Trisemestre on Methods of Proof Theory in Mathematics (Seminar Talks)

Max Planck Institute for Mathematics, Bonn

Multi-Summation and D-finite sequences

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May 2, 2007

Multi-summation approaches

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Sigma: contains a specialized (but optimized) version of Chyzak's approach

Example 1: Indefinite summation

⏩ Example 2

GIVEN

$$S(n) := \sum_{k=1}^{n} \frac{2k+1}{k+1} \overbrace{\sum_{j=0}^{k} \frac{(-k)_j (k+1)_j (2)_{k-j}}{j!k!} \left(\frac{1-x}{2}\right)^j}^{=: P(k) (= P^{(1,-1)}(x))}.$$

where

$$(k)_j = k(k+1)\dots(k+j-1).$$

FIND a closed form for $S(n)$.

Arose in joint cooperation with the JKU-Finite Element group.

GIVEN

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Intermediate step: FIND a recurrence for $P(k)$.

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$$P(k+2) = \frac{(2k+3)x}{k+2} P(k+1) - \frac{k}{k+1} P(k).$$

FIND a closed form for S(n).

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$$P(k+2) = \frac{(2k+3)x}{k+2} P(k+1) - \frac{k}{k+1} P(k).$$

FIND $g(k)$:

$$\boxed{g(k+1) - g(k) = \frac{2k+1}{k+1} P(k)}$$

for all $1 \leq k \leq n$ and all $n \geq 1$.

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$$P(k+2) = \frac{(2k+3)x}{k+2} P(k+1) - \frac{k}{k+1} P(k).$$

Sigma computes $g(k) = \frac{1+k-x-2kx}{(x-1)(k+1)} P(k) + \frac{1}{x-1} P(k+1)$:

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GIVEN

$$\color{red}{P(k+2)} = \frac{(2k+3)x}{k+2} P(k+1) - \frac{k}{k+1} P(k).$$

Sigma computes $g(k) = \frac{1+k-x-2kx}{(x-1)(k+1)} P(k) + \frac{1}{x-1} P(k+1)$:

$$\boxed{g(k+1) - g(k) = \frac{2k+1}{k+1} P(k)}$$

for all $1 \leq k \leq n$ and all $n \geq 1$.

VERIFICATION:

$$g(k) \xrightarrow{\text{depends on}} P(k), P(k+1)$$

$$g(k+1) \xrightarrow{\text{depends on}} P(k+1), \color{red}{P(k+2).}$$

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$$P(k+2) = \frac{(2k+3)x}{k+2} P(k+1) - \frac{k}{k+1} P(k).$$

Sigma computes $g(k) = \frac{1+k-x-2kx}{(x-1)(k+1)} P(k) + \frac{1}{x-1} P(k+1)$:

$$\boxed{g(k+1) - g(k) = \frac{2k+1}{k+1} P(k)}$$

for all $1 \leq k \leq n$ and all $n \geq 1$.

Summing this equation over k from 1 to n gives:

$$\sum_{k=1}^{n} \frac{2k+1}{k+1} P(k) = -\frac{x+1}{x-1} - \frac{nP(n)}{(n+1)(x-1)} + \frac{P(n+1)}{x-1}.$$

GIVEN

$$P(k+2)=\frac{(2k+3)x}{k+2}P(k+1)-\frac{k}{k+1}P(k).$$

ANSATZ: $g(k)=g_0(k)P(k)+g_1(k)P(k+1)$ such that

$$g(k+1)-g(k)=\frac{2k+1}{k+1}P(k).$$

GIVEN

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ANSATZ: $g(k) = g_0(k)P(k) + g_1(k)P(k+1)$ such that

$$g(k+1) - g(k) = \frac{2k+1}{k+1}P(k).$$

$$\Updownarrow$$

$$\Big[g_0(k+1)P(k+1) + g_1(k+1)P(k+2)\Big] - \Big[g_0(k)P(k) + g_1(k)P(k+1)\Big] = \frac{2k+1}{k+1}P(k)$$

GIVEN

$$\mathsf{P(k{+}2)} = \frac{(2k+3)x}{k+2}P(k+1) - \frac{k}{k+1}P(k).$$

ANSATZ: $g(k) = g_0(k)P(k) + g_1(k)P(k+1)$ such that

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$$\Big[g_0(k+1)P(k+1) + g_1(k+1)\mathsf{P(k{+}2)}\Big] - \Big[g_0(k)P(k) + g_1(k)P(k+1)\Big] = \frac{2k+1}{k+1}P(k)$$

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$$\Updownarrow$$

$$P(k)\big[-\frac{k}{k+1}g_1(k+1) - g_0(k) - \frac{2k+1}{k+1}\big]$$
$$+P(k+1)\big[g_0(k+1) + \frac{(2k+3)x}{k+2}g_1(k+1) - g_1(k)\big] = 0$$

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$$P(k+2)=\frac{(2k+3)x}{k+2}P(k+1)-\frac{k}{k+1}P(k).$$

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$$\Updownarrow$$

$$\Big[g_0(k+1)P(k+1)+g_1(k+1)P(k+2)\Big]-\Big[g_0(k)P(k)+g_1(k)P(k+1)\Big]=\frac{2k+1}{k+1}P(k)$$

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$$P(k)\Big[-\frac{k}{k+1}g_1(k+1)-g_0(k)-\frac{2k+1}{k+1}\Big]$$

$$+P(k+1)\Big[g_0(k+1)+\frac{(2k+3)x}{k+2}g_1(k+1)-g_1(k)\Big]=0$$

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$$P(k+2) = \frac{(2k+3)x}{k+2}P(k+1) - \frac{k}{k+1}P(k).$$

ANSATZ: $g(k) = g_0(k)P(k) + g_1(k)P(k+1)$ such that

$$g(k+1) - g(k) = \frac{2k+1}{k+1}P(k).$$

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$$g_0(k) = -\frac{k}{k+1}g_1(k+1) - \frac{2k+1}{k+1}$$

$$g_0(k+1) + \frac{(2k+3)x}{k+2}g_1(k+1) - g_1(k) = 0$$

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$$\downarrow \mathcal{S}_k g_0(k)$$

$$g_0(k+1) + \frac{(2k+3)x}{k+2}g_1(k+1) - g_1(k) = 0$$

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Sigma computes:

$$g_1(k) = \frac{1}{x-1},$$

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ANSATZ: $g(k) = g_0(k)P(k) + g_1(k)P(k+1)$ such that

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$$\Updownarrow$$

$$g_0(k) = -\frac{k}{k+1} g_1(k+1) - \frac{2k+1}{k+1}$$

$$-\frac{k+1}{k+2} g_1(k+2) + \frac{(2k+3)x}{k+2} g_1(k+1) - g_1(k) = \frac{2k+3}{k+2}$$

