Parameterized telescoping proves algebraic independence of sums

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Abstract: Usually creative telescoping is used to derive recurrences for sums. In this article we show that the non-existence of a creative telescoping solution, and more generally, of a parameterized telescoping solution, proves algebraic independence of certain types of sums. Combining this fact with summation-theory shows transcendence of whole classes of sums.

Parameterized telescoping

Given $f_1(k),\dots,f_d(k)$ over a field[a] $\mathbb{K}$;
find constants $c_1,\dots,c_d\in\mathbb{K}$ and $g(k)$ such that

$$g(k+1)-g(k)=c_1f_1(k)+\dots+c_df_d(k). \qquad (1)$$

If one succeeds, one gets the sum-relation

$$g(n+1)-g(0)=c_1\sum_{k=0}^{n}f_1(k)+\dots+c_d\sum_{k=0}^{n}f_d(k).$$

Telescoping: Restrict to $d=1$.
Zeilberger's creative telescoping: Take a bivariate sequence $f(m,k)$ and set $f_i(k):=f(m+i-1,k)$ in (1).

In the summation package Sigma [Sch07b] parameterized telescoping can be solved in Karr's $\Pi\Sigma^*$-fields: the $f_i(k)$ can be indefinite nested sums and products.

a All rings and fields contain $\mathbb{Q}$.

$\Pi\Sigma^*$-extensions and sequences

Example. Let $\mathbb{F}:=\mathbb{Q}(m)(k)(h)(b)$ be a rational function field and define the field automorphism σ by

$$\sigma(c)=c\ \ \forall c\in\mathbb{Q}(m),$$
$$\sigma(k)=k+1,$$
$$\sigma(h)=h+\frac{1}{k+1}, \qquad H_{k+1}=H_k+\frac{1}{k+1},$$
$$\sigma(b)=\frac{m-k}{k+1}b, \qquad \binom{m}{k+1}=\frac{m-k}{k+1}\binom{m}{k}.$$

$(\mathbb{F},\sigma)$ is a difference field, more precisely, a $\Pi\Sigma^*$-field.

Difference rings and fields: A *difference field* $(\mathbb{F},\sigma)$ is a ring (resp. field) $\mathbb{F}$ together with a ring (resp. field) automorphism $\sigma:\mathbb{F}\to\mathbb{F}$; the *constant ring* (resp. *constant field*) is given by $\text{const}_\sigma\mathbb{F}:=\{f\in\mathbb{F}|\sigma(f)=f\}$.

$\Pi\Sigma^*$-field: A difference field $(\mathbb{F}(t),\sigma')$ is a *Σ^*-extension (resp. Π-extension)* of a difference field $(\mathbb{F},\sigma)$:⇔

1. t is transcendental over $\mathbb{F}$,
2. $\sigma'(f)=\sigma(f)$ for all $f\in\mathbb{F}$,
3. $\sigma'(t)=t+f$ (resp. $\sigma'(t)=f\,t$) for some $f\in\mathbb{F}^*$,
4. the constant field remains unchanged, i.e., $\text{const}_{\sigma'}\mathbb{F}(t)=\text{const}_\sigma\mathbb{F}$.

A *$\Pi\Sigma^*$-extension* is either a Π- or a Σ^*-extension. $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ is a $\Pi\Sigma^*$-extension (resp. Σ^*-extension or Π-extension) if it is a tower of such extensions. $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ is a $\Pi\Sigma^*$-field over $\mathbb{F}$, if $\mathbb{F}=\text{const}_\sigma\mathbb{F}$.

Example. Each of the extensions k,h,b forms a $\Pi\Sigma^*$-extension over the field below. In particular, $\text{const}_\sigma\mathbb{Q}(m)(k)(h)(b)=\mathbb{Q}(m)$.

Ring of sequences: The set of sequences over a field $\mathbb{K}$ is denoted by

$$S(\mathbb{K}):=\{(a_n)_{n\geq0}|a_i\in\mathbb{K}\};$$

we identity two sequences if they agree from a certain point on. The difference ring $(S(\mathbb{K}),S)$ with the shift operation (ring automorphism)

$$S:\langle a_0,a_1,a_2,\dots\rangle\mapsto\langle a_1,a_2,a_3,\dots\rangle$$

is called the *ring of sequences*.

