Parameterized telescoping proves algebraic independence of sums

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Abstract: Usually creative telescoping is used to derive recurrences for sums. In this article we show that the non-existence of a creative telescoping solution, and more generally, of a parameterized telescoping solution, proves algebraic independence of certain types of sums. Combining this fact with summation-theory shows transcendence of whole classes of sums.

Parameterized telescoping

Given $f_1(k), \dots, f_d(k)$ over a field[a] $\mathbb{K}$;
find constants $c_1, \dots, c_d \in \mathbb{K}$ and $g(k)$ such that

$$g(k+1) - g(k) = c_1 f_1(k) + \dots + c_d f_d(k). \qquad (1)$$

If one succeeds, one gets the sum-relation

$$g(n+1) - g(0) = c_1 \sum_{k=0}^{n} f_1(k) + \dots + c_d \sum_{k=0}^{n} f_d(k).$$

Telescoping: Restrict to $d = 1$.

Zeilberger's creative telescoping: Take a bivariate sequence $f(m,k)$ and set $f_i(k) := f(m+i-1,k)$ in (1).

In the summation package Sigma [Sch07b] parameterized telescoping can be solved in Karr's $\Pi\Sigma^*$-fields: the $f_i(k)$ can be indefinite nested sums and products.

a All rings and fields contain $\mathbb{Q}$.

$\Pi\Sigma^*$-extensions and sequences

Example. Let $\mathbb{F} := \mathbb{Q}(m)(k)(h)(b)$ be a rational function field and define the field automorphism σ by

$$\sigma(c) = c \ \ \forall c \in \mathbb{Q}(m),$$
$$\sigma(k) = k+1,$$
$$\sigma(h) = h + \frac{1}{k+1}, \qquad H_{k+1} = H_k + \frac{1}{k+1},$$
$$\sigma(b) = \frac{m-k}{k+1} b, \qquad \binom{m}{k+1} = \frac{m-k}{k+1}\binom{m}{k}.$$

$(\mathbb{F}, \sigma)$ is a difference field, more precisely, a $\Pi\Sigma^*$-field.

Difference rings and fields: A *difference field* $(\mathbb{F}, \sigma)$ is a ring (resp. field) $\mathbb{F}$ together with a ring (resp. field) automorphism $\sigma : \mathbb{F} \to \mathbb{F}$; the *constant ring* (resp. *constant field*) is given by $\mathrm{const}_\sigma \mathbb{F} := \{f \in \mathbb{F} | \sigma(f) = f\}$.

$\Pi\Sigma^*$**-field:** A difference field $(\mathbb{F}(t), \sigma')$ is a Σ^*-*extension (resp. Π-extension)* of a difference field $(\mathbb{F}, \sigma)$:⇔

1. t is transcendental over $\mathbb{F}$,
2. $\sigma'(f) = \sigma(f)$ for all $f \in \mathbb{F}$,
3. $\sigma'(t) = t + f$ (resp. $\sigma'(t) = f\,t$) for some $f \in \mathbb{F}^*$,
4. the constant field remains unchanged, i.e., $\mathrm{const}_{\sigma'}\mathbb{F}(t) = \mathrm{const}_\sigma \mathbb{F}$.

A $\Pi\Sigma^*$-*extension* is either a Π- or a Σ^*-extension. $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ is a $\Pi\Sigma^*$-extension (resp. Σ^*-extension or Π-extension) if it is a tower of such extensions. $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ is a $\Pi\Sigma^*$-field over $\mathbb{F}$, if $\mathbb{F} = \mathrm{const}_\sigma\mathbb{F}$.

Example. Each of the extensions k, h, b forms a $\Pi\Sigma^*$-extension over the field below. In particular, $\mathrm{const}_\sigma \mathbb{Q}(m)(k)(h)(b) = \mathbb{Q}(m)$.

Ring of sequences: The set of sequences over a field $\mathbb{K}$ is denoted by

$$S(\mathbb{K}) := \{(a_n)_{n\geq 0} | a_i \in \mathbb{K}\};$$

we identity two sequences if they agree from a certain point on. The difference ring $(S(\mathbb{K}), S)$ with the shift operation (ring automorphism)

$$S : \langle a_0, a_1, a_2, \dots \rangle \mapsto \langle a_1, a_2, a_3, \dots \rangle$$

is called the *ring of sequences*.