Sigma computes:

$$g_1(k) = \frac{1}{x-1}, \qquad g_0(k) = \frac{1+k-x-2kx}{(x-1)(k+1)}.$$

Telescoping

GIVEN $f(k) := h(k)P(k)$

and

$$P(k+s+1) = a_0(k)P(k) + \cdots + a_s(k)P(k+s).$$

FIND $g(k) = g_0(k)P(k) + \cdots + g_s(k)P(k+s)$:

$$g(k+1) - g(k) = f(k).$$

Telescoping

Example 2

GIVEN $f(k) := h(k)P(k)$

and

$$P(k+s+1) = a_0(k)P(k) + \cdots + a_s(k)P(k+s).$$

FIND $g(k) = g_0(k)P(k) + \cdots + g_s(k)P(k+s)$:

$$g(k+1) - g(k) = f(k).$$

1. FIND a solution $g_s(k)$ for

$$\boxed{\sum_{j=0}^{s} a_{s-j}(k+j)g_s(k+j+1) - g_s(k) = h(k+1).}$$

2. COMPUTE the remaining $g_0, \ldots, g_{s-1}$:

$$g_0(k) = a_0(k)g_s(k+1) - h(k),$$
$$g_r(k) = a_r(k)g_s(k+1) + g_{r-1}(k+1), \qquad 0 < r < s.$$

Slightly extended telescoping

Example 2

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and

$$P(k+s+1) = a_0(k)P(k) + \cdots + a_s(k)P(k+s).$$

FIND $g(k) = g_0(k)P(k) + \cdots + g_s(k)P(k+s)$:

$$g(k+1) - g(k) = f(k).$$

1. FIND a solution $g_s(k)$ for

$$\sum_{j=0}^{s} a_{s-j}(k+j)g_s(k+j+1) - g_s(k) = \sum_{j=0}^{s} h_{s-j}(k+j).$$

2. COMPUTE the remaining $g_0, \dots, g_{s-1}$:

$$g_0(k) = a_0(k)g_s(k+1) - h_0(k),$$
$$g_r(k) = a_r(k)g_s(k+1) + g_{r-1}(k+1) - h_r(k), \qquad 0 < r < s.$$

Parameterized telescoping

Example 2

GIVEN $f_0(k) := h_0^{(0)}(k)P(k) + \cdots + h_s^{(0)}(k)P(k+s), \dots,$
$f_d(k) := h_0^{(d)}(k)P(k) + \cdots + h_s^{(d)}(k)P(k+s)$

and

$$P(k+s+1) = a_0(k)P(k) + \cdots + a_s(k)P(k+s).$$

FIND $g(k) = g_0(k)P(k) + \cdots + g_s(k)P(k+s)$ and $c_0, \dots, c_d$:

$$g(k+1) - g(k) = c_0 f_0(k) + \cdots + c_d f(k).$$

1. FIND a solution $g_s(k)$ and $c_0, \dots, c_d$ for

$$\sum_{j=0}^{s} a_{s-j}(k+j)g_s(k+j+1) - g_s(k) = \sum_{i=0}^{d} c_i \sum_{j=0}^{s} h_{s-j}^{(i)}(k+j).$$

2. COMPUTE the remaining $g_0, \dots, g_{s-1}$:

$$g_0(k) = a_0(k)g_s(k+1) - \sum\nolimits_{i=0}^{d} c_i h_0^{(i)}(k),$$

$$g_r(k) = a_r(k)g_s(k+1) + g_{r-1}(k+1) - \sum\nolimits_{i=0}^{d} c_i h_r^{(i)}(k), \qquad 0 < r < s.$$

Example 2: Definite summation

Example 3

Definite summation

Show that

$$\sum_{k=0}^{n}\underbrace{\sum_{s=0}^{n}(-1)^{n+k+s}\binom{n}{k}\binom{n}{s}\binom{n+k}{k}\binom{n+s}{s}\binom{2n-s-k}{n}}_{=P(n,k)}=\sum_{k=0}^{n}\binom{n}{k}^{4}.$$

($A = B$, M. Petkovšek, H.S. Wilf, and D. Zeilberger)

Definite summation

Show that

$$\underbrace{\sum_{k=0}^{n}\sum_{s=0}^{n}(-1)^{n+k+s}\binom{n}{k}\binom{n}{s}\binom{n+k}{k}\binom{n+s}{s}\binom{2n-s-k}{n}}_{=P(n,k)}=\sum_{k=0}^{n}\binom{n}{k}^4.$$

Proof. We COMPUTE for both sides the same recurrence

$$-4(1+n)(3+4n)(5+4n)S(n)\\ -2(3+2n)(7+9n+3n^2)S(1+n)+(2+n)^3S(2+n)=0.$$

Both sides agree at $n=0,1$. □

Definite summation

Show that

$$\sum_{k=0}^{n}\underbrace{\sum_{s=0}^{n}(-1)^{n+k+s}\binom{n}{k}\binom{n}{s}\binom{n+k}{k}\binom{n+s}{s}\binom{2n-s-k}{n}}_{=P(n,k)}=\sum_{k=0}^{n}\binom{n}{k}^4.$$

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$$-4(1+n)(3+4n)(5+4n)S(n)$$
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Both sides agree at $n=0,1$. □

Chyzak's approach (2500s) and Wegschaider's approach (500s) are much slower on that.

telescoping for $S(n)=\sum_{k=0}^{n}P(n,k)$

GIVEN

$$P(n,k+2)=a_0(n,k)P(n,k)+a_1(n,k)P(n,k+1)$$

FIND $g(n,k)=g_0(n,k)P(n,k)+g_1(n,k)P(n,k+1)$:

$$g(n,k+1)-g(n,k)=P(n,k)$$

telescoping for $S(n)=\sum_{k=0}^{n}P(n,k)$

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$$g(n,k+1)-g(n,k)=P(n,k)$$

Sigma:

No solution

Creative telescoping for $S(n) = \sum_{k=0}^{n} P(n,k)$

GIVEN

$$P(n,k+2) = a_0(n,k)P(n,k) + a_1(n,k)P(n,k+1)$$
$$P(n+1,k) = b_0(n,k)P(n,k) + b_1(n,k)P(n,k+1).$$

FIND $g(n,k) = g_0(n,k)P(n,k) + g_1(n,k)P(n,k+1)$, $c_1(n)$:

$$g(n,k+1) - g(n,k) = P(n,k) + c_1(n)P(n+1,k)$$

Sigma:

Creative telescoping for $S(n) = \sum_{k=0}^{n} P(n,k)$

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$$P(n,k+2) = a_0(n,k)P(n,k) + a_1(n,k)P(n,k+1)$$
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FIND $g(n,k) = g_0(n,k)P(n,k) + g_1(n,k)P(n,k+1)$, $c_1(n)$:

$$g(n,k+1) - g(n,k) = P(n,k) + c_1(n)\Big[b_0(n,k)P(n,k) + b_1(n,k)P(n,k+1)\Big]$$