Goal: Embed, e.g., $\mathbb{Q}(m)(k)[h,b]$ into $(S(\mathbb{Q}(m)),S)$.

Embedding example

We construct step by an embedding $(\mathbb{Q}(m)(k)[h][b],\sigma)$ into $(S(\mathbb{Q}(m)),S)$.

- Start with $\tau_0:\mathbb{Q}(m)\to S(\mathbb{Q}(m))$ where

$$\tau_0(c)=\langle c,c,c,\dots\rangle \qquad \forall c\in\mathbb{Q}(m).$$

- Next, define the ring homomorphism $\tau_1:\mathbb{Q}(m)(k)\to S(\mathbb{Q}(m))$ with $\tau_1(\frac{p}{q})=\langle F(k)\rangle_{k\geq0}$ where

$$F(k)=\begin{cases}0 & q(k)=0\\ \frac{p(k)}{q(k)} & q(k)\neq0.\end{cases}$$

Note that

$$\tau_1(\sigma(f))=S(\tau_1(f)),\quad \forall f\in\mathbb{Q}(m)(k).$$

τ_1 is injective: Since $p(k),q(k)$ have only finitely many roots, $\tau_1(\frac{p}{q})=0$ if and only if $\frac{p(k)}{q(k)}=0$. Hence τ_1 is injective.

- Define the ring homomorphism $\tau_2:\mathbb{Q}(m)(k)[h]\to S(\mathbb{Q}(m))$ with $\tau_2(h)=\langle H_n\rangle_{n\geq0}$ and

$$\tau_2(\sum_{i=0}^{d}f_ih^i)=\sum_{i=0}^{d}\tau_1(f_i)\tau_2(h)^i.$$

τ_2 is injective: If not, take $f=\sum_{i=0}^{d}f_ih^i\in\mathbb{Q}(m)(k)[h]^*$ with $\deg(f)=d$ minimal such that $\tau_2(f)=0$. Note that $f\notin\mathbb{Q}(m)(k)$ (otherwise, $0=\tau_2(f)=\tau_1(f)$; since τ_1 is injective, $f=0$). Define

$$g:=\sigma(f_d)f-f_d\sigma(f)\in\mathbb{Q}(m)(k)[h].$$

Note: $\deg(g)<d$ by construction. Moreover,

$$\tau_2(g)=\tau_1(\sigma(f_n))\underbrace{\tau_2(f)}_{=0}-\tau_1(f_n)\underbrace{\tau_2(\sigma(f))}_{S(\tau_2(f))=S(0)=0}=0.$$

By the minimality of $\deg(f)$, $g=0$, i.e.,

$$\sigma(f_d)f-f_d\sigma(f)=0.$$

Equivalently,

$$\frac{\sigma(f)}{f}=\frac{\sigma(f_d)}{f_d}\in\mathbb{Q}(m)(k).$$

With $f\notin\mathbb{Q}(m)(k)$ this contradicts to [Kar81].

- To this end, take the ring homomorphism $\tau_3:\mathbb{Q}(m)(k)[h][b]\to S(\mathbb{Q}(m))$ with $\tau_3(b)=\langle\binom{m}{n}\rangle_{n\geq0}$ and

$$\tau_3(\sum_{i=0}^{d}f_ib^i)=\sum_{i=0}^{d}\tau_2(f_i)\tau_3(b)^i.$$

By similar arguments, τ_3 is injective.

Result 1: The embedding into the ring of sequences

A *generalized d'Alembertian extension* $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ is a $\Pi\Sigma^*$-extension such that for all $1\leq i\leq e$,

$$\sigma(t_i)-t_i\in\mathbb{F}[t_1,\dots,t_{i-1}]\quad\text{or}\quad\sigma(t_i)/t_i\in\mathbb{F};$$

note that the t_i are transcendental and $\mathbb{K}:=\text{const}_\sigma\mathbb{F}(t_1)\dots(t_e)=\text{const}_\sigma\mathbb{F}$.

