ACAT 2007, Nikhef, Amsterdam

The summation package Sigma evaluates Feynman integrals

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I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

GIVEN

$$F(N)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Bigg(\underbrace{\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(\varepsilon)\Gamma(1-\varepsilon)\frac{\Gamma(j+1-2\varepsilon)\Gamma(j+1+\varepsilon)\Gamma(k+j+1+N)}{\Gamma(j+1-\varepsilon)\Gamma(j+2+N)\Gamma(k+j+2)} +\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(-\varepsilon)\Gamma(1+\varepsilon)\frac{\Gamma(j+1+2\varepsilon)\Gamma(j+1-\varepsilon)\Gamma(k+j+1+\varepsilon+N)}{\Gamma(j+1)\Gamma(j+2+\varepsilon+N)\Gamma(k+j+2+\varepsilon)}}_{f(N,k,j)}\Bigg).$$

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

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$$F(N)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\left(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(\varepsilon)\Gamma(1-\varepsilon)\frac{\Gamma(j+1-2\varepsilon)\Gamma(j+1+\varepsilon)\Gamma(k+j+1+N)}{\Gamma(j+1-\varepsilon)\Gamma(j+2+N)\Gamma(k+j+2)}\right.
+\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(-\varepsilon)\Gamma(1+\varepsilon)\frac{\Gamma(j+1+2\varepsilon)\Gamma(j+1-\varepsilon)\Gamma(k+j+1+\varepsilon+N)}{\Gamma(j+1)\Gamma(j+2+\varepsilon+N)\Gamma(k+j+2+\varepsilon)}\left.\right)}_{f(N,k,j)}.$$

FIND the ϵ-expansion

$$F(N)=F_0(N)+\varepsilon F_1(N)+\dots$$

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GIVEN

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FIND the ϵ-expansion

$$F(N)=F_0(N)+\varepsilon F_1(N)+\dots$$

with Sigma: Brand-new algorithms (1999-2007) based on Karr's $\Pi\Sigma^*$-difference fields (1981)

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

GIVEN

$$F(N)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(\varepsilon)\Gamma(1-\varepsilon)\frac{\Gamma(j+1-2\varepsilon)\Gamma(j+1+\varepsilon)\Gamma(k+j+1+N)}{\Gamma(j+1-\varepsilon)\Gamma(j+2+N)\Gamma(k+j+2)} + \frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(-\varepsilon)\Gamma(1+\varepsilon)\frac{\Gamma(j+1+2\varepsilon)\Gamma(j+1-\varepsilon)\Gamma(k+j+1+\varepsilon+N)}{\Gamma(j+1)\Gamma(j+2+\varepsilon+N)\Gamma(k+j+2+\varepsilon)}\Bigg)}_{f(N,k,j)}.$$

Step 1: FIND the ϵ-expansion

$$f(N,k,j) = \qquad f_0(N,k,j) + \qquad f_1(N,k,j)+\dots$$

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

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$$F(N)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(\varepsilon)\Gamma(1-\varepsilon)\frac{\Gamma(j+1-2\varepsilon)\Gamma(j+1+\varepsilon)\Gamma(k+j+1+N)}{\Gamma(j+1-\varepsilon)\Gamma(j+2+N)\Gamma(k+j+2)} +\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(-\varepsilon)\Gamma(1+\varepsilon)\frac{\Gamma(j+1+2\varepsilon)\Gamma(j+1-\varepsilon)\Gamma(k+j+1+\varepsilon+N)}{\Gamma(j+1)\Gamma(j+2+\varepsilon+N)\Gamma(k+j+2+\varepsilon)}\Bigg)}_{f(N,k,j)}.$$

Step 2: Simplify the sums in

$$\underbrace{\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f(N,k,j)}_{=\,F(N)}=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)+\dots$$

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

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$$F(N)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(\varepsilon)\Gamma(1-\varepsilon)\frac{\Gamma(j+1-2\varepsilon)\Gamma(j+1+\varepsilon)\Gamma(k+j+1+N)}{\Gamma(j+1-\varepsilon)\Gamma(j+2+N)\Gamma(k+j+2)} +\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(-\varepsilon)\Gamma(1+\varepsilon)\frac{\Gamma(j+1+2\varepsilon)\Gamma(j+1-\varepsilon)\Gamma(k+j+1+\varepsilon+N)}{\Gamma(j+1)\Gamma(j+2+\varepsilon+N)\Gamma(k+j+2+\varepsilon)}\Bigg)}_{f(N,k,j)}.$$