Goal: Embed, e.g., $\mathbb{Q}(m)(k)[h,b]$ into $(S(\mathbb{Q}(m)), S)$.

Embedding example

We construct step by an embedding $(\mathbb{Q}(m)(k)[h][b], \sigma)$ into $(S(\mathbb{Q}(m)), S)$.

• Start with $\tau_0 : \mathbb{Q}(m) \to S(\mathbb{Q}(m))$ where

$$\tau_0(c) = \langle c, c, c, \dots \rangle \qquad \forall c \in \mathbb{Q}(m).$$

• Next, define the ring homomorphism $\tau_1 : \mathbb{Q}(m)(k) \to S(\mathbb{Q}(m))$ with $\tau_1(\frac{p}{q}) = \langle F(k) \rangle_{k\geq 0}$ where

$$F(k) = \begin{cases} 0 & q(k) = 0 \\ \frac{p(k)}{q(k)} & q(k) \neq 0. \end{cases}$$

Note that

$$\tau_1(\sigma(f)) = S(\tau_1(f)), \quad \forall f \in \mathbb{Q}(m)(k).$$

τ_1 **is injective:** Since $p(k), q(k)$ have only finitely many roots, $\tau_1(\frac{p}{q}) = 0$ if and only if $\frac{p(k)}{q(k)} = 0$. Hence τ_1 is injective.

• Define the ring homomorphism $\tau_2 : \mathbb{Q}(m)(k)[h] \to S(\mathbb{Q}(m))$ with $\tau_2(h) = \langle H_n \rangle_{n\geq 0}$ and

$$\tau_2(\sum_{i=0}^{d} f_i h^i) = \sum_{i=0}^{d} \tau_1(f_i)\tau_2(h)^i.$$

τ_2 **is injective:** If not, take $f = \sum_{i=0}^{d} f_i h^i \in \mathbb{Q}(m)(k)[h]^*$ with $\deg(f) = d$ minimal such that $\tau_2(f) = 0$. Note that $f \notin \mathbb{Q}(m)(k)$ (otherwise, $0 = \tau_2(f) = \tau_1(f)$; since τ_1 is injective, $f = 0$). Define

$$g := \sigma(f_d) f - f_d \sigma(f) \in \mathbb{Q}(m)(k)[h].$$

Note: $\deg(g) < d$ by construction. Moreover,

$$\tau_2(g) = \tau_1(\sigma(f_n)) \underbrace{\tau_2(f)}_{=0} - \tau_1(f_n) \underbrace{\tau_2(\sigma(f))}_{S(\tau_2(f))=S(0)=0} = 0.$$

By the minimality of $\deg(f)$, $g = 0$, i.e.,

$$\sigma(f_d) f - f_d \sigma(f) = 0.$$

Equivalently,

$$\frac{\sigma(f)}{f} = \frac{\sigma(f_d)}{f_d} \in \mathbb{Q}(m)(k).$$

With $f \notin \mathbb{Q}(m)(k)$ this contradicts to [Kar81].

• To this end, take the ring homomorphism $\tau_3 : \mathbb{Q}(m)(k)[h][b] \to S(\mathbb{Q}(m))$ with $\tau_3(b) = \langle \binom{m}{n} \rangle_{n\geq 0}$ and

$$\tau_3(\sum_{i=0}^{d} f_i b^i) = \sum_{i=0}^{d} \tau_2(f_i)\tau_3(b)^i.$$

By similar arguments, τ_3 is injective.

Result 1: The embedding into the ring of sequences

A *generalized d'Alembertian extension* $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ of $(\mathbb{F}, \sigma)$ is a $\Pi\Sigma^*$-extension such that for all $1 \leq i \leq e$,

$$\sigma(t_i) - t_i \in \mathbb{F}[t_1, \dots, t_{i-1}] \quad \text{or} \quad \sigma(t_i)/t_i \in \mathbb{F};$$

note that the t_i are transcendental and $\mathbb{K} := \mathrm{const}_\sigma \mathbb{F}(t_1)\dots(t_e) = \mathrm{const}_\sigma \mathbb{F}$.

