Formal Power Series and Algebraic Combinatorics
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Computer Algebra and Power Series with Positive Coefficients

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ABSTRACT. We consider the question whether all the coefficients in the series expansions of some specific rational functions are positive, and we demonstrate how computer algebra can help answering questions arising in this context. By giving partial computer proofs, we provide new evidence in support of some longstanding open conjectures. Also two new conjectures are made.

1. Introduction

Proving that all the coefficients in the series expansion of some given multivariate rational function are positive can be quite a difficult task. There are difficult papers on this subject by Szegö [**14**], Askey and Gasper [**2**], Koornwinder [**10**], and others. Gillis, Reznick and Zeilberger [**8**] have pointed out that some seemingly difficult positivity results can be proven also by elementary means. In this paper, we make an attempt at going one step further: We ask to which extent positivity results can be proven automatically using computer algebra. Two results from the literature and two longstanding open conjectures related to them are considered. For none of the latter we are able to provide full proofs, but we give partial proofs that add new evidence in support of these conjectures.

Also Zeilberger [**15**] addresses the question of proving positivity results with the aid of computer algebra. He proposes a method using positivity-preserving transformations, which is independent of our approach described below. The approach we take is a continuation of our previous work on treating special function inequalities via symbolic computation [**5, 6, 9, 1**].

While it is easy to show that there can be no algorithm which for a given multivariate rational function decides whether all its series coefficients are positive, computer algebra is nevertheless useful for deciding subproblems that may arise in the construction of a positivity proof. Our proofs follow a common pattern: We first determine recurrence equations for the coefficients (using computer algebra, see Section 1.1), and then prove a suitable quantified formula about polynomial inequalities (using computer algebra, see Section 1.2) which together with the recurrence equations implies the desired positivity result. Sometimes this method is successful, sometimes it is not.

Before entering the subject, let us briefly summarize the two main techniques from computer algebra that are used throughout the rest of the paper.

1.1. Guessing and Proving Recurrence Relations.

Let $a(n,m)$ be a sequence whose value can be computed for every particular point $(n,m) \in \mathbb{N}^2$. If the sequence satisfies a recurrence equation with polynomial coefficients, then this equation can be found easily by making an ansatz. To this end, we first

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choose a finite set $S \subseteq \mathbb{N}^4$. A recurrence corresponds to an array of constants α_i ($i \in S$) with

$$\sum_{i=(i_1,i_2,i_3,i_4)\in S} \alpha_i n^{i_1} m^{i_2} a(n+i_3, m+i_4) = 0.$$

In order to find candidates for α_i, we evaluate the above expression for a couple of specific sample points $(n,m) \in \mathbb{N}^2$ and undetermined coefficients α_i, thus obtaining a linear system for the coefficients α_i. Each coefficient vector $(\alpha_i : i \in S)$ corresponding to an actual recurrence will belong to the solution space of the system, but there might be additional solutions not corresponding to recurrence equations. This is why the method is referred to as *automated guessing* [**12**].

If the $a(n,m)$ arise as coefficients in the series expansion of a rational function $r(x,y)$, then it is easy to decide for a conjectured recurrence whether it is true or false. It suffices to transform the conjectured recurrence into a PDE for its generating function (this can be done automatically) and check whether $r(x,y)$ solves that differential equation. It will do so if and only if the conjectured recurrence holds true. We can thus repeat the guessing procedure with bigger and bigger sets of sample points until the recurrences delivered by the guessing method are all found to be true.

It is of course immaterial that $a(n,m)$ is a bivariate sequence; the same algorithm is applicable for any arity. All the recurrence equations claimed in this paper have been found and verified in this way, unless otherwise stated.