Step 2: Simplify the sums in

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f(N,k,j)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)+\dots$$

$$||$$

$$F(N)$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{j=0}^{a} f_0(N,k,j) =$$ Sigma

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{j=0}^{a} f_0(N,k,j) = \frac{(a+1)!(k-1)!(a+k+N+1)!(S_1(a)-S_1(a+k)-S_1(a+N)+S_1(a+k+N))}{N(a+k+1)!(a+N+1)!(k+N+1)!}$$
$$+\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}+\frac{(2a+k+N+2)a!k!(a+k+N)!}{(a+k+1)(a+N+1)(a+k+1)!(a+N+1)!(k+N+1)!}$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{j=0}^{\infty} f_0(N,k,j) = \frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j) = \sum_{k=1}^{\infty}\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j) = \sum_{k=1}^{\infty}\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

$$=$$ Sigma

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j) = \sum_{k=1}^{\infty}\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

$$= \frac{S_1(N)^2+S_2(N)}{2N(N+1)!}$$

where

$$S_2(N)=\sum_{i=1}^{N}\frac{1}{i^2}$$

GIVEN

$$F(N)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(\varepsilon)\Gamma(1-\varepsilon)\frac{\Gamma(j+1-2\varepsilon)\Gamma(j+1+\varepsilon)\Gamma(k+j+1+N)}{\Gamma(j+1-\varepsilon)\Gamma(j+2+N)\Gamma(k+j+2)}$$
$$+\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(-\varepsilon)\Gamma(1+\varepsilon)\frac{\Gamma(j+1+2\varepsilon)\Gamma(j+1-\varepsilon)\Gamma(k+j+1+\varepsilon+N)}{\Gamma(j+1)\Gamma(j+2+\varepsilon+N)\Gamma(k+j+2+\varepsilon)}\Bigg).$$
$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)+\dots,$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)=\frac{S_1(N)^2+3S_1(N)}{2N(N+1)!}.$$

GIVEN

$$F(N)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(\varepsilon)\Gamma(1-\varepsilon)\frac{\Gamma(j+1-2\varepsilon)\Gamma(j+1+\varepsilon)\Gamma(k+j+1+N)}{\Gamma(j+1-\varepsilon)\Gamma(j+2+N)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(-\varepsilon)\Gamma(1+\varepsilon)\frac{\Gamma(j+1+2\varepsilon)\Gamma(j+1-\varepsilon)\Gamma(k+j+1+\varepsilon+N)}{\Gamma(j+1)\Gamma(j+2+\varepsilon+N)\Gamma(k+j+2+\varepsilon)}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)+\dots,$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)=\frac{S_1(N)^2+3S_1(N)}{2N(N+1)!}.$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)=\frac{-S_1(N)^3-3S_2(N)S_1(N)-8S_3(N)}{6N(N+1)!}.$$

Diff

A. Mitov, S. Moch; *QCD corrections to semi-inclusive hadron production in electron-positron annihilation at two loops*. 2006

$$\sum_{i=1}^{N}\Bigg(\frac{-40(357\zeta(2)+58\zeta(3)-1584)i^4-8(336\zeta(2)+29\zeta(3)-2936)i^3+(5920-210\zeta(2))i^2+916i+65}{128i^5(2i+1)^5}$$
$$+\frac{-256(63\zeta(2)+29\zeta(3)-98)i^2-64(630\zeta(2)+203\zeta(3)-1154)i-32(1113\zeta(2)+261\zeta(3)-2894)}{128(2i+1)^5}$$
$$+\frac{(96i^3+144i^2+44i+5)}{8i^3(2i+1)^3}\big(S_2(2i)-\tfrac{1}{2}S_2(i)\big)+\frac{(4i+1)}{i^2(2i+1)^2}\big(S_3(2i)-\tfrac{3}{8}S_3(i)\big)\Bigg)$$

A. Mitov, S. Moch; *QCD corrections to semi-inclusive hadron production in electron-positron annihilation at two loops*. 2006