Embedding: Suppose that $(\mathbb{F}, \sigma)$ describes the rational case ($\mathbb{F} = \mathbb{K}(k)$ with $\sigma(k) = k+1$), the q-rational case or the mixed case.
Then there is an injective ring homomorphism

$$\tau : \mathbb{F}[t_1 \dots, t_e] \to S(\mathbb{K})$$

with

$$\tau(c) = \langle c, c, c, \dots \rangle \quad \forall c \in \mathbb{K}$$

such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{F}[t_1 \dots, t_e] \arrow[r, "\tau"] \arrow[d, "\sigma"'] & S(\mathbb{K}) \arrow[d, "S"] \\ \mathbb{F}[t_1 \dots, t_e] \arrow[r, "\tau"] & S(\mathbb{K}) \end{tikzcd}$$

We call such an embedding a $\mathbb{K}$-*monomorphism*.

Consequence:

$$\mathbb{F}[t_1, \dots, t_e] \cong \tau(\mathbb{F})[\tau(t_1), \dots, \tau(t_e)].$$

In particular, the sequences $\tau(t_1), \dots, \tau(t_e)$ are algebraically independent over $\tau(\mathbb{F})$.

Result 2: Parameterized telescoping and $\Pi\Sigma^*$-extension

Let $(\mathbb{F}, \sigma)$ be a difference field with constants $\mathbb{K}$ and $f_1, \dots, f_d \in \mathbb{F}^*$. Then:

There are no $0 \neq (c_1, \dots, c_d) \in \mathbb{K}^d$ and $g \in \mathbb{F}$ with

$$\sigma(g) - g = c_1 f_1 + \dots + c_d f_d. \qquad (2)$$

$$\Updownarrow$$

There is a Σ^*-extension $(\mathbb{F}(t_1)\dots(t_d), \sigma)$ of $(\mathbb{F}, \sigma)$ with $\sigma(t_i) = t_i + f_i$ for $1 \leq i \leq d$.

There are no $0 \neq (c_1, \dots, c_d) \in \mathbb{Z}^d$ and $g \in \mathbb{F}^*$ with

$$\frac{\sigma(g)}{g} = f_1^{c_1} \dots f_d^{c_d}. \qquad (3)$$

$$\Updownarrow$$

There is a Π-extension $(\mathbb{F}(t_1, \dots, t_d), \sigma)$ of $(\mathbb{F}, \sigma)$ with $\sigma(t_i) = f_i t_i$ for $1 \leq i \leq d$.

Remark. The case $d = 1$ (telescoping) has been worked out in [Kar81].
As a consequence, one can check with telescoping that, e.g., the difference field $(\mathbb{Q}(m)(k)(h)(b), \sigma)$ is a $\Pi\Sigma^*$-field.

Result 3: A criterion for algebraic independence of sums and products

Combining the ideas from Result 1 and Result 2 gives the following main result:

Let $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ be a generalized d'Alembertian-extension of $(\mathbb{F}, \sigma)$ with $\mathbb{K} := \mathrm{const}_\sigma \mathbb{F}$.
Let $\tau : \mathbb{F}[t_1, \dots, t_e] \to S(\mathbb{K})$ be a $\mathbb{K}$-monomorphism.

Let $f_1, \dots, f_d \in \mathbb{F}[t_1, \dots, t_e]^*$ with $\langle F_i(k) \rangle_{k\geq 0} := \tau(f_i)$. Then:

There are no $g \in \mathbb{F}[t_1, \dots, t_e]$, $0 \neq (c_1, \dots, c_d) \in \mathbb{K}^d$ with (2).

$$\Updownarrow$$

The sequences $\{(S_1(n))_{n\geq 0}, \dots, (S_d(n))_{n\geq 0}\}$ given by[a]

$$S_1(n) := \sum_{k=r}^{n} F_1(k), \dots, S_d(n) := \sum_{k=r}^{n} F_d(k) \qquad (4)$$

are algebraically independent over $\tau(\mathbb{F}[t_1, \dots, t_e])$.

Let $f_1, \dots, f_d \in \mathbb{F}^*$ with $\langle F_i(k) \rangle_{k\geq 0} := \tau(f_i)$. Then:

There are no $g \in \mathbb{F}^*$ and $0 \neq (c_1, \dots, c_d) \in \mathbb{Z}^d$ with (3).

$$\Updownarrow$$

The sequences $\{(S_1(n))_{n\geq 0}, \dots, (S_d(n))_{n\geq 0}\}$ given by

$$S_1(n) := \prod_{k=r}^{n} F_1(k), \dots, S_d(n) := \prod_{k=r}^{n} F_d(k) \qquad (5)$$

are algebraically independent over $\tau(\mathbb{F})$.

a The lower bound r is chosen big enough.

