma(j+1-\varepsilon)\Gamma(k+j+1+\varepsilon+N)}{\Gamma(j+1)\Gamma(j+2+\varepsilon+N)\Gamma(k+j+2+\varepsilon)}\right)}_{f(N,k,j)}$$

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006.

GIVEN

$$F(N)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Bigg(\underbrace{\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(\varepsilon)\Gamma(1-\varepsilon)\frac{\Gamma(j+1-2\varepsilon)\Gamma(j+1+\varepsilon)\Gamma(k+j+1+N}{\Gamma(j+1-\varepsilon)\Gamma(j+2+N)\Gamma(k+j+2)} +\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(-\varepsilon)\Gamma(1+\varepsilon)\frac{\Gamma(j+1+2\varepsilon)\Gamma(j+1-\varepsilon)\Gamma(k+j+1+\varepsilon+N)}{\Gamma(j+1)\Gamma(j+2+\varepsilon+N)\Gamma(k+j+2+\varepsilon)}}_{f(N,k,j)}\Bigg)$$

FIND the ϵ-expansion

$$F(N)=F_0(N)+\varepsilon F_1(N)+\dots$$

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006.

GIVEN

$$F(N) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(\varepsilon)\Gamma(1-\varepsilon)\frac{\Gamma(j+1-2\varepsilon)\Gamma(j+1+\varepsilon)\Gamma(k+j+1+N}{\Gamma(j+1-\varepsilon)\Gamma(j+2+N)\Gamma(k+j+2)} + \frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(-\varepsilon)\Gamma(1+\varepsilon)\frac{\Gamma(j+1+2\varepsilon)\Gamma(j+1-\varepsilon)\Gamma(k+j+1+\varepsilon+N)}{\Gamma(j+1)\Gamma(j+2+\varepsilon+N)\Gamma(k+j+2+\varepsilon)}\Bigg)}_{f(N,k,j)}$$

Step 1: FIND the ϵ-expansion

$$f(N,k,j) = \qquad f_0(N,k,j) + \qquad \varepsilon f_1(N,k,j) + \dots$$

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$$F(N)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(\varepsilon)\Gamma(1-\varepsilon)\frac{\Gamma(j+1-2\varepsilon)\Gamma(j+1+\varepsilon)\Gamma(k+j+1+N}{\Gamma(j+1-\varepsilon)\Gamma(j+2+N)\Gamma(k+j+2)} +\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(-\varepsilon)\Gamma(1+\varepsilon)\frac{\Gamma(j+1+2\varepsilon)\Gamma(j+1-\varepsilon)\Gamma(k+j+1+\varepsilon+N)}{\Gamma(j+1)\Gamma(j+2+\varepsilon+N)\Gamma(k+j+2+\varepsilon)}\Bigg)}_{f(N,k,j)}$$

Step 2: Simplify the sums in

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f(N,k,j)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)+\dots$$

$$\|$$

$$F(N)$$

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$$F(N) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(\varepsilon)\Gamma(1-\varepsilon)\frac{\Gamma(j+1-2\varepsilon)\Gamma(j+1+\varepsilon)\Gamma(k+j+1+N}{\Gamma(j+1-\varepsilon)\Gamma(j+2+N)\Gamma(k+j+2)}$$
$$\underbrace{+\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(-\varepsilon)\Gamma(1+\varepsilon)\frac{\Gamma(j+1+2\varepsilon)\Gamma(j+1-\varepsilon)\Gamma(k+j+1+\varepsilon+N)}{\Gamma(j+1)\Gamma(j+2+\varepsilon+N)\Gamma(k+j+2+\varepsilon)}}_{f(N,k,j)}\Bigg)$$

Step 2: Simplify the sums in

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f(N,k,j) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j) + \varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_1(N,k,j) + \dots$$

$$||$$

$$F(N)$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{j=0}^{a} f_0(N,k,j) =$$ Sigma

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{j=0}^{a} f_0(N,k,j) = \frac{(a+1)!(k-1)!(a+k+N+1)!(H_a - H_{a+k} - H_{a+N} + H_{a+k+N})}{N(a+k+1)!(a+N+1)!(k+N+1)!}$$
$$+ \frac{H_k + S_N - H_{k+N}}{kN(k+N+1)N!} + \frac{(2a+k+N+2)a!k!(a+k+N)!}{(a+k+1)(a+N+1)(a+k+1)!(a+N+1)!(k+N+1)!}$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{j=0}^{\infty} f_0(N,k,j) = \frac{H_k + H_N - H_{k+N}}{kN(k+N+1)N!}$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j) = \sum_{k=1}^{\infty}\frac{H_k + H_N - H_{k+N}}{kN(k+N+1)N!}$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j) = \sum_{k=1}^{\infty}\frac{H_k + H_N - H_{k+N}}{kN(k+N+1)N!}$$

$$= \text{Sigma}$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j) = \sum_{k=1}^{\infty}\frac{H_k + H_N - H_{k+N}}{kN(k+N+1)N!}$$

$$= \frac{H_N^2 + H_N^{(2)}}{2N(N+1)!}$$

where

$$H_N^{(2)} = \sum_{i=1}^{N}\frac{1}{i^2}$$

GIVEN

$$F(N) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(\varepsilon)\Gamma(1-\varepsilon)\frac{\Gamma(j+1-2\varepsilon)\Gamma(j+1+\varepsilon)\Gamma(k+j+1+N}{\Gamma(j+1-\varepsilon)\Gamma(j+2+N)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(-\varepsilon)\Gamma(1+\varepsilon)\frac{\Gamma(j+1+2\varepsilon)\Gamma(j+1-\varepsilon)\Gamma(k+j+1+\varepsilon+N)}{\Gamma(j+1)\Gamma(j+2+\varepsilon+N)\Gamma(k+j+2+\varepsilon)}\Bigg)$$

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j) + \varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_1(N,k,j) + \dots,$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j) = \frac{H_N^2+3H_N}{2N(N+1)!}.$$

GIVEN

$$F(N)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(\varepsilon)\Gamma(1-\varepsilon)\frac{\Gamma(j+1-2\varepsilon)\Gamma(j+1+\varepsilon)\Gamma(k+j+1+N}{\Gamma(j+1-\varepsilon)\Gamma(j+2+N)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(-\varepsilon)\Gamma(1+\varepsilon)\frac{\Gamma(j+1+2\varepsilon)\Gamma(j+1-\varepsilon)\Gamma(k+j+1+\varepsilon+N)}{\Gamma(j+1)\Gamma(j+2+\varepsilon+N)\Gamma(k+j+2+\varepsilon)}\Bigg)$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)+\dots,$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)=\frac{H_N^2+3H_N}{2N(N+1)!}.$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)=\frac{-H_N^3-3H_NH_N^{(2)}-8H_N}{6N(N+1)!}.$$