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$\downarrow$ Sigma

$$-\frac{(-5732+735z2+116z3)-28(-1048+168z2+29z3)-4(-15658+2793z2+522z3)}{4(2N+1)^5}$$
$$-\frac{20(-796+147z2+29z3)+8(-796+147z2+29z3)}{(2N+1)^5}-\frac{63}{4}S_4(2N)-\frac{63}{4}S_5(2N)$$
$$+\frac{\big(12(7\zeta(2)-22)N^2+12(7\zeta(2)-22)N+21\zeta(2)-50\big)}{8(2N+1)^2}S_3(2N)+24S_{-5}(2N)+\frac{91}{2}S_{-4}(2N)$$
$$-\frac{3\big(12(7\zeta(2)-22)N^2+12(7\zeta(2)-22)N+21\zeta(2)-74\big)}{4(2N+1)^2}S_{-3}(2N)$$
$$-S_{-2}(2N)\frac{\big(8(147\zeta(2)+58\zeta(3)-262)N^3+12(147\zeta(2)+58\zeta(3)-262)N^2+2(441\zeta(2)+174\zeta(3)-842)N+147\zeta(2)-}{8(2N+1)^3}$$
$$-\frac{7}{2}S_{-2}(2N)^2-2\left(S_3(2N)+3S_{-3}(2N)\right)S_{-2}(2N)+\frac{3}{8}\left(\sum_{i=1}^{N}\frac{S_{-2,2}(i)}{i}+8\sum_{i=1}^{N}\frac{S_{-2,2}(i)}{(2i+1)^3}\right);$$

In[1]:= $<<$ **Sigma.m**

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Back

In[1]:= $<<$ **Sigma.m**

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In[2]:= **summand =**

$$\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(\varepsilon)\Gamma(1-\varepsilon)\frac{\Gamma(j+1-2\varepsilon)\Gamma(j+1+\varepsilon)\Gamma(k+j+1+N)}{\Gamma(j+1-\varepsilon)\Gamma(j+2+N)\Gamma(k+j+2)}$$
$$+\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(-\varepsilon)\Gamma(1+\varepsilon)\frac{\Gamma(j+1+2\varepsilon)\Gamma(j+1-\varepsilon)\Gamma(k+j+1+\varepsilon+N)}{\Gamma(j+1)\Gamma(j+2+\varepsilon+N)\Gamma(k+j+2+\varepsilon)};$$

Back

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$$+\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(-\varepsilon)\Gamma(1+\varepsilon)\frac{\Gamma(j+1+2\varepsilon)\Gamma(j+1-\varepsilon)\Gamma(k+j+1+\varepsilon+N)}{\Gamma(j+1)\Gamma(j+2+\varepsilon+N)\Gamma(k+j+2+\varepsilon)};$$

In[3]:= **summand0 = SeriesForProduct[summand, e, 0, 0, j][[1]]**

Back

In[1]:= $<<$ **Sigma.m**

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In[2]:= **summand =**

$$\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(\varepsilon)\Gamma(1-\varepsilon)\frac{\Gamma(j+1-2\varepsilon)\Gamma(j+1+\varepsilon)\Gamma(k+j+1+N)}{\Gamma(j+1-\varepsilon)\Gamma(j+2+N)\Gamma(k+j+2)} + \frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(-\varepsilon)\Gamma(1+\varepsilon)\frac{\Gamma(j+1+2\varepsilon)\Gamma(j+1-\varepsilon)\Gamma(k+j+1+\varepsilon+N)}{\Gamma(j+1)\Gamma(j+2+\varepsilon+N)\Gamma(k+j+2+\varepsilon)};$$

In[3]:= **summand0 = SeriesForProduct[summand, e, 0, 0, j][[1]]**

3.865 Second

Out[3]=

$$\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\left(-S_1[j]+S_1[j+k]+S_1[j+N]-S_1[j+k+N]\right)}{(j+k+1)!(j+N+1)!(k+N+1)!}$$

Back

In[1]:= $<<$ **Sigma.m**

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In[2]:= $$\mathbf{summand} = \frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(\varepsilon)\Gamma(1-\varepsilon)\frac{\Gamma(j+1-2\varepsilon)\Gamma(j+1+\varepsilon)\Gamma(k+j+1+N)}{\Gamma(j+1-\varepsilon)\Gamma(j+2+N)\Gamma(k+j+2)} + \frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(-\varepsilon)\Gamma(1+\varepsilon)\frac{\Gamma(j+1+2\varepsilon)\Gamma(j+1-\varepsilon)\Gamma(k+j+1+\varepsilon+N)}{\Gamma(j+1)\Gamma(j+2+\varepsilon+N)\Gamma(k+j+2+\varepsilon)};$$

In[3]:= $\mathbf{summand0 = SeriesForProduct[summand, e, 0, 0, j][[1]]}$

3.865 Second

Out[3]= $$\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\,(-S_1[j]+S_1[j+k]+S_1[j+N]-S_1[j+k+N])}{(j+k+1)!(j+N+1)!(k+N+1)!}$$