Sigma:

Creative telescoping for $S(n) = \sum_{k=0}^{n} P(n,k)$

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FIND $g(n,k) = g_0(n,k)P(n,k) + g_1(n,k)P(n,k+1)$, $c_1(n)$:

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Sigma:

No solution

Creative telescoping for $S(n) = \sum_{k=0}^{n} P(n,k)$

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$$P(n,k+2) = a_0(n,k)P(n,k) + a_1(n,k)P(n,k+1)$$
$$P(n+1,k) = b_0(n,k)P(n,k) + b_1(n,k)P(n,k+1).$$

FIND $g(n,k) = g_0(n,k)P(n,k) + g_1(n,k)P(n,k+1)$, $c_1(n)$, $c_2(n)$:

$$\begin{aligned} g(n,k+1) - g(n,k) = {} & P(n,k) + \\ & c_1(n)P(n+1,k) + \\ & c_2(n)P(n+2,k) \end{aligned}$$

Sigma:

Creative telescoping for $S(n)=\sum_{k=0}^{n}P(n,k)$

GIVEN

$$P(n,k+2)=a_0(n,k)P(n,k)+a_1(n,k)P(n,k+1)$$
$$P(n+1,k)=b_0(n,k)P(n,k)+b_1(n,k)P(n,k+1).$$

FIND $g(n,k)=g_0(n,k)P(n,k)+g_1(n,k)P(n,k+1)$, $c_1(n)$, $c_2(n)$:

$$\begin{aligned}g(n,k+1)-g(n,k)=P(n,k)+&\\ &c_1(n)\Big[b_0(n,k)P(n,k)+b_1(n,k)P(n,k+1)\Big]+\\ &c_2(n)\Big[\beta_0(n,k)P(n,k)+\beta_1(n,k)P(n,k+1)\Big]\end{aligned}$$

Sigma:

Creative telescoping for $S(n)=\sum_{k=0}^{n}P(n,k)$

GIVEN

$$P(n,k+2)=a_0(n,k)P(n,k)+a_1(n,k)P(n,k+1)$$
$$P(n+1,k)=b_0(n,k)P(n,k)+b_1(n,k)P(n,k+1).$$

FIND $g(n,k)=g_0(n,k)P(n,k)+g_1(n,k)P(n,k+1)$, $c_1(n)$, $c_2(n)$:

$$\begin{aligned}g(n,k+1)-g(n,k)=P(n,k)+&\\ &c_1(n)\Big[b_0(n,k)P(n,k)+b_1(n,k)P(n,k+1)\Big]+\\ &c_2(n)\Big[\beta_0(n,k)P(n,k)+\beta_1(n,k)P(n,k+1)\Big]\end{aligned}$$

Sigma:

Solution

Creative telescoping for $S(n) = \sum_{k=0}^{n} P(n,k)$

GIVEN

$$P(n,k+2) = a_0(n,k)P(n,k) + a_1(n,k)P(n,k+1)$$
$$P(n+1,k) = b_0(n,k)P(n,k) + b_1(n,k)P(n,k+1).$$

FIND $g(n,k) = g_0(n,k)P(n,k) + g_1(n,k)P(n,k+1)$, $c_1(n)$, $c_2(n)$:

$$\begin{aligned} g(n,k+1) - g(n,k) = {} & P(n,k) + \\ & c_1(n)P(n+1,k) + \\ & c_2(n)P(n+2,k) \end{aligned}$$

$$S(n) = \sum_{k=0}^{n} P(n,k)$$

Summing over k gives:

$$0 = S(n) + c_1(n)S(n+1) + c_2(n)S(n+2)$$

Example 3: Stembridge's TSPP Theorem

(G. Andrews, P. Paule, Sigma)

Example 4

Plane partitions

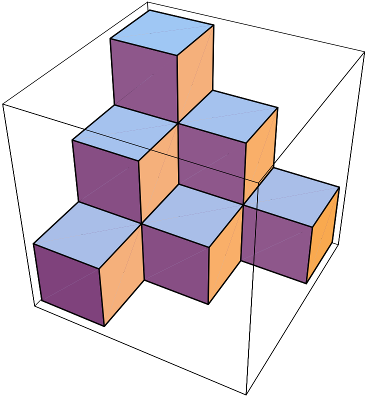
A plane partition with largest part $\leq n$ is

$$
\begin{array}{llllllll}
n \geq & & & & & & & \\
& a_{11} & \geq & a_{12} & \geq & a_{13} & \geq & \dots & a_{1r} \\
& \vee\!| & & \vee\!| & & & & & \vee\!| \\
& a_{21} & \geq & a_{22} & & & & \dots & a_{2r} \\
& \vee\!| & & & & & & & \vee\!| \\
& a_{31} & & & & & & & a_{3r} \\
& \vdots & & & & & & & \vdots \\
& & & & & & & & \vee\!| \\
& a_{s1} & \geq & a_{s2} & \geq & a_{s3} & \geq & \dots \geq & a_{s,r} \\
& & & & & & & & & \geq 0.
\end{array}
$$

Totally Symmetric Plane Partition (TSPP)

Example:

$$\begin{matrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{matrix} \longleftrightarrow$$



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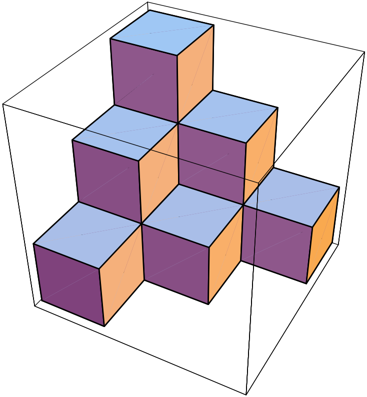

Conjecture (Andrews,Macdonald,Stanley; in 80th). Let T_n be the number of totally symmetric plane partitions with largest part $\leq n$. Then for $n \geq 1$

$$T_n = \prod_{1 \leq i \leq j \leq k \leq n} \frac{i+j+k-1}{i+j+k-2}.$$

Totally Symmetric Plane Partition (TSPP)

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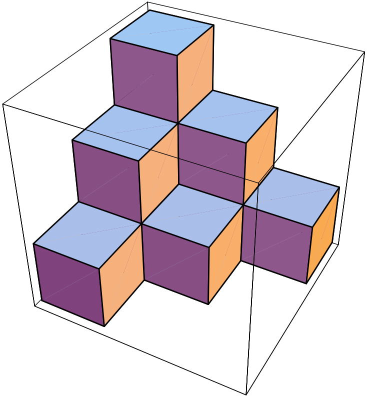

