Gröbner Bases:
An Introduction

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■ What is the Purpose of GB?

■ A Uniform Approach

The method (theory plus algorithms) of Gröbner bases provides a uniform approach to solving a wide range of problems expressed in terms of sets of multivariate polynomials.

$$x y - 2 y z - z = 0$$
$$y^2 - x^2 z + x z = 0$$
$$z^2 - y^2 x + x = 0$$

Example: kinematic equations of a robot.

■ Areas (see GB-98 Proceedings: Tutorials)

- algebraic geometry, commutative algebra, polynomial ideal theory
- invariant theory
- automated geometrical theorem proving

- coding theory
- integer programming
- partial differential equations
- hypergeometric functions
- symbolic summation
- statistics
- systems theory
- numerics (e.g. wavelets) →

■ Example Commutative Algebra (see GB-98: Intro to GB)

- solvability and solving of algebraic systems of equations
- ideal and radical membership
- effective computation in residue class rings modulo polynomial ideals
- linear diophantine equations with polynomial coefficients ("syzygies")
- Hilbert functions
- algebraic relations among polynomials
- implicitization
- inverse polynomial mappings

...

↑

■ How Can GB be Applied?

The general strategy: Given a set F of polynomials in $K[x_1, \ldots, x_n]$ (that describes some problem, e.g. a system of equations):

- We transform F into another set G of polynomials "with certain nice properties" (called a "Gröbner basis") such that F and G are "equivalent" (i.e. generate the same ideal).

- From the theory of GBs we know that this transformation can be carried out by an algorithm.

- From the theory of GBs we know that, because of the "nice properties of GBs, many problems that are difficult for general F are "easy" for Gröbner bases G.

- We solve the problem for G and transform the solution back to F.

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■ What are GB?

■ Division ("Reduction") of Multivariate Polynomials

```
g = x^2 y^3 + 3 x y^2 - 5 x;
```

```
f1 = x y - 2 y;   f2 = 2 y^2 - x^2;   F = {f1, f2};
```

One possible division ("reduction") step:

```
h = g - (3 x y^2 / (x y)) f1                // Expand
```

$-5x + 6y^2 + x^2 y^3$

We write

$$g \longrightarrow_F h$$

for this situation. →

Note that we presuppose that we have fixed some ordering of the power products in polynomials so that we can speak about the "leading power product" of polynomials.

(There are infinitely orderings that are "admissible" for the GB theory. They can be characterized by two simple axioms.)

■ In General, Many Reductions are Possible

```
g = x^2 y^3 + 3 x y^2 - 5 x;
```

```
f1 = x y - 2 y;  f2 = 2 y^2 - x^2;  F = {f1, f2};
```

Another possible division ("reduction") step:

```
h2 = g - (x^2 y^3 / (x y)) f1                // Expand
```

$-5x + 3xy^2 + 2xy^3$

```
h1 = -5 x + 6 y^2 + x^2 y^3
```

$-5x + 6y^2 + x^2 y^3$

Another possible division ("reduction") step:

```
h3 = g - (x^2 y^3 / (2 y^2)) f2                // Expand
```

$-5x + \frac{x^4 y}{2} + 3xy^2$

■ Multivariate Polynomial Division Always Terminates But is Not Unique

We write

$$g \longrightarrow^{*}_{F} h$$

if *g* reduces to *h* by finitely many reduction steps w.r.t. *F.*

Fact (Termination): Given *g* and *F*, any sequence of reduction steps will always arrive at a polynomial *h* that is in reduced form w.r.t. *F*.

Fact (Non-uniqueness): Given *g* and *F*, it may well be that

$$\underline{h}_F \longleftarrow_F^{*} g \longrightarrow_F^{*} \underline{k}_F$$

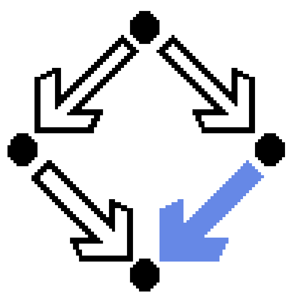
but $h \neq k$.

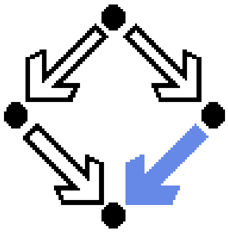
■ Definition of Gröbner Bases

Definition:

F is a Gröbner basis $:\Longleftrightarrow$ $\longrightarrow_F$ is unique.

$$\underset{g,h,k}{\forall} \left(\underline{h}_F \longleftarrow_F^{*} g \longrightarrow_F^{*} \underline{k}_F \Longrightarrow h = k \right).$$





■ How Can GB be Constructed?

■ The Problem

Given *F*, find *G* s.t. Ideal(*F*) = Ideal(*G*) and *G* is a Gröbner basis.

■ An Algorithm

This problem can be solved by the following algorithm:

Start with G:= F.
For any pair of polynomials $f_1, f_2 \in G$:

Compute the "S-polynomial" of f_1, f_2 and reduce it to a reduced form *h* w.r.t. G.