In[4]:= $\mathbf{S_1[j]//GetDefinition}$

Out[4]= $$\sum_{i=1}^{j}\frac{1}{i}$$

Back

In[5]:= $\mathbf{innerSum = SigmaSum[summand0, \{j, 0, a\}];}$

Back

In[5]:= **innerSum = SigmaSum[summand0, {j, 0, a}];**

Out[5]= $$\sum_{j=0}^{a}\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\,(-S_1[j]+S_1[j+k]+S_1[j+N]-S_1[j+k+N])}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)$$

Back

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Out[5]= $$\sum_{j=0}^{a}\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\,(-S_1[j]+S_1[j+k]+S_1[j+N]-S_1[j+k+N])}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)$$

In[6]:= $\mathbf{SigmaReduce[innerSum]}$

Back

In[5]:= **innerSum = SigmaSum[summand0, {j, 0, a}];**

Out[5]=
$$\sum_{j=0}^{a}\left(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!}+\frac{j!k!(j+k+N)!\,(-S_1[j]+S_1[j+k]+S_1[j+N]-S_1[j+k+N])}{(j+k+1)!(j+N+1)!(k+N+1)!}\right)$$

In[6]:= **SigmaReduce[innerSum]**

6.47 Second

Out[6]=
$$\frac{(a+1)!(k-1)!(a+k+N+1)!\,(S_1[a]-S_1[a+k]-S_1[a+N]+S_1[a+k+N])}{N(a+k+1)!(a+N+1)!(k+N+1)!}+\frac{S_1[k]+S_1[N]-S_1[k+N]}{kN(k+N+1)N!}+\frac{(2a+k+N+2)a!k!(a+k+N)!}{(a+k+1)(a+N+1)(a+k+1)!(a+N+1)!(k+N+1)!}$$

Back

In[7]:= $$\mathbf{doubleSum} = \sum_{k=1}^{a} \frac{S_1[k] + S_1[N] - S_1[k+N]}{kN(k+N+1)N!};$$

Back

In[7]:= $$\mathbf{doubleSum} = \sum_{k=1}^{a} \frac{S_1[k] + S_1[N] - S_1[k+N]}{kN(k+N+1)N!};$$

In[8]:= $$\mathbf{rec} = \mathbf{GenerateRecurrence}[\mathbf{doubleSum}, N][[1]]$$

Back

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In[8]:= $\mathbf{rec} = \mathbf{GenerateRecurrence}[\mathbf{doubleSum}, N][[1]]$

1.422 Second

Out[8]= $$-N\,SUM[N] + (1+N)(2+N)SUM[1+N] == \frac{(a+1)\left(S_1[a] + S_1[N] - S_1[a+N]\right)}{(N+1)^2(a+N+2)N!} + \frac{a(a+1)}{(N+1)^3(a+N+1)(a+N+2)N!}$$

Back

In[7]:= $\mathbf{doubleSum} = \sum_{k=1}^{\infty} \frac{S_1[k] + S_1[N] - S_1[k+N]}{kN(k+N+1)N!};$

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In[9]:= $\mathbf{rec} = \mathbf{LimitRec[rec, SUM[N], a]}$

Back

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In[9]:= $\mathbf{rec} = \mathbf{LimitRec}[\mathbf{rec}, SUM[N], a]$

0.962 Second

Out[9]= $-N\,SUM[N] + (1+N)(2+N)SUM[1+N] == \frac{(N+1)S_1[N] + 1}{(N+1)^3 N!}$

Back

In[7]:= $\mathbf{doubleSum} = \sum_{k=1}^{\infty} \frac{S_1[k] + S_1[N] - S_1[k+N]}{kN(k+N+1)N!};$

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In[10]:= $\mathbf{recSol} = \mathbf{SolveRecurrence[rec, SUM[N]]}$

Back

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In[10]:= $\mathbf{recSol} = \mathbf{SolveRecurrence}[\mathbf{rec}, SUM[N]]$

0.761 Second

Out[10]= $\{\{0, \frac{1}{N(N+1)N!}\}, \{1, \frac{S_1[N]^2 + \sum_{i=1}^{N} \frac{1}{i^2}}{2N(N+1)N!}\}\}$

Back

Using the integral representation of harmonic sums, differentiate!

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Out[12]= $-\frac{S(1,N)^3}{6N(N+1)!} - \frac{\zeta(2)S(1,N)}{2N(N+1)!} - \frac{4\zeta(3)}{3N(N+1)!} + \frac{S(1,N)D_N^{(1)}[S(1,N)]}{2N(N+1)!} - \frac{2D_N^{(2)}[S(1,N)]}{3N(N+1)!}$

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