Embedding: Suppose that $(\mathbb{F},\sigma)$ describes the rational case ($\mathbb{F}=\mathbb{K}(k)$ with $\sigma(k)=k+1$), the q-rational case or the mixed case.
Then there is an injective ring homomorphism

$$\tau:\mathbb{F}[t_1\dots,t_e]\to S(\mathbb{K})$$

with

$$\tau(c)=\langle c,c,c,\dots\rangle\quad\forall c\in\mathbb{K}$$

such that the following diagram commutes:

$$\begin{array}{ccc}\mathbb{F}[t_1\dots,t_e] & \xrightarrow{\ \tau\ } & S(\mathbb{K})\\ \downarrow\sigma & & \downarrow S\\ \mathbb{F}[t_1\dots,t_e] & \xrightarrow{\ \tau\ } & S(\mathbb{K})\end{array}$$

We call such an embedding a *$\mathbb{K}$-monomorphism*.

Consequence:

$$\mathbb{F}[t_1,\dots,t_e]\cong\tau(\mathbb{F})[\tau(t_1),\dots,\tau(t_e)].$$

In particular, the sequences $\tau(t_1),\dots,\tau(t_e)$ are algebraically independent over $\tau(\mathbb{F})$.

Result 2: Parameterized telescoping and $\Pi\Sigma^*$-extension

Let $(\mathbb{F},\sigma)$ be a difference field with constants $\mathbb{K}$ and $f_1,\dots,f_d\in\mathbb{F}^*$. Then:

There are no $0\neq(c_1,\dots,c_d)\in\mathbb{K}^d$ and $g\in\mathbb{F}$ with

$$\sigma(g)-g=c_1f_1+\dots+c_df_d. \qquad (2)$$

⇕

There is a Σ^*-extension $(\mathbb{F}(t_1)\dots(t_d),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(t_i)=t_i+f_i$ for $1\leq i\leq d$.

There are no $0\neq(c_1,\dots,c_d)\in\mathbb{Z}^d$ and $g\in\mathbb{F}^*$ with

$$\frac{\sigma(g)}{g}=f_1^{c_1}\cdots f_d^{c_d}. \qquad (3)$$

⇕

There is a Π-extension $(\mathbb{F}(t_1,\dots,t_d),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(t_i)=f_it_i$ for $1\leq i\leq d$.

Remark. The case $d=1$ (telescoping) has been worked out in [Kar81].
As a consequence, one can check with telescoping that, e.g., the difference field $(\mathbb{Q}(m)(k)(h)(b),\sigma)$ is a $\Pi\Sigma^*$-field.

Result 3: A criterion for algebraic independence of sums and products

Combining the ideas from Result 1 and Result 2 gives the following main result:

Let $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ be a generalized d'Alembertian-extension of $(\mathbb{F},\sigma)$ with $\mathbb{K}:=\text{const}_\sigma\mathbb{F}$.
Let $\tau:\mathbb{F}[t_1,\dots,t_e]\to S(\mathbb{K})$ be a $\mathbb{K}$-monomorphism.

Let $f_1,\dots,f_d\in\mathbb{F}[t_1,\dots,t_e]^*$ with $\langle F_i(k)\rangle_{k\geq0}:=\tau(f_i)$. Then:

There are no $g\in\mathbb{F}[t_1,\dots,t_e]$, $0\neq(c_1,\dots,c_d)\in\mathbb{K}^d$ with (2).

⇕

The sequences $\{(S_1(n))_{n\geq0},\dots,(S_d(n))_{n\geq0}\}$ given by[a]

$$S_1(n):=\sum_{k=r}^{n}F_1(k),\dots,S_d(n):=\sum_{k=r}^{n}F_d(k) \qquad (4)$$

are algebraically independent over $\tau(\mathbb{F}[t_1,\dots,t_e])$.

Let $f_1,\dots,f_d\in\mathbb{F}^*$ with $\langle F_i(k)\rangle_{k\geq0}:=\tau(f_i)$. Then:

There are no $g\in\mathbb{F}^*$ and $0\neq(c_1,\dots,c_d)\in\mathbb{Z}^d$ with (3).

⇕

The sequences $\{(S_1(n))_{n\geq0},\dots,(S_d(n))_{n\geq0}\}$ given by

$$S_1(n):=\prod_{k=r}^{n}F_1(k),\dots,S_d(n):=\prod_{k=r}^{n}F_d(k) \qquad (5)$$

are algebraically independent over $\tau(\mathbb{F})$.

a The lower bound r is chosen big enough.