Theorem (Stembridge, 1995). Let T_n be the number of totally symmetric plane partitions with largest part $\leq n$. Then for $n \geq 1$

$$T_n = \prod_{1\leq i\leq j\leq k\leq n} \frac{i+j+k-1}{i+j+k-2}.$$

Okada's Theorem (1989). For $n \geq 3$,

$$T_{n-2}^2 = \begin{cases} \det\big(M(n)\big)x^{-1} & \text{if } n \text{ is odd,} \\ \det\big(M(n)\big) & \text{if } n \text{ is even,} \end{cases}$$

with $M(n) = \big(\mu_1(i,j)\big)_{0 \leq i,j \leq n-1}$

where

$$\mu(i,j) = \begin{cases} 0 & \text{if } j \leq i, \\ 2^{j-1} + (-1)^{j-1} & \text{if } i = 0,\ i < j, \\ (-1)^{j-i-1} + \sum_{s=i}^{j-1} \binom{i+j-2}{s} & \text{if } 0 < i < j, \end{cases}$$

$$\mu_1(i,j) = \begin{cases} x & \text{if } i = j = 0, \\ (-1)^{j-1} & \text{if } i = 0,\ j > 0, \\ (-1)^{i} & \text{if } j = 0,\ i > 0, \\ 0 & \text{if } i = j > 0, \\ \mu(i-1, j-1) & \text{if } j > i \geq 1, \\ -\mu(j-1, i-1) & \text{if } 1 \leq j < i. \end{cases}$$

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PROBLEM: Evaluate $\det(M)$.

Okada's Theorem (1989). For $n \geq 3$,

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with $M(n) = \big(\mu_1(i,j)\big)_{0 \leq i,j \leq n-1}$

G. Andrews guessed (1990)

$$W = \begin{pmatrix} * & \dots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \quad \text{with} \quad \det(W) = 1 \quad MW = \begin{pmatrix} * & & 0 \\ \vdots & \ddots & \\ * & \dots & * \end{pmatrix} =: U.$$

($M = W^{-1}U$ is "LU-decomposition")

Okada's Theorem (1989). For $n \geq 3$,

$$T_{n-2}^2 = \begin{cases} \det\big(M(n)\big)x^{-1} & \text{if } n \text{ is odd,} \\ \det\big(M(n)\big) & \text{if } n \text{ is even,} \end{cases}$$

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G. Andrews guessed (1990)

$$W = \begin{pmatrix} * & \dots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \quad \text{with} \quad \det(W) = 1 \quad MW = \begin{pmatrix} * & & 0 \\ \vdots & \ddots & \\ * & \dots & * \end{pmatrix}.$$

THEN

$$\det(M) = \det(M)\det(W) = \det(M\,W)$$

$$= \text{product of diagonal elements of } M\,W$$

Define

$$\left\{ \begin{matrix} x \\ n \end{matrix} \right\} = \frac{1}{2}\left(\binom{x}{n} + \binom{x-1}{n}\right);$$

$$t_1(n) = \begin{cases} 1 & \text{if } n = 0, \\ \frac{(n+1)(n+3)\dots(3n-1)}{(n)_n} & \text{if } n > 0; \end{cases}$$

$$t(n) = \begin{cases} 1 & \text{if } n = 0, \\ \frac{t_1(n)}{t_1(n-1)} & \text{if } n > 0. \end{cases}$$

$$r_3(s,j) = 4^{-s} \sum_{k=0}^{s} \frac{(j-k)(j)_k(-3j-1)_k}{jk!(-2j+\frac{1}{2})_k},$$

$$f_1(c,j) = (-1)^c \sum_{s=0}^{\lfloor \frac{c}{2} \rfloor} \frac{(-1)^s \binom{j-1-s}{c-2s} (j)_s(-3j+1)_s(3j-3s-1)}{4^s s!(-2j+\frac{3}{2})_s(3j-1)},$$

$$f_2(c,j) = (-1)^c \sum_{s=0}^{\lfloor \frac{c}{2} \rfloor} (-1)^s \left\{ \begin{matrix} j-s \\ c-2s \end{matrix} \right\} r_3(s,j),$$

$$r_2(j) = \begin{cases} \frac{t_1(j-1)}{2} & \text{if } j \text{ even}, \\ \frac{t_1(j-1)}{2} + \frac{f_2(j-2,\frac{j-1}{2})}{2} & \text{if } j \text{ odd}, \end{cases}$$

$$r_1(j) = \begin{cases} t_1(j-1) & \text{if } j \text{ even}, \\ 0 & \text{if } j \text{ odd}, \end{cases}$$

$$e_1(i,j) = \begin{cases} 0 & \text{if } i > j, \\ 1 & \text{if } i = j, \\ r_1(j) & \text{if } i = 0,\ i < j, \\ r_2(j) & \text{if } i = 1,\ i < j, \\ f_1(j-i, \frac{j}{2}) & \text{if } 2 \le i < j,\ j \text{ even}, \\ f_2(j-i, \frac{j-1}{2}) & \text{if } 2 \le i < j,\ j \text{ odd}, \end{cases}$$

$$e(i,j) = \begin{cases} 0 & \text{if } i > j, \\ 1 & \text{if } i = j, \\ e_1(i,j) - t(j-1) \times \\ \quad \times x^{(-1)^j} e_1(i,j-1) & \text{if } i < j, \end{cases}$$

$$W(n) = \big(e(i,j)\big)_{0 \le i,j \le n-1}$$

Conjecture (G. Andrews, 1990) For each $n \ge 1$,

$$M(n)W(n) = \begin{pmatrix} x & 0 & 0 & 0 & \dots & 0 \\ A_{10} & \frac{t_1(0)^2}{x} & 0 & 0 & \dots & 0 \\ A_{20} & A_{21} & t_1(1)^2 x & 0 & \dots & 0 \\ \vdots & & & & \ddots & \vdots \\ A_{n-1,0} & A_{n-1,1} & \dots & \dots & \dots & t_1(n-2)^2 x^{\pm 1} \end{pmatrix}.$$

Define

$$\left\{ {x \atop n} \right\} = \frac{1}{2}\left(\binom{x}{n} + \binom{x-1}{n}\right);$$

$$t_1(n) = \begin{cases} 1 & \text{if } n = 0, \\ \frac{(n+1)(n+3)\dots(3n-1)}{(n)_n} & \text{if } n > 0; \end{cases}$$

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$$f_1(c,j) = (-1)^c \sum_{s=0}^{\lfloor \frac{c}{2} \rfloor} \frac{(-1)^s \binom{j-1-s}{c-2s} (j)_s(-3j+1)_s(3j-3s-1)}{4^s s!(-2j+\frac{3}{2})_s(3j-1)},$$

$$f_2(c,j) = (-1)^c \sum_{s=0}^{\lfloor \frac{c}{2} \rfloor} (-1)^s \left\{ {j-s \atop c-2s} \right\} r_3(s,j),$$

$$r_2(j) = \begin{cases} \frac{t_1(j-1)}{2} & \text{if } j \text{ even}, \\ \frac{t_1(j-1)}{2} + \frac{f_2(j-2,\frac{j-1}{2})}{2} & \text{if } j \text{ odd}, \end{cases}$$