1.2. Proving and Finding Polynomial Inequalities. Our second ingredient is the CAD algorithm [**4**]. This algorithm operates on quantified formulas about polynomial inequalities over the reals. Formally, a quantifier-free formula $\Phi(x_1,\dots,x_n)$ is

- a logical constant, i.e., True or False,
- or an atomic formula, i.e., $p(x_1,\dots,x_n) \diamondsuit 0$ for a polynomial $p \in k[x_1,\dots,x_n]$ and a relation $\diamondsuit \in \{=,\neq,>,<,\geq,\leq\}$; k being the field of real algebraic numbers,
- or a boolean combination of other quantifier-free formulas, i.e.,

  $$\Psi_1(x_1,\dots,x_n) \diamondsuit \Psi_2(x_1,\dots,x_n)$$

  with $\diamondsuit \in \{\wedge,\vee,\Rightarrow,\Leftrightarrow\}$, or $\neg\Psi_1(x_1,\dots,x_n)$, for some subformulas Ψ_1, Ψ_2.

A quantified formula $\Phi(x_1,\dots,x_n)$ is a formula of the form

$$Q_1y_1\ Q_2y_2 \dots Q_my_m : \Psi(x_1,\dots,x_n,y_1,\dots,y_m),$$

where Ψ is a quantifier-free formula and $Q_1,\dots,Q_m \in \{\forall,\exists\}$. The $x_1,\dots,x_n$ are referred to as free variables, as opposed to the $y_1,\dots,y_m$ which are called bound variables.

The CAD algorithm is able to perform quantifier elimination over such formulas, i.e., given any quantified formula $\Phi(x_1,\dots,x_n)$, it computes a quantifier-free formula $\Phi'(x_1,\dots,x_n)$ such that

$$\forall x_1,x_2,\dots,x_n \in \mathbb{R} : \big(\Phi(x_1,\dots,x_n) \iff \Phi'(x_1,\dots,x_n)\big).$$

The formula $\Phi'(x_1,\dots,x_n)$ describes the condition that the $x_1,\dots,x_n$ have to satisfy in order to make Φ true. If CAD is applied to a formula Φ with no free variables, it will deliver as quantifier-free formula Φ' either True (then this is a rigorous proof for Φ) or False (then this is a rigorous proof for $\neg\Phi$).

All the quantified formulas claimed in this paper have been proven with Mathematica's implementation of CAD [**13**], unless otherwise stated.

2. A Result of Askey and Gasper, and a Conjectured Variation

THEOREM 2.1. [**3**] *Let $a(n,m,k)$ be such that*

$$\frac{1}{1-(x+y+z)+4xyz} = \sum_{n,m,k\geq 0} a(n,m,k)x^n y^m z^k.$$

Then $a(n,m,k) > 0$ for all $n,m,k \geq 0$.

First we determine all constants $\alpha_{i,u,v,w}$ such that

$$\sum_{u,v,w\in\{0,1\}} (\alpha_{0,u,v,w} + \alpha_{1,u,v,w} n + \alpha_{2,u,v,w} m + \alpha_{3,u,v,w} k) a(n+u, m+v, k+w) = 0$$

for all $n, m, k \geq 0$. It turns out that the possible choices for $\alpha_{i,u,v,w}$ form a vector space of dimension 10; the algorithm described in the introduction gives us a basis for this vector space, say the basis elements correspond to the recurrence equations

$$\sum_{u,v,w\in\{0,1\}} p_{i,u,v,w}(n,m,k) a(n+u, m+v, k+w) = 0 \quad (i = 1, \ldots, 10), \tag{2.1}$$

for certain linear polynomials $p_{i,u,v,w}(n,m,k)$ that we know explicity.