The (q–)rational case

Theorem. Let $f_1(k),\dots,f_d(k)\in\mathbb{K}(k)$. If there are no $g(k)\in\mathbb{K}(k)$ and $c_1,\dots,c_d\in\mathbb{K}$ with (1), then the sequences (4) are algebraically independent over $\mathbb{K}(n)$; i.e., $\nexists P(x_1,\dots,x_d)\in\mathbb{K}(n)[x_1,\dots,x_d]^*$ with

$$P(S_1(n),\dots,S_d(n))=0\quad\forall n\geq0.$$

Corollary. Let $p_1(k),p_2(k),\dots\in\mathbb{K}[k]^*$, $u_1(k),u_2(k),\dots\in\mathbb{K}[k]^*$ and $v\in\mathbb{K}[k]$ with $\deg(v)>0$ where

$$\gcd(p_i,v)=\gcd(u_i,v)=1\quad\forall i\geq1,$$
$$\gcd(v(k),v(k+r))=1\quad\forall r\in\mathbb{N}^*,$$
$$v(r)\neq0\quad\forall r\in\mathbb{N}^*.$$

Then the sums

$$S_1(n):=\sum_{k=1}^{n}u_1(k)\left(\frac{p_1(k)}{v(k)}\right),S_2(n):=\sum_{k=1}^{n}u_2(k)\left(\frac{p_2(k)}{v(k)}\right)^2,\dots$$

are algebraically independent over $\mathbb{K}(n)$.

Proof. Denote $f_i(k):=u_i(\frac{p_i}{v})^i$. Suppose there are $g(k)\in\mathbb{K}(k)$ and $c_i\in\mathbb{K}$ with (1) where $d\geq1$ is minimal. Then

$$g(k+1)-g(k)=\frac{c_1u_1p_1v^{d-1}+c_2u_2p_2^2v^{d-2}+\dots+c_du_dp_d^d}{v^d}=:\frac{w}{v^d}.$$

Since $c_d\neq0$, $\gcd(v,c_du_dp_d^d)=1$. Hence $\gcd(w,v)=1$, and thus $\gcd(w,v^d)=1$. By [Abr71, Pau95] such a $g(k)\in\mathbb{K}(k)$ cannot exist; a contradiction. □

Example. Choosing $p_i=u_i=1,v=k$ in the Corollary proves that the generalized harmonic numbers $\{H_n^{(i)}|i\geq1\}$ are algebraically independent over $\mathbb{K}(n)$.

Example. Similarly, the q-harmonic numbers $\{\sum_{k=1}^{n}\frac{1}{(1-q^k)^i}|i\geq1\}$ (or $\{\sum_{k=1}^{n}(\frac{q^k}{1-q^k})^i|i\geq1\}$) are algebraically independent over $\mathbb{K}(q^k)$.

The (q–)hypergeometric case

Let $f(k)$ be hypergeometric, i.e., for all r big enough,

$$\alpha(r)=\frac{f(r+1)}{f(r)}$$

for some $\alpha(k)\in\mathbb{K}(k)$.
We restrict to the case that there are no $v(k)\in\mathbb{K}(k)$ and no root of unity $\gamma\in\mathbb{K}$ with $f(k)=\gamma^kv(k)$. Then there is a $\Pi\Sigma^*$-field $(\mathbb{K}(k)(t),\sigma)$ over $\mathbb{K}$ with

$$\sigma(k)=k+1\quad\text{and}\quad\sigma(t)=\alpha t,$$

and a $\mathbb{K}$-monomorphism $\tau:\mathbb{K}(k)[t]\to S(\mathbb{K})$.