The (q–)rational case

Theorem. Let $f_1(k), \dots, f_d(k) \in \mathbb{K}(k)$. If there are no $g(k) \in \mathbb{K}(k)$ and $c_1, \dots, c_d \in \mathbb{K}$ with (1), then the sequences (4) are algebraically independent over $\mathbb{K}(n)$; i.e., $\nexists P(x_1, \dots, x_d) \in \mathbb{K}(n)[x_1, \dots, x_d]^*$ with

$$P(S_1(n), \dots, S_d(n)) = 0 \quad \forall n \geq 0.$$

Corollary. Let $p_1(k), p_2(k), \dots \in \mathbb{K}[k]^*$, $u_1(k), u_2(k), \dots \in \mathbb{K}[k]^*$ and $v \in \mathbb{K}[k]$ with $\deg(v) > 0$ where

$$\gcd(p_i, v) = \gcd(u_i, v) = 1 \quad \forall i \geq 1,$$
$$\gcd(v(k), v(k+r)) = 1 \quad \forall r \in \mathbb{N}^*,$$
$$v(r) \neq 0 \quad \forall r \in \mathbb{N}^*.$$

Then the sums

$$S_1(n) := \sum_{k=1}^{n} u_1(k)\left(\frac{p_1(k)}{v(k)}\right), S_2(n) := \sum_{k=1}^{n} u_2(k)\left(\frac{p_2(k)}{v(k)}\right)^2, \dots$$

are algebraically independent over $\mathbb{K}(n)$.

Proof. Denote $f_i(k) := u_i(\frac{p_i}{v})^i$. Suppose there are $g(k) \in \mathbb{K}(k)$ and $c_i \in \mathbb{K}$ with (1) where $d \geq 1$ is minimal. Then

$$g(k+1) - g(k) = \frac{c_1 u_1 p_1 v^{d-1} + c_2 u_2 p_2^2 v^{d-2} + \dots + c_d u_d p_d^d}{v^d} =: \frac{w}{v^d}.$$

Since $c_d \neq 0$, $\gcd(v, c_d u_d p_d^d) = 1$. Hence $\gcd(w, v) = 1$, and thus $\gcd(w, v^d) = 1$. By [Abr71, Pau95] such a $g(k) \in \mathbb{K}(k)$ cannot exist; a contradiction. □

Example. Choosing $p_i = u_i = 1, v = k$ in the Corollary proves that the generalized harmonic numbers $\{H_n^{(i)} | i \geq 1\}$ are algebraically independent over $\mathbb{K}(n)$.

Example. Similarly, the q-harmonic numbers $\{\sum_{k=1}^{n} \frac{1}{(1-q^k)^i} | i \geq 1\}$ (or $\{\sum_{k=1}^{n} \left(\frac{q^k}{1-q^k}\right)^i | i \geq 1\}$) are algebraically independent over $\mathbb{K}(q^k)$.

The (q–)hypergeometric case

Let $f(k)$ be hypergeometric, i.e., for all r big enough,

$$\alpha(r) = \frac{f(r+1)}{f(r)}$$

for some $\alpha(k) \in \mathbb{K}(k)$.
We restrict to the case that there are no $v(k) \in \mathbb{K}(k)$ and no root of unity $\gamma \in \mathbb{K}$ with $f(k) = \gamma^k v(k)$. Then there is a $\Pi\Sigma^*$-field $(\mathbb{K}(k)(t), \sigma)$ over $\mathbb{K}$ with

$$\sigma(k) = k+1 \quad \text{and} \quad \sigma(t) = \alpha t,$$

and a $\mathbb{K}$-monomorphism $\tau : \mathbb{K}(k)[t] \to S(\mathbb{K})$.