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$$\boxed{W(n) = \big(e(i,j)\big)_{0 \le i,j \le n-1}}$$

Theorem (Andrews, Paule, Sigma, 2005) For each $n \ge 1$,

$$M(n)W(n) = \begin{pmatrix} x & 0 & 0 & 0 & \dots & 0 \\ A_{10} & \frac{t_1(0)^2}{x} & 0 & 0 & \dots & 0 \\ A_{20} & A_{21} & t_1(1)^2 x & 0 & \dots & 0 \\ \vdots & & & & \ddots & \vdots \\ A_{n-1,0} & A_{n-1,1} & \dots & \dots & \dots & t_1(n-2)^2 x^{\pm 1} \end{pmatrix}.$$

Reduction to summation problems (P. Paule):

E.g., define

$$h(k,m) := \sum_{s=0}^{\lfloor \frac{2m-k}{2} \rfloor -1} \frac{k}{m-s} \binom{m-s}{2m-2s-k} \frac{(-1)^{s+k}}{2m\,4^s} \sum_{r=0}^{s} \frac{(m-r)(m)_r(-3m-1)_r}{r!(\frac{1}{2}-2m)_r}$$

and

FIND Rec

$$A_0(i,m) := \sum_{k=0}^{2m} \binom{i+k-3}{i-2} h(k,m),$$

$$A_2(i,m) := \sum_{k=i}^{2m} (-1)^k h(k,m).$$

Theorem. For all $m \geq 1$ and $3 \leq i \leq 2m+1$,

$$2h(i-2,m) - 5h(i-1,m) - A_0(i,m) + 6(-1)^i A_2(i,m) - 3(-1)^i \prod_{s=1}^{2m-1} \frac{2(m+s-1)}{2m+s-2} = 0.$$

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$$2h(i-2,m) - 5h(i-1,m)$$

$$- A_0(i,m) + 6(-1)^i A_2(i,m) - 3(-1)^i \prod_{s=1}^{2m-1} \frac{2(m+s-1)}{2m+s-2} = 0.$$

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Theorem. For all $m \geq 1$ and $3 \leq i \leq 2m+1$,

ADD

$$2h(i-2,m) - 5h(i-1,m) - A_0(i,m) + 6(-1)^i A_2(i,m) - 3(-1)^i \prod_{s=1}^{2m-1} \frac{2(m+s-1)}{2m+s-2} = 0.$$

Zeilberger's Opinion 65:

Seeing all the details, (that nowadays can (and should!) be easily relegated to the computer), even if they are extremely hairy, is a hang-up that traditional mathematicians should learn to wean themselves from. A case in point is the excellent but unnecessarily long-winded recent article

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You don't need 30 pages, and frankly all this EXPLICIT LANGUAGE of hairy computer output is almost pornographic.

Example 4: Handbook of Mathematical Functions (Abramowitz and Stegun)

National Institute of Standards and Technology "NIST"
Digital Library of Mathematical Functions (DLMF)

Example 5

From: Frank W. J. Olver [University of Maryland]
To: Peter Paule

"...The writing of DLMF Chapter Bessel Functions by Leonard Maximon and myself is now largely complete;...
However, a problem has arisen in connection with about a dozen formulas from Chapter 10 of Abramowitz and Stegun for which we have not yet tracked down proofs,
and the author of this chapter, Henry Antosiewiecz, died about a year ago.

Since it is the editorial policy for the DLMF not to state formulas without indications of proofs,
I am hoping that you will be willing to step into the breach and supply verifications by computer algebra methods..."

My favorite of the dozen problems

Verify with computer algebra the identity

$$J_0(z\sin\theta)=\sum_{n=0}^{\infty}(4n+1)\frac{(2n)!}{2^{2n}n!^2}j_{2n}(z)P_{2n}(\cos\theta) \qquad (10.1.48)$$

where

$P_n(z)$: Legendre polynomials,
$J_n(z)$: Bessel functions of the first kind,
$j_n(z)$: spherical Bessel functions of the first kind.

My favorite of the dozen problems

Verify with computer algebra the identity

$$J_0(z\sin\theta) = \sum_{n=0}^{\infty}(4n+1)\frac{(2n)!}{2^{2n}n!^2}j_{2n}(z)P_{2n}(\cos\theta) \qquad (10.1.48)$$

$$t := \cos\theta \qquad \sin(\theta) = \sqrt{1-t^2}$$

where

$P_n(z)$: Legendre polynomials,
$J_n(z)$: Bessel functions of the first kind,
$j_n(z)$: spherical Bessel functions of the first kind.

My favorite of the dozen problems

Verify with computer algebra the identity

$$J_0(z\sqrt{1-t^2}) = \sum_{n=0}^{\infty}(4n+1)\frac{(2n)!}{2^{2n}n!^2}j_{2n}(z)P_{2n}(t)$$

where

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Verify with computer algebra the identity

$$J_0(z\sqrt{1-t^2}) = \sum_{n=0}^{\infty}(4n+1)\frac{(2n)!}{2^{2n}n!^2}j_{2n}(z)P_{2n}(t)$$

By hypergeometric series representation we get

$$J_0(z\sqrt{1-t^2}) = \sum_{k=0}^{\infty}\frac{(-\frac{1}{2}z^2)^k}{k!^2}\left(\frac{1-t^2}{2}\right)^k,$$

$$P_{2n}(t) = \sum_{k=0}^{\infty}\frac{(-2n)_k(2n+1)_k}{k!^2}\left(\frac{1-t}{2}\right)^k,$$

$$j_{2n}(z) = \sum_{i=0}^{\infty}\frac{\sqrt{\pi}}{2}\frac{(-z^2/4)^i(z/2)^{2n}}{i!(2n+i+1/2)!}.$$

Show that

$$\sum_{k=0}^{\infty} a_k \left(\frac{1-t^2}{2}\right)^k = \sum_{k=0}^{\infty} b_k \left(\frac{1-t}{2}\right)^k$$

where

$$a_k = \frac{(-\frac{1}{2}z^2)^k}{k!^2},$$

$$b_k = \sum_{n=0}^{\infty}(4n+1)\frac{(2n)!}{2^{2n}n!}\frac{(-2n)_k(2n+1)_k}{k!^2}\overbrace{\sum_{i=0}^{\infty}\frac{\sqrt{\pi}}{2}\frac{(-z^2/4)^i(z/2)^{2n}}{i!(2n+i+1/2)!}}^{=:\,P(n)}.$$

Show that

$$\sum_{k=0}^{\infty} a_k \left(\frac{1-t^2}{2}\right)^k = \sum_{k=0}^{\infty} b_k \left(\frac{1-t}{2}\right)^k$$

where

$$a_k = \frac{(-\frac{1}{2}z^2)^k}{k!^2},$$

$$b_k = \sum_{n=0}^{\infty}(4n+1)\frac{(2n)!}{2^{2n}n!}\frac{(-2n)_k(2n+1)_k}{k!^2}\overbrace{\sum_{i=0}^{\infty}\frac{\sqrt{\pi}}{2}\frac{(-z^2/4)^i(z/2)^{2n}}{i!(2n+i+1/2)!}}^{=:\,P(n)}.$$

FIND recurrences for a_k and b_k.