We want to construct a recurrence equation

$$\sum_{u,v,w\in\{0,1\}} q_{u,v,w}(n,m,k) a(n+u, m+v, k+w) = 0$$

with $q_{1,1,1}(n,m,k) < 0$ and $q_{u,v,w}(n,m,k) \geq 0$ $((u,v,w) \neq (1,1,1))$ for all $n \geq m \geq k \geq 0$. From such a recurrence, the positivity of the $a(n,m,k)$ easily follows by induction. (It is sufficient to consider $n \geq m \geq k \geq 0$ by the symmetry of $a(n,m,k)$.) If such a recurrence exists, then it must be a linear combination of the basis equations in (2.1), so we can make an ansatz with undetermined coefficients:

$$q_{u,v,w}(n,m,k) := \beta_1 p_{1,u,v,w}(n,m,k) + \beta_2 p_{2,u,v,w}(n,m,k) + \cdots + \beta_{10} p_{10,u,v,w}(n,m,k). \tag{2.2}$$

Next, we apply CAD to the formula

$$\forall n \geq m \geq k \geq 0 : q_{1,1,1}(n,m,k) < 0 \wedge \bigwedge_{\substack{u,v,w\in\{0,1\} \\ (u,v,w)\neq(1,1,1)}} q_{u,v,w}(n,m,k) \geq 0.$$

The resulting condition depends on the basis elements (2.1). For our basis, we obtained that the formula becomes true if and only if the β_i are chosen such that

$$\begin{gathered}
\beta_1 \leq 0 \wedge \beta_2 \geq 0 \wedge \beta_3 \geq 0 \wedge \beta_4 = 0 \wedge \beta_5 \geq 0 \wedge \beta_6 \geq 0 \wedge 2\beta_1 + 2\beta_2 + 2\beta_3 + \beta_4 + \beta_5 + \beta_6 \leq 0 \\
\wedge\, \beta_3 \geq \beta_7 \wedge 2\beta_1 + \beta_3 + \beta_6 + \beta_7 \leq 0 \wedge \beta_2 + \beta_8 = 0 \wedge \beta_3 + \beta_4 \geq 2\beta_7 + \beta_8 + \beta_9 \\
\wedge\, \beta_9 \leq 0 \wedge 3\beta_1 + \beta_2 + \beta_3 + \beta_5 + \beta_6 + \beta_7 + \beta_9 \leq 0 \wedge 2\beta_1 + \beta_5 + \beta_6 + 2(\beta_7 + \beta_9) \geq \beta_4 \\
\wedge\, \beta_4 \leq \beta_5 + \beta_6 + 2\beta_{10} \wedge \beta_1 + \beta_3 + \beta_4 \geq \beta_7 + \beta_8 + \beta_{10} \wedge \beta_1 + \beta_2 + \beta_7 + \beta_8 \geq \beta_{10} \\
\wedge\, \beta_{10} < 0 \wedge 2\beta_1 + \beta_2 + \beta_3 + \beta_5 + \beta_6 + \beta_{10} \leq 0 \wedge (\beta_1 + \beta_7 = 0 \vee (\beta_1 + \beta_7 < 0 \wedge \beta_1 + \beta_7 + \beta_9 \leq 0)).
\end{gathered}$$

It is an easy matter to find a tuple $(\beta_1, \ldots, \beta_{10})$ that satisfies this condition, if desired, we can have the computer find one. For instance, $(\beta_1, \ldots, \beta_{10}) = (-1, 0, 0, 0, 0, 2, 0, 0, 0, -1)$ satisfies the condition, and, if plugged into the ansatz equation (2.2), yields

$$2(n+m-k)a(n,m,k+1) + (1+n-m+k)a(n,m+1,k+1) - (1+n)a(n+1,m+1,k+1) = 0.$$

Together with the initial value $a(0,0,0) = 1 > 0$, this recurrence forms an easy induction proof of Theorem 2.1.

The recurrence above was first observed by Gillis and Kleeman [**7**]. They do, however, not remark on how they discovered this recurrence. As we have shown above, it is possible to derive it in a systematic way using computer algebra.