If $h = 0$, consider the next pair.

If $h \neq 0$, add *h* to *G* and iterate. →

■ S-Polynomials

```
f1 = x y - 2 y;  f2 = 2 y^2 - x^2;
```

```
S-polynomial[f1, f2] = y f1 - 1/2 x f2    // Expand
```

$\frac{x^3}{2} + (-2)\, y^2$

Note that the least common multiple of the "leading power products" of $f_1$and f_2 does not any more occur in the S-polynomial of $f_1$and f_2!

> The notion of S-polynomials is the nucleus of algorithmic Gröbner bases theory.

■ Specializations

The Gröbner bases algorithm,

for linear polynomials, specializes to Gauss' algorithm, and

for univariate polynomials, specializes to Euclid's algorithm.

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■ Why Does This Work?

■ Termination of the Algorithm

An application of "Dickson's Lemma".

■ Correctness of the Algorithm

The correctness of the algorithm is based on the following **"Main Theorem of Gröbner Bases Theory":**

> F is a Gröbner basis $:\Longleftrightarrow$
> $\underset{f_1, f_2 \in F}{\forall}$ RF[F, S–polynomial[f_1, f_2]] = 0.

Proof: Nontrivial. Combinatorial.
All details see:

- BB, PhD thesis 1965, published in aequationes mathematicae 1970, (English translation in the appendix of the Proceedings of the GB Conference 1998, see below).

- BB, Introduction to GB: Proceedings of the GB Conference 1998, see below.

The power of the Gröbner bases method is contained in this proof.

■ Examples

■ A Simple Set of Equations

■ Computation of Groebner bases

```
f1 = x y - 2 y;
f2 = 2 y^2 - x^2;
F = {f1, f2};
```

Call of the Groebner bases algorithm in *Mathematica*:

```
G = GroebnerBasis[F]
```

$\{-2x^2 + x^3, -2y + xy, -x^2 + 2y^2\}$

This is the (uniquely determined reduced) Groebner basis w.r.t. lexicographic ordering with $x \prec y$. The ordering can also be indicated explicitly:

```
G = GroebnerBasis[F, {y, x}]
```

$\{-2x^2 + x^3, -2y + xy, -x^2 + 2y^2\}$

This is the (uniquely determined reduced) Groebner basis w.r.t. lexicographic ordering with $y \prec x$:

```
GroebnerBasis[F, {x, y}]
```

$\{2y - y^3, -2y + xy, -x^2 + 2y^2\}$

Gröbner bases have - among many "nice" properties - the "elimination property"!

■ Solution of Systems Using Groebner Bases

```
g1 = G[[1]]
```

$-2x^2 + x^3$

```
Solve[{g1 == 0}, {x}]
```

{{x → 0}, {x → 0}, {x → 2}}

```
g2 = G[[2]] /. {x → 2}
```

0

```
g3 = G[[3]] /. {x → 2}
```

$-4 + 2y^2$

```
G
```

$\{-2x^2 + x^3, -2y + xy, -x^2 + 2y^2\}$

```
Solve[g3 == 0, y]
```

$\{\{y \to -\sqrt{2}\}, \{y \to \sqrt{2}\}\}$

→

■ A More Complicated Set of Equations

```
f1 = x y - 2 y z - z;
f2 = y^2 - x^2 z + x z;
f3 = z^2 - y^2 x + x;

F = {f1, f2, f3};
```

```
{time, G} = GroebnerBasis[F] // Timing
```

$\{0.03\,\text{Second}, \{z + 4z^3 - 17z^4 + 3z^5 - 45z^6 + 60z^7 - 29z^8 + 124z^9 - 48z^{10} + 64z^{11} - 64z^{12},$
$-22001z + 14361yz + 16681z^2 + 26380z^3 + 226657z^4 + 11085z^5 -$
$90346z^6 - 472018z^7 - 520424z^8 - 139296z^9 - 150784z^{10} + 490368z^{11},$
$43083y^2 - 11821z + 267025z^2 - 583085z^3 + 663460z^4 - 2288350z^5 + 2466820z^6 -$
$3008257z^7 + 4611948z^8 - 2592304z^9 + 2672704z^{10} - 1686848z^{11},$
$43083x - 118717z + 69484z^2 + 402334z^3 + 409939z^4 + 1202033z^5 - 2475608z^6 +$
$354746z^7 - 6049080z^8 + 2269472z^9 - 3106688z^{10} + 3442816z^{11}\}\}$

```
zsol = NSolve[G[[1]] == 0, z]
```

{{z → -0.331304 - 0.586934 i}, {z → -0.331304 + 0.586934 i},
{z → -0.296413 - 0.705329 i}, {z → -0.296413 + 0.705329 i}, {z → -0.163124 - 0.37694 i},
{z → -0.163124 + 0.37694 i}, {z → 0.}, {z → 0.0248919 - 0.89178 i},
{z → 0.0248919 + 0.89178 i}, {z → 0.468852}, {z → 0.670231}, {z → 1.39282}}

```
Gsub = G /. zsol[[1]]
```