Show that

$$\sum_{k=0}^{\infty} a_k \left(\frac{1-t^2}{2}\right)^k = \sum_{k=0}^{\infty} b_k \left(\frac{1-t}{2}\right)^k$$

where

$$z^2 a_k + 2(k+1)^3 a_{k+1} = 0$$

and

$$\begin{aligned}4(k+3)z^2 b_k &- 2(2k+5)z^2 b_{k+1}\\ &- (k+2)(k^2+5k+6-z^2)b_{k+2} + (k+2)(k+3)^2 b_{k+3} = 0.\end{aligned}$$

Show that

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where

$$z^2 a_k + 2(k+1)^3 a_{k+1} = 0$$

and

$$4(k+3)z^2 b_k - 2(2k+5)z^2 b_{k+1} \\ -(k+2)(k^2+5k+6-z^2)b_{k+2} + (k+2)(k+3)^2 b_{k+3} = 0.$$

FIND ODEs for

$$A(t) = \sum_{k=0}^{\infty} a_k t^k \quad \text{and} \quad B(t) = \sum_{k=0}^{\infty} b_k t^k.$$

Show that

$$\sum_{k=0}^{\infty} a_k \left(\frac{1-t^2}{2}\right)^k = \sum_{k=0}^{\infty} b_k \left(\frac{1-t}{2}\right)^k$$

We know that

$$A(t) = \sum_{k=0}^{\infty} a_k t^k \quad \text{and} \quad B(t) = \sum_{k=0}^{\infty} b_k t^k :$$

are solutions of

$$z^2 A(t) + 2A'(t) + 2tA''(t) = 0$$

and

$$\begin{aligned} &12z^2 B(t) + 10(2tz^2 - z^2)B'(t) \\ &+ (4t^2z^2 - 4tz^2 + z^2 - 6)B''(t) - 3(2t-1)B^{(3)}(t) - t(t-1)B^{(4)}(t) = 0. \end{aligned}$$

Show that

$$\overbrace{\sum_{k=0}^{\infty} a_k \left(\frac{1-t^2}{2}\right)^k}^{=A(\frac{1-t^2}{2})} = \overbrace{\sum_{k=0}^{\infty} b_k \left(\frac{1-t}{2}\right)^k}^{=B(\frac{1-t}{2})}$$

We know that SUBSTITUTE

$$A(t) = \sum_{k=0}^{\infty} a_k t^k \quad \text{and} \quad B(t) = \sum_{k=0}^{\infty} b_k t^k :$$

are solutions of

$$z^2 A(t) + 2A'(t) + 2tA''(t) = 0$$

and

$$12z^2 B(t) + 10(2tz^2 - z^2)B'(t)$$
$$+ (4t^2z^2 - 4tz^2 + z^2 - 6)B''(t) - 3(2t-1)B^{(3)}(t) - t(t-1)B^{(4)}(t) = 0.$$

Show that

$$\overbrace{\sum_{k=0}^{\infty} a_k \left(\frac{1-t^2}{2}\right)^k}^{=:A(t)} = \overbrace{\sum_{k=0}^{\infty} b_k \left(\frac{1-t}{2}\right)^k}^{=:B(t)}$$

We know that $A(t)$ and $B(t)$ are solutions of

$$-t^3 z^2 A(t) + (t^2+1)A'(t) + t(t+1)(t-1)A''(t) = 0$$

and

$$-3z^2 B(t) - 5tz^2 B'(t) - (t^2 z^2 - 6)B''(t) + 6tB^{(3)}(t) + (t+1)(t-1)B^{(4)}(t) = 0.$$

ADD

Show that

$$\overbrace{\sum_{k=0}^{\infty} a_k \left(\frac{1-t^2}{2}\right)^k}^{=:A(t)} = \overbrace{\sum_{k=0}^{\infty} b_k \left(\frac{1-t}{2}\right)^k}^{=:B(t)}$$

We know that

$$C(t) := A(t) - B(t)$$

is a solution of

$$-3z^2C(t)-5tz^2C'(t)-(t^2z^2-6)C''(t)+6tC^{(3)}(t)+(t+1)(t-1)C^{(4)}(t)=0.$$

Show that

$$\overbrace{\sum_{k=0}^{\infty} a_k \left(\frac{1-t^2}{2}\right)^k}^{=:A(t)} = \overbrace{\sum_{k=0}^{\infty} b_k \left(\frac{1-t}{2}\right)^k}^{=:B(t)}$$

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By inspection we get the initial conditions

$$a_0 = b_0 \qquad \boxed{b_1 = 2a_1}, \qquad b_2 = -2a_1 + 4a_2.$$

Proving them shows that

$$C(t) = 0.$$

Show that

$$\underbrace{\sum_{n=0}^{\infty} -(4n+1)n\frac{(2n+1)!}{n!^2 2^{2n-1}} \overbrace{\sum_{i=0}^{\infty} \frac{\sqrt{\pi}}{2} \frac{(-z^2/4)^i (z/2)^{2n}}{i!(2n+i+1/2)!}}^{P(n)}}_{=:\,S(z)} = -z^2.$$

- FIND an ODE.

Show that

$$\underbrace{\sum_{n=0}^{\infty} -(4n+1)n\frac{(2n+1)!}{n!^2 2^{2n-1}} \overbrace{\sum_{i=0}^{\infty} \frac{\sqrt{\pi}}{2} \frac{(-z^2/4)^i (z/2)^{2n}}{i!(2n+i+1/2)!}}^{P(n)}}_{=:\,S(z)} = -z^2.$$

- We get

$$2S(z) - zS'(z) = 0.$$

Show that

$$\underbrace{\sum_{n=0}^{\infty} -(4n+1)n\frac{(2n+1)!}{n!^2 2^{2n-1}} \overbrace{\sum_{i=0}^{\infty} \frac{\sqrt{\pi}}{2} \frac{(-z^2/4)^i (z/2)^{2n}}{i!(2n+i+1/2)!}}^{P(n)}}_{=:\,S(z)} = -z^2.$$

- We get
$$2S(z) - zS'(z) = 0.$$
- Hence $S(z) = c\,z^2$ for some constant c.