Theorem. For $1\leq i\leq d$, let $v_i\in\mathbb{K}(k)$ and $f_i:=v_it$. If there are no $c_i\in\mathbb{K}$ and $w\in\mathbb{K}(k)$ with $g:=w\,t$ such that (2), then the sequences

$$f(n),\ S_1(n)=\sum_{k=r}^{n}v_1(k)f(k),\dots,S_d(n)=\sum_{k=r}^{n}v_d(k)f(k)$$

(r big enough) are algebraically independent over $\mathbb{K}(n)$; i.e., $\nexists P(x_0,x_1,\dots,x_d)\in\mathbb{K}(n)[x_0,x_1,\dots,x_d]^*$ with

$$P(f(n),S_1(n),\dots,S_d(n))=0\quad\forall n\geq0.$$

Corollary Let $f(m,k)$ be hypergeometric in m,k where $f\neq\gamma^kv$ for all $v\in\mathbb{K}(m,k)$ and all roots of unity $\gamma\in\mathbb{K}\setminus\{1\}$. If **Z's algorithm** [Zei91] fails to solve (1) with $f_i:=f(m+i-1,k)$, then $S_0(n)=f(m,n)$ and

$$S_1(n)=\sum_{k=r}^{n}f(m,k),\dots,S_d(n)=\sum_{k=r}^{n}f(m+d-1,k)$$

(r big enough) are transcendental over $\mathbb{K}(m)(n)$.

Example. For the Apéry-sum

$$A(m)=\sum_{k=0}^{m}\binom{m}{k}^2\binom{m+k}{k}$$

Zeilberger's algorithm finds a recurrence of order 2, but not a smaller one. Hence, the sequences

$$S_0(n)=\binom{m}{n}^2\binom{m+n}{n},$$
$$S_1(n)=\sum_{k=0}^{n}\binom{m}{k}^2\binom{m+k}{k},$$
$$S_2(n)=\sum_{k=0}^{n}\binom{m+1}{k}^2\binom{m+k+1}{k}$$

are algebraically independent over $\mathbb{Q}(m)(n)$.

Example. In [Abr03] a criterion is given when Zeilberger's algorithm fails to find a creative telescoping solution for a hypergeometric input summand $f(m,k)$. If $f(m,k)$ satisfies this criterion, then all the sequences

$$\{f(m,n)\}\cup\{\sum_{k=r}^{n}f(m+i,k)|i\geq0\}$$

in n (r big enough) are algebraically independent over $\mathbb{K}(m)$. A typical example [Abr03, Exp. 2] is

$$f(m,k)=\frac{1}{mk+1}(-1)^k\binom{m+1}{k}\binom{2m-2k-1}{m-1}.$$

Remark. Analogously, all ideas can be carried over to the q–hypergeometric case.

The nested sum case

Consider the following sum from [PS03]:

$$S(m,n):=\sum_{k=0}^{n}f(m,k)=\sum_{k=0}^{n}(1+5H_k(m-2k))\binom{m}{k}^5.$$

Then the package Sigma shows that the sequences

$$\binom{m}{n},\ H_n,\ S(m,n),\ S(m+1,n),\ S(m+2,n),\ S(m+3,n)$$

in n are algebraically independent over $\mathbb{K}(m)(n)$.

Proof. Sigma constructs the $\Pi\Sigma^*$-field $(\mathbb{Q}(m)(k)(h)(b),\sigma)$ and designs the $\mathbb{Q}(m)$-monomorphism from above.
Then it sets

$$f_1=b^5(1+5h(m-2k)),$$
$$f_2=\frac{b^5(m+1)^5(5h(-2k+m+1)+1)}{(-k+m+1)^5},$$
$$f_3=\frac{b^5(m+1)^5(m+2)^5(5h(-2k+m+2)+1)}{(-k+m+1)^5(-k+m+2)^5},$$
$$f_4=\frac{b^5(m+1)^5(m+2)^5(m+3)^5(5h(-2k+m+3)+1)}{(-k+m+1)^5(-k+m+2)^5(-k+m+3)^5};$$

this is motivated by $\binom{m+1}{k}=\frac{m+1}{m+1-k}\binom{m}{k}$ which shows that

$$\tau(f_i)=\langle f(m+i-1,k)\rangle_{k\geq0}.$$

Finally, Sigma proves algorithmically that there are no $g\in\mathbb{Q}(m)(k)(h)(h)$ and $c_i\in\mathbb{Q}(m)$ with (2). Hence the transcendence of the sequences follows by Result 3. □

Remark. Note that the situation $m=n$ for the sum $S(m,n)$ is completely different. We get

$$S(m,m)=\sum_{k=0}^{m}\binom{m}{k}^2\binom{m+k}{k}.$$

In particular, $S(m,m)$ satisfies a recurrence relation of minimal order 2.

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