Let us now turn to an open problem. Gillis, Reznick and Zeilberger [**8**] have raised the conjecture that for any $r \geq 4$, the series coefficients of $1/(1 - (x_1 + x_2 + \cdots + x_r) + r! x_1 x_2 \cdots x_r)$ are nonnegative. According to their Proposition 3, in order to confirm the conjecture, it suffices to prove nonnegativity of the diagonal coefficients $a_r(n,n,\ldots,n)$ which are given by the binomial sum

$$a_r(n,n,\ldots,n) = \sum_{k=0}^{n} (-1)^k \frac{(rn-(r-1)k)!(r!)^k}{(n-k)!^r k!}.$$

As computational evidence for their conjecture, they verified that $a_r(n,n,\ldots,n) \geq 0$ for $r = 4$ and $0 \leq n \leq 220$. We next prove the conjecture for $r = 4, 5, 6$ and arbitrary n.

THEOREM 2.2.

(1) *Let $a(n,m,k,l)$ be such that*

$$\frac{1}{1-(x+y+z+w)+4!xyzw} = \sum_{n,m,k,l\geq 0} a(n,m,k,l)x^n y^m z^k w^l.$$

Then $a(n,m,k,l)\geq 0$ for all $n,m,k,l\geq 0$.

(2) *Let $a(n,m,k,i,j)$ be such that*

$$\frac{1}{1-(x+y+z+u+v)+5!xyzuv} = \sum_{n,m,k,i,j\geq 0} a(n,m,k,i,j)x^n y^m z^k u^i v^j.$$

Then $a(n,m,k,i,j)\geq 0$ for all $n,m,k,i,j\geq 0$.

(3) *Let $a(n,m,k,i,j,l)$ be such that*

$$\frac{1}{1-(x+y+z+u+v+w)+6!xyzuvw} = \sum_{n,\,m,\,k,\,i,\,j\,\geq 0} a(n,m,k,i,j)x^n y^m z^k u^i v^j w^l.$$

Then $a(n,m,k,i,j,l)\geq 0$ for all $n,m,k,i,j,l\geq 0$.

PROOF. We prove the first statement, the others can be done in the same manner. According to Prop. 3 of [**8**], it suffices to show that $a(n)\geq 0$ for all $n\geq 0$, where

$$a(n)=\sum_{k=0}^{n}(-1)^k\frac{(4n-3k)!4!^k}{(n-k)!^4 k!}.$$

With Zeilberger's algorithm [**16, 11**] we obtained the recurrence equation

$$\begin{aligned}
&331776(2n+7)(4n+11)(4n+15)(n+1)^3a(n)\\
&+13824(4n+15)\left(32n^5+344n^4+1424n^3+2855n^2+2801n+1085\right)a(n+1)\\
&+576\left(192n^6+3072n^5+20108n^4+68918n^3+130513n^2+129613n+52815\right)a(n+2)\\
&-8(n+3)(4n+7)(4n+13)\left(40n^3+380n^2+1193n+1240\right)a(n+3)\\
&+(n+4)^3(2n+5)(4n+7)(4n+11)a(n+4)=0
\end{aligned}$$

for $a(n)$. Consider the formula

$$\begin{aligned}
&\forall\ A_0,A_1,A_2,A_3,A_4\in\mathbb{R}\ \forall\ n\geq 0:\\
&\quad\big(A_3\geq\beta A_2\wedge A_2\geq\beta A_1\wedge A_1\geq\beta A_0\geq 0\wedge p_0(n)A_0+\cdots+p_4(n)A_4=0\big)\Longrightarrow A_4\geq\beta A_3,
\end{aligned}$$

where $p_i(n)$ denotes the polynomial appearing as coefficient of $a(n+i)$ in the above recurrence $(i=0,\dots,4)$. Using CAD, we find that this formula is valid if and only if $\beta\geq\beta_0$, where β_0 is the real root of the polynomial

$$x^4-160x^3+3456x^2+55296x+331776$$

whose approximate value is 42.04.

We prove $a(n+1)\geq 43a(n)\geq 0$ for $n\geq 1$ by induction on n. The induction step follows from the formula above. As for the induction base, it suffices to check that $a(n+1)\geq 43a(n)\geq 0$ for $n=1,2,3,4$, which is trivial.