Show that

$$\underbrace{\sum_{n=0}^{\infty} -(4n+1)n\frac{(2n+1)!}{n!^2 2^{2n-1}} \overbrace{\sum_{i=0}^{\infty} \frac{\sqrt{\pi}}{2} \frac{(-z^2/4)^i (z/2)^{2n}}{i!(2n+i+1/2)!}}^{P(n)}}_{=:\, S(z)} = -z^2.$$

- We get

$$2S(z) - zS'(z) = 0.$$

- Hence $S(z) = c\,z^2$ for some constant c.
- By inspection we get $S(z) = -z^2$. □

Creative telescoping for $S(z) = \sum_{n=0}^{\infty} P(z,n)$

GIVEN

$$P(z,n+2) = -\frac{4n+7}{4n+3}P(z,n) + \frac{(4n+5)(16n^2+40n-2z^2+21)}{(4n+3)z^2}P(z,n+1)$$

$$\frac{d}{dz}P(z,n) = \frac{6n+8n^2-z^2}{4n+3}P(z,n) - \frac{z^2}{4n+3}P(z,n+1).$$

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FIND $g(z,n) = g_0(z,n)P(z,n) + g_1(z,n)P(z,n+1)$ and $c_1(z)$:

$$g(z,n+1) - g(z,n) = P(z,n) + c_1(z)\frac{d}{dz}P(z,n)$$

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Sigma: $c_1=-\frac{z}{2}$, $g(z,n)=$

$$\frac{(2n)!}{n!^2 2^{2n+1}}\frac{4(n-1)n(2n+1)}{4n+3}\big[(z^2-16n^2-16n-3)P(z,n)+z^2P(z,n+1)\big].$$

Creative telescoping for $S(z)=\sum_{n=0}^{\infty}P(z,n)$

GIVEN

$$P(z,n+2)=-\frac{4n+7}{4n+3}P(z,n)+\frac{(4n+5)(16n^2+40n-2z^2+21)}{(4n+3)z^2}P(z,n+1)$$

$$\frac{d}{dz}P(z,n)=\frac{6n+8n^2-z^2}{4n+3}P(z,n)-\frac{z^2}{4n+3}P(z,n+1).$$

FIND $g(z,n)=g_0(z,n)P(z,n)+g_1(z,n)P(z,n+1)$ and $c_1(z)$:

$$g(z,n+1)-g(z,n)=P(z,n)+ c_1(z)\frac{d}{dz}P(z,n)$$

Summing over k gives

$$0=S(z)-\frac{z}{2}S'(z).$$

Example 5: Summation with unspecified/generic sequences

(joint work with M. Kauers)

Identity 1

$$\sum_{k=0}^{n} kk! = (n+1)! - 1$$

$$\sum_{k=0}^{n} \frac{1-q^k}{1-q} \prod_{i=1}^{k} \frac{1-q^i}{1-q} = \frac{1}{q}\Big(\prod_{i=1}^{n+1} \frac{1-q^i}{1-q} - 1\Big)$$

$$\sum_{k=0}^{n} \frac{kH_k + H_k - k}{k+1} \prod_{i=1}^{k} H_i = \prod_{i=1}^{n+1} H_i - 1$$

Identity 1

$$\sum_{k=0}^{n} k k! = (n+1)! - 1 \qquad X_k := k$$

$$\sum_{k=0}^{n} \frac{1-q^k}{1-q} \prod_{i=1}^{k} \frac{1-q^i}{1-q} = \frac{1}{q}\Big(\prod_{i=1}^{n+1} \frac{1-q^i}{1-q} - 1\Big) \qquad X_k := \frac{1-q^k}{1-q}$$

$$\sum_{k=0}^{n} \frac{kH_k + H_k - k}{k+1} \prod_{i=1}^{k} H_i = \prod_{i=1}^{n+1} H_i - 1 \qquad X_k := H_k$$

$$\boxed{\sum_{k=0}^{n} (X_{k+1} - 1) \prod_{i=1}^{k} X_i = \prod_{k=1}^{n+1} X_k - 1}$$

Sigma

Identity 1

Graham, Knuth, Patashnik. *Concrete Mathematics*, Bonus Problem 5.93:

$$\sum_{k=1}^{n} \frac{1}{X_k} \prod_{i=1}^{k} \frac{X_i}{a+X_i} = \frac{1}{a}\Big(1 - \prod_{k=1}^{n} \frac{X_k}{a+X_k}\Big)$$

van der Poorten. A proof that Euler missed...Apéry's proof of the irrationality of $\zeta(3)$, 1979:

$$\sum_{k=0}^{n} \frac{\prod_{j=1}^{k} \frac{(X_j+\alpha)}{X_j}}{X_j+\alpha} = \frac{1}{\alpha}\Big(\prod_{j=1}^{k} \frac{\prod_{j=1}^{n}(X_j+\alpha)}{X_j} - \frac{X_0}{X_0+\alpha}\Big)$$

$$\boxed{\sum_{k=0}^{n}(X_{k+1}-1)\prod_{i=1}^{k} X_i = \prod_{k=1}^{n+1} X_k - 1}$$

Sigma

Identity 2

$$\sum_{k=1}^{n} k^2 \sum_{i=1}^{k} X_i =$$ Sigma

Identity 2

$$\sum_{k=1}^{n} k^2 \sum_{i=1}^{k} X_i = \frac{1}{6}\Big(n(n+1)(2n+1)\sum_{k=1}^{n} X_k - \sum_{k=1}^{n} k X_k + 3\sum_{k=1}^{n} k^2 X_k - 2\sum_{k=1}^{n} k^3 X_k\Big)$$

Identity 2

$$\sum_{k=1}^{n} k^2 \sum_{i=1}^{k} X_i = \tfrac{1}{6}\Big(n(n{+}1)(2n{+}1)\sum_{k=1}^{n} X_k - \sum_{k=1}^{n} k X_k + 3\sum_{k=1}^{n} k^2 X_k - 2\sum_{k=1}^{n} k^3 X_k\Big)$$

$$\downarrow X_k = \frac{1}{n+k}$$

$$\sum_{k=1}^{n} k^2 \sum_{i=1}^{k} \frac{1}{n+i} = \frac{1}{36} n(n+1)(1 - 10n + 12(2n+1)\sum_{k=1}^{n} \frac{1}{n+k})$$

Identity 2

$$\sum_{k=1}^{n} k^2 \sum_{i=1}^{k} X_i = \tfrac{1}{6}\Big(n(n+1)(2n+1)\sum_{k=1}^{n} X_k - \sum_{k=1}^{n} k X_k + 3\sum_{k=1}^{n} k^2 X_k - 2\sum_{k=1}^{n} k^3 X_k\Big)$$