Theorem. For $1 \leq i \leq d$, let $v_i \in \mathbb{K}(k)$ and $f_i := v_i t$.
If there are no $c_i \in \mathbb{K}$ and $w \in \mathbb{K}(k)$ with $g := w\,t$ such that (2), then the sequences

$$f(n),\ S_1(n) = \sum_{k=r}^{n} v_1(k) f(k), \dots, S_d(n) = \sum_{k=r}^{b} v_d(k) f(k)$$

(r big enough) are algebraically independent over $\mathbb{K}(n)$; i.e., $\nexists P(x_0, x_1, \dots, x_d) \in \mathbb{K}(n)[x_0, x_1, \dots, x_d]^*$ with

$$P(f(n), S_1(n), \dots, S_d(n)) = 0 \quad \forall n \geq 0.$$

Corollary Let $f(m,k)$ be hypergeometric in m, k where $f \neq \gamma^k v$ for all $v \in \mathbb{K}(m,k)$ and all roots of unity $\gamma \in \mathbb{K} \setminus \{1\}$. If **Z's algorithm** [Zei91] fails to solve (1) with $f_i := f(m+i-1,k)$, then $S_0(n) = f(m,n)$ and

$$S_1(n) = \sum_{k=r}^{n} f(m,k), \dots, S_d(n) = \sum_{k=r}^{b} f(m+d-1,k)$$

(r big enough) are transcendental over $\mathbb{K}(m)(n)$.

Example. For the Apéry-sum

$$A(m) = \sum_{k=0}^{m} \binom{m}{k}^2 \binom{m+k}{k}$$

Zeilberger's algorithm finds a recurrence of order 2, but not a smaller one. Hence, the sequences

$$S_0(n) = \binom{m}{n}^2 \binom{m+n}{n},$$
$$S_1(n) = \sum_{k=0}^{n} \binom{m}{k}^2 \binom{m+k}{k},$$
$$S_2(n) = \sum_{k=0}^{n} \binom{m+1}{k}^2 \binom{m+k+1}{k}$$

are algebraically independent over $\mathbb{Q}(m)(n)$.

Example. In [Abr03] a criterion is given when Zeilberger's algorithm fails to find a creative telescoping solution for a hypergeometric input summand $f(m,k)$. If $f(m,k)$ satisfies this criterion, then all the sequences

$$\{f(m,n)\} \cup \{\sum_{k=r}^{n} f(m+i,k) | i \geq 0\}$$

in n (r big enough) are algebraically independent over $\mathbb{K}(m)$. A typical example [Abr03, Exp. 2] is

$$f(m,k) = \frac{1}{mk+1}(-1)^k \binom{m+1}{k}\binom{2m-2k-1}{m-1}.$$

Remark. Analogously, all ideas can be carried over to the q–hypergeometric case.

The nested sum case

Consider the following sum from [PS03]:

$$S(m,n) := \sum_{k=0}^{n} f(m,k) = \sum_{k=0}^{n} (1 + 5 H_k(m-2k))\binom{m}{k}^5.$$

Then the package Sigma shows that the sequences

$$\binom{m}{n},\ H_n,\ S(m,n),\ S(m+1,n),\ S(m+2,n),\ S(m+3,n)$$

in n are algebraically independent over $\mathbb{K}(m)(n)$.

Proof. Sigma constructs the $\Pi\Sigma^*$-field $(\mathbb{Q}(m)(k)(h)(b), \sigma)$ and designs the $\mathbb{Q}(m)$-monomorphism from above.
Then it sets

$$f_1 = b^5(1 + 5h(m-2k)),$$
$$f_2 = \frac{b^5(m+1)^5(5h(-2k+m+1)+1)}{(-k+m+1)^5},$$
$$f_3 = \frac{b^5(m+1)^5(m+2)^5(5h(-2k+m+2)+1)}{(-k+m+1)^5(-k+m+2)^5},$$
$$f_4 = \frac{b^5(m+1)^5(m+2)^5(m+3)^5(5h(-2k+m+3)+1)}{(-k+m+1)^5(-k+m+2)^5(-k+m+3)^5};$$

this is motivated by $\binom{m+1}{k} = \frac{m+1}{m+1-k}\binom{m}{k}$ which shows that

$$\tau(f_i) = \langle f(m+i-1,k) \rangle_{k\geq 0}.$$

Finally, Sigma proves algorithmically that there are no $g \in \mathbb{Q}(m)(k)(h)(h)$ and $c_i \in \mathbb{Q}(m)$ with (2). Hence the transcendence of the sequences follows by Result 3. □

Remark. Note that the situation $m = n$ for the sum $S(m,n)$ is completely different. We get

$$S(m,m) = \sum_{k=0}^{m} \binom{m}{k}^2 \binom{m+k}{k}.$$

In particular, $S(m,m)$ satisfies a recurrence relation of minimal order 2.

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