The nonnegativity of $a(n)$ for $n\geq 1$ is hence established. Furthermore, $a(0)=1\geq 0$, so the proof is complete. $\square$

The proof for parts 2 and 3 of the theorem proceeds along the the same lines, of course with different recurrence equations, and consequently with different values for β_0. Their approximate values for $r=5$ and $r=6$ are $\beta_0\approx 138.9$ and $\beta_0\approx 715.5$, respectively. We believe that for any specific value of r it is possible to obtain a similar proof, but the runtime requirements for the computations grow drastically and with currently available machines we were not able to go beyond $r=6$ with reasonable effort. The runtime required for completing the cases $r=4,5,6$ on a 2.4GHz Linux machine was, however, no more than a few minutes.

3. A Result of Szegö, and a Conjectured Variation

THEOREM 3.1. *Let $a(n,m,k)$ be such that*

$$\frac{1}{1-(x+y+z)+\frac{3}{4}(xy+xz+yz)} = \sum_{n,m,k\geq 0} a(n,m,k)x^n y^m z^k.$$

Then $a(n,m,k) > 0$ for all $n,m \geq 0$ and (at least) $k = 0,1,2,\ldots,16$.

PROOF. Because of symmetry it suffices to consider the case $n \geq m \geq k \geq 0$. For all the claimed values of k, we find a recurrence equation

$$p_{0,k}(n,m)a(n,m,k) + p_{1,k}(n,m)a(n+1,m,k) + p_{2,k}(n,m)a(n+2,m,k) = 0,$$

with $p_{i,k}(n,m)$ being polynomials in n and m for which the formula

$$\begin{aligned}&\forall A_0, A_1, A_2 \in \mathbb{R}\ \forall\ n \geq m \geq k :\\ &\quad \big(A_1 \geq A_0 > 0 \wedge p_{0,k}(n,m)A_0 + p_{1,k}(n,m)A_1 + p_{2,k}(n,m)A_2 = 0\big) \Longrightarrow A_2 \geq A_1\end{aligned}$$

is true. This gives the induction step for proving $a(n+1,m,k,l) \geq a(n,m,k,l) > 0$ for all $n \geq m \geq k \geq 0$. Checking the induction base is trivial. $\square$

We believe that the proof technique described above succeeds for every specific value of k; again we were not able to verify this for $k > 16$ owing to the extensive runtime requirements of the algorithms.

Szegö [**14**] has shown by a rather complicated derivation that indeed $a(n,m,k) > 0$ for all $n,m,k \geq 0$. Later, Askey and Gasper [**2**] have given a different, but still complicated proof for the same fact. Both proofs rely on finding an integral representation for $a(n,m,k)$ and then applying arguments from the theory of special function for showing that the integrals are always positive.

Comparison to the simple recurrence equation obtained in Section 2 that asserts positivity of the coefficients $(1-(x+y+z)+4xyz)^{-1}$ leads naturally to the question whether a similar recurrence can be given for the coefficients of $(1-(x+y+z)+\frac{3}{4}(xy+xz+yz))^{-1}$. No such recurrence has been published so far. By applying the procedure described in the proof of Theorem 2.1 to the present example, we have also not found such a recurrence. This, however, means that no such recurrence exists at all—a result that is perhaps not as easy to prove without computer assistence.

PROPOSITION 3.1. *Let $a(n,m,k)$ be as in Theorem 3.1. Then there does not exist a recurrence equation*

$$\sum_{u,v,w\in\{0,1\}} q_{u,v,w}(n,m,k)a(n+u,m+v,k+w) = 0 \qquad (n,m,k \geq 0)$$

with linear polynomials $q_{u,v,w}(n,m,k)$ $(u,v,w \in \{0,1\})$ such that

$$q_{1,1,1}(n,m,k) < 0 \text{ and } q_{u,v,w}(n,m,k) \geq 0 \ ((u,v,w) \neq (1,1,1))$$

for all $n \geq m \geq k \geq 0$.