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$$\sum_{k=1}^{n} k^2 \sum_{i=1}^{k} \frac{1}{n+i} = \frac{1}{36} n(n+1)(1-10n+12(2n+1)\sum_{k=1}^{n}\frac{1}{n+k})$$

$$\downarrow H_{n+k} = H_n + \sum_{i=1}^{k}\frac{1}{n+i}$$

$$\sum_{k=1}^{n} k^2 H_{n+k} = \tfrac{1}{3} n(n+\tfrac{1}{2})(n+1)(2H_{2n}-H_n) - \tfrac{1}{36}(10n^2+9n-1)$$

(The warmup example)

Identity 3

$$\sum_{k=1}^{n} a^k \sum_{j=1}^{k} X_j = \text{Sigma}$$

Identity 3

$$\sum_{k=1}^{n} a^k \sum_{j=1}^{k} X_j = \frac{a^{n+1} \sum_{k=1}^{n} X_k - \sum_{k=1}^{n} a^k X_k}{a-1}, \quad a \neq 1$$

Identity 3

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- $X_j := \frac{1}{j}$:

$$\sum_{k=1}^{n} a^k H_k = \frac{1}{a-1}\left[a^{n+1} H_n - \sum_{k=1}^{n} \frac{a^k}{k}\right].$$

Identity 3

$$\sum_{k=1}^{n} a^k \sum_{j=1}^{k} X_j = \frac{a^{n+1}\sum_{k=1}^{n} X_k - \sum_{k=1}^{n} a^k X_k}{a-1}, \quad a \neq 1$$

- $X_j := \frac{1}{j}$:

$$\sum_{k=1}^{n} a^k H_k = \frac{1}{a-1}\Big[a^{n+1} H_n - \sum_{k=1}^{n} \frac{a^k}{k}\Big].$$

- $X_j = \binom{m}{j-1}$, $a=-1$, and $n := m+1$:

$$\sum_{k=0}^{m} (-1)^{k+1} \sum_{j=0}^{k} \binom{m}{j} = \frac{1}{2}(-1)^{m+1} 2^m.$$

Z. Zhang. A kind of binomial identity. *Discrete Math.*, 1999.

a=1

$$\sum_{k=1}^{n}\sum_{j=1}^{k} X_j = \text{Sigma}$$

a=1

$$\sum_{k=1}^{n}\sum_{j=1}^{k} X_j = (n+1)\sum_{k=1}^{n} X_k - \sum_{k=1}^{n} k X_k$$

- $X_j := \frac{1}{j^2}$:

$$\sum_{k=1}^{n} H_k^{(2)} = (n+1)H_n^{(2)} - H_k.$$

a=1

$$\sum_{k=1}^{n}\sum_{j=1}^{k}X_j=(n+1)\sum_{k=1}^{n}X_k-\sum_{k=1}^{n}kX_k$$

- $X_j=\binom{m}{j}$:

$$\sum_{k=0}^{n}\sum_{j=0}^{k}\tbinom{m}{j}=(n+1)\sum_{k=0}^{n}\binom{m}{k}-\sum_{k=0}^{n}k\binom{m}{k}$$

a=1

$$\sum_{k=1}^{n}\sum_{j=1}^{k}X_j=(n+1)\sum_{k=1}^{n}X_k-\sum_{k=1}^{n}kX_k$$

- $X_j=\binom{m}{j}$:

$$\sum_{k=0}^{n}\sum_{j=0}^{k}\tbinom{m}{j}=(n+1)\sum_{k=0}^{n}\binom{m}{k}-\sum_{k=0}^{n}k\binom{m}{k}$$
$$=\frac{1}{2}(m-n)\binom{m}{n}+(2n-m+2)\sum_{i=0}^{n}\binom{m}{i}$$

a=1

$$\sum_{k=1}^{n}\sum_{j=1}^{k} X_j = (n{+}1)\sum_{k=1}^{n} X_k - \sum_{k=1}^{n} k X_k$$

- $X_j = \binom{m}{j}$:

$$\begin{aligned}\sum_{k=0}^{n}\sum_{j=0}^{k}\tbinom{m}{j} &= (n+1)\sum_{k=0}^{n}\binom{m}{k} - \sum_{k=0}^{n} k\binom{m}{k}\\ &= \frac{1}{2}(m-n)\binom{m}{n} + (2n-m+2)\sum_{i=0}^{n}\binom{m}{i}\\ &\overset{(m=n)}{=} \frac{m+2}{2}2^m\end{aligned}$$

Andrews, Paule. MacMahon's Partition Analysis IV: Hypergeometric Multisums. SLC'99

X-Collection

$$\sum_{k=1}^{a}(-1)^k\Big(\sum_{j=1}^{k}X_j-\frac{X_k}{2}\Big)^2=\frac{1}{2}(-1)^a\Big(\sum_{k=1}^{a}X_k\Big)^2-\frac{1}{4}\sum_{k=1}^{a}(-1)^kX_k^2,$$

$$\sum_{k=1}^{a}\Big(\sum_{j=1}^{k}X_j+X_k(k-1)\Big)^2=n\Big(\sum_{k=1}^{a}X_k\Big)^2-\sum_{k=1}^{a}kX_k^2+\sum_{k=1}^{a}k^2X_k^2\Big),$$

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}X_{n-i}\right)^2=2\sum_{k=0}^{n}X_k\sum_{j=0}^{k}jX_{j-1}+\sum_{k=0}^{n}X_k^2+\sum_{k=0}^{n}kX_k^2,$$

$$\sum_{k=1}^{a}(-1)^k\binom{n}{k}\sum_{j=1}^{k}X_j=\frac{1}{n}\Big[(n-a)\binom{n}{a}(-1)^a\sum_{k=1}^{a}X_k+\sum_{k=1}^{a}(-1)^kk\binom{n}{k}X_k\Big].$$

Radical expressions

$$X_k^2 = k$$

$$\sum_{k=0}^{n} \frac{1}{\sqrt{k+1}+\sqrt{k}} = \sqrt{n+1},$$

$$\sum_{k=0}^{n} (k-\sqrt{k}+1)\sqrt{k!} = (n+1)\sqrt{n!},$$

$$\sum_{k=0}^{n} ((k-\sqrt{k}+1)H_k+1)\sqrt{k!} = (1+(n+1)H_n)\sqrt{n!},$$

$$\sum_{k=0}^{n} \frac{n-\sqrt{k}\sqrt{k+1}-k}{\sqrt{k+1}}\binom{n}{k} = 0,$$

$$\sum_{k=1}^{n} \left(2k+2k^2+k^3-\sqrt{k^2+1}\right)(k-1)!\prod_{i=1}^{k}\sqrt{i^2+1}$$

$$= (2+2n+n^2)n!\prod_{k=1}^{n}\sqrt{k^2+1}-2,$$