Despite some effort, we have also not been able to find a computer proof for general k by other means. We did find some simple recurrences for $a(n,m,k)$ but we did not succeed in constructing a positivity proof from any of them.

A variation of Theorem 3.1 arises as conjecture in the article of Askey and Gasper [**2**]: Are the coefficients in the series expansion of $(1-(x+y+z+w)+\frac{2}{3}(xy+xz+xw+yz+yw+zw))^{-1}$ all positive? This conjecture is still open; the techniques of this paper are also insufficient for giving a complete proof. We can only offer additional evidence by supplying proofs for the situation where two indices are set to specific integers.

THEOREM 3.2. *Let $a(n,m,k,l)$ be such that*

$$\frac{1}{1-(x+y+z+w)+\frac{2}{3}(xy+xz+xw+yz+yw+zw)} = \sum_{n,m,k,l\geq 0} a(n,m,k,l)x^n y^m z^k w^l$$

Then $a(n,m,k,l) > 0$ for all $n,m \geq 0$ and (at least) all k,l with $0 \leq k+l \leq 14$.

PROOF. Because of symmetry it suffices to consider the case $n \geq m \geq k \geq l \geq 0$. For all the claimed values of k and l, we find a recurrence equation

$$p_{0,k,l}(n,m)a(n,m,k,l) + p_{1,k,l}(n,m)a(n+1,m,k,l) + p_{2,k,l}(n,m)a(n+2,m,k,l) = 0$$

where $p_{i,k,l}(n,m)$ are some polynomials for which the formula

$$\begin{aligned}&\forall A_0, A_1, A_2 \in \mathbb{R}\ \forall\ n \geq m \geq k :\\ &\quad\big(A_1 \geq A_0 > 0 \wedge p_{0,k,l}(n,m)A_0 + p_{1,k,l}(n,m)A_1 + p_{2,k,l}(n,m)A_2 = 0\big) \Longrightarrow A_2 \geq A_1\end{aligned}$$

is true. This gives the induction step for proving $a(n+1,m,k,l) \geq a(n,m,k,l) > 0$ for all $n \geq m \geq k \geq l \geq 0$. Checking the induction base is trivial. □

Further evidence can be obtained by considering diagonals.

THEOREM 3.3. *Let $a(n,m,k,l)$ be as in the previous theorem. Then $a(n,n+u,n+v,n+w) > 0$ for all $n \geq 0$ and (at least) all u,v,w with $0 \leq u,v,w \leq 12$.*

PROOF. We abreviate $a_{u,v,w}(n) := a(n,n+u,n+v,n+w)$. For all the claimed values of u,v,w, we find a recurrence equation

$$p_{0,u,v,w}(n)a_{u,v,w}(n) + p_{1,u,v,w}(n)a_{u,v,w}(n+1) + p_{2,u,v,w}(n)a_{u,v,w}(n+2) = 0$$

where $p_{i,u,v,w}(n)$ are some polynomials in n and m for which the formula

$$\begin{aligned}&\forall A_0, A_1, A_2 \in \mathbb{R}\ \forall\ n \geq 3(u+v+w+2) :\\ &\quad\big(A_1 \geq \tfrac{64}{9}A_0 > 0 \wedge p_{0,u,v,w}(n)A_0 + p_{1,u,v,w}(n)A_1 + p_{2,u,v,w}(n)A_2 = 0\big) \Longrightarrow A_2 \geq \tfrac{64}{9}A_1\end{aligned}$$

is true. This gives the induction step for proving $a_{u,v,w}(n+1) \geq \frac{64}{9}a_{u,v,w}(n) > 0$. The proof is completed by checking $a_{u,v,w}(3(u+v+w+2)+1) \geq \frac{64}{9}a_{u,v,w}(3(u+v+w+2)) > 0$ as induction base, and $a_{u,v,w}(n) > 0$ for the points $0 \leq n < 3(u+v+w+2)$ not covered by the induction argument. □

The mysterious constant $\frac{64}{9}$ that is needed in the induction step formula of the proof above was obtained in the same way as in the proof of Theorem 2.2. The lower bound $3(n+v+w+2)$ was found by experimenting and would probably have to be adjusted for values of u,v,w outside the range that we have considered.

4. Two new Conjectures

We conclude this paper with two rational functions for which we conjecture that all their series coefficients are positive. Computational experiments have led us to these conjectures. As evidence in support of the conjectures, we provide partial proofs. We shall not state any opinion about the difficulty of proving the conjectures in full generality.

THEOREM 4.1. *Let $a(n,m,k)$ be such that*

$$\frac{1}{1-(x+y+z)+\frac{1}{4}(x^2+y^2+z^2)} = \sum_{n,m,k\geq 0} a(n,m,k)x^n y^m z^k.$$

Then $a(n,m,k) > 0$ for all $n,m \geq 0$ and (at least) $k = 0,1,2,\ldots,9$.

PROOF. Because of symmetry it suffices to consider the case $n \geq m \geq k \geq 0$. For all the claimed values of k, we find a recurrence

$$p_{0,k}(n,m)a(n,m,k) + p_{1,k}(n,m)a(n+1,m,k) + p_{2,k}(n,m)a(n+2,m,k) = 0,$$

with $p_{i,k}(n,m)$ being polynomials in n and m for which the formula

$$\begin{aligned}&\forall A_0, A_1, A_2 \in \mathbb{R}\ \forall\ n \geq m \geq 0 :\\ &\quad\big(A_1 \geq \tfrac{1}{2}A_0 > 0 \wedge p_{0,k}(n,m)A_0 + p_{1,k}(n,m)A_1 + p_{2,k}(n,m)A_2 = 0\big) \Longrightarrow A_2 \geq \tfrac{1}{2}A_1.\end{aligned}$$

is true. This is the induction step for proving $a(n+1,m,k) \geq \frac{1}{2}a(n,m,k) > 0$ for all $n \geq m \geq k \geq 0$. Checking the induction base is trivial. □

CONJECTURE 4.1. *Let $a(n,m,k)$ be as in Theorem 4.1. Then $a(n,m,k) > 0$ for all $n,m,k \geq 0$.*

Computer experiments suggest furthermore that the conjecture becomes false if the constant $\frac{1}{4}$ in the denominator be replaced by $\frac{1}{4}+\varepsilon$ for any $\varepsilon>0$.

THEOREM 4.2. *Let $a(n,m,k,l)$ be such that*

$$\frac{1}{1-(x+y+z+w)+\frac{64}{27}(xyz+xyw+xzw+yzw)}=\sum_{n,m,k,l\geq 0}a(n,m,k,l)x^ny^mz^kw^l.$$

Then $a(n,m,k,l)>0$ for all $n,m\geq 0$ and (at least) all k,l with $0\leq k+l\leq 12$.

PROOF. For all the claimed values of k and l, we find that $a(n,m,k,l)$ is hypergeometric with respect to n, i.e., we find rational functions $r_{k,l}(n,m)$ such that

$$\forall n,m\geq 0: a(n+1,m,k,l)=r_{k,l}(n,m)a(n,m,k,l).$$

A CAD computation confirms that $r_{k,l}(n,m)>0$ for all $n,m\geq k+l$, so positivity of the $a(n,m,k,l)$ follows from the positivity of $a(n,m,k,l)$ for $n,m\leq k+l$, which is easily verified. $\square$

CONJECTURE 4.2. *Let $a(n,m,k,l)$ be as in Theorem 4.2. Then $a(n,m,k,l)>0$ for all $n,m,k,l\geq 0$.*

Also for this example, computer experiments suggest furthermore that the conjecture becomes false if the constant $\frac{64}{27}$ in the denominator be replaced by $\frac{64}{27}+\varepsilon$ for any $\varepsilon>0$.

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