{6.66134×10^{-16} - 2.35922×10^{-16} i, (-523.519 - 4967.65 i) - (4757.86 + 8428.97 i) y,
(-7846.9 - 8372.06 i) + 43083 y^2, (-16311.7 + 16611. i) + 43083 x}

```
PolynomialGCD[Gsub[[2]], Gsub[[3]]]
```

1

```
ysol = NSolve[ Gsub[[2]] == 0, y]
```

{{y → -0.473535 - 0.205184 i}}

```
ysol = NSolve[ Gsub[[3]] == 0, y]
```

{{y → -0.473535 - 0.205184 i}, {y → 0.473535 + 0.205184 i}}

```
xsol = NSolve[ Gsub[[4]] == 0, x]
```

{{x → 0.378611 - 0.385558 i}}

```
F /. zsol[[1]] /. ysol[[1]] /. xsol[[1]]
```

{1.94289×10^{-15} + 6.80012×10^{-16} i,
-1.44329×10^{-15} - 7.91034×10^{-16} i, 2.05391×10^{-15} - 4.44089×10^{-16} i}

```
S = Solve[{f1 == 0, f2 == 0, f3 == 0}, {x, y, z}];
```

```
S
```

$\{\{z\to 0, y\to 0, x\to 0\}, \{z\to 0, y\to 0, x\to 0\},$
$\{z\to \frac{1}{2111}(\text{Root}[1 - 8\#1^2 + 20\#1^3 - 17\#1^4 + 15\#1^5 - 39\#1^6 + 41\#1^7 - 20\#1^8 + 20\#1^9 - 16\#1^{10} + 4\#1^{11}\&, 1]$
$(-2100 + 5808\,\text{Root}[1 - 8\#1^2 + 20\#1^3 - 17\#1^4 + 15\#1^5 - 39\#1^6 + 41\#1^7 - 20\#1^8 + 20\#1^9 - 16\#1^{10} + 4\#1^{11}\&, 1] - 747\,\text{Root}[1 - 8\#1^2 + 20\#1^3 - 17\#1^4 + 15\#1^5 - 39\#1^6 + 41\#1^7 - 20\#1^8 + 20\#1^9 - 16\#1^{10} + 4\#1^{11}\&, 1]^2 +$
$4783\,\text{Root}[1 - 8\#1^2 + 20\#1^3 - 17\#1^4 + 15\#1^5 - 39\#1^6 + 41\#1^7 - 20\#1^8 + 20\#1^9 - 16\#1^{10} + 4\#1^{11}\&, 1]^3 - 14296\,\text{Root}[1 - 8\#1^2 + 20\#1^3 - 17\#1^4 + 15\#1^5 - 39\#1^6 + 41\#1^7 - 20\#1^8 + 20\#1^9 - 16\#1^{10} + 4\#1^{11}\&, 1]^4 +$

```
    2924 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 1]^5 - 3920 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
          15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 1]^6 +
    5804 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 1]^7 - 880 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
          15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 1]^8 -
    176 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 1]^10)),
y -> 1/2111 (-1056 - 264 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
        41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 1] +
   2049 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 1]^2 -
   1081 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 1]^3 +
   266 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 1]^4 -
   2052 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 1]^5 +
   562 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 1]^6 +
   120 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 1]^7 +
   40 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 1]^8 +
   8 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 1]^10),
x -> Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
   20 #1^9 - 16 #1^10 +
   4 #1^11 &, 1]},
{z -> 1/2111 (Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 -
       20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 2]
   (-2100 + 5808 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
          41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 2] -
    747 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
          16 #1^10 + 4 #1^11 &, 2]^2 + 4783 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
          15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 2]^3 -
    14296 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 2]^4 + 2924 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
          15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 2]^5 -
    3920 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 2]^6 + 5804 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
          15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 2]^7 -
    880 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
          16 #1^10 + 4 #1^11 &, 2]^8 - 176 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
          15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 2]^10)),
y -> 1/2111 (-1056 - 264 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
        41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 2] +
```

```
    2049 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 2]^2 -
    1081 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 2]^3 +
    266 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 2]^4 -
    2052 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 2]^5 +
    562 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 2]^6 +
    120 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 2]^7 +
    40 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 2]^8 +
    8 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 2]^10),
  x -> Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
    20 #1^9 - 16 #1^10 +
    4 #1^11 &, 2]},
 {z -> 1/2111 (Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 -
        20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 3]
    (-2100 + 5808 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
          41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 3] -
      747 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
            16 #1^10 + 4 #1^11 &, 3]^2 + 4783 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
            15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 3]^3 -
      14296 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
            20 #1^9 - 16 #1^10 + 4 #1^11 &, 3]^4 + 2924 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
            15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 3]^5 -
      3920 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
            20 #1^9 - 16 #1^10 + 4 #1^11 &, 3]^6 + 5804 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
            15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 3]^7 -
      880 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
            16 #1^10 + 4 #1^11 &, 3]^8 - 176 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
            15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 3]^10)),
  y -> 1/2111 (-1056 - 264 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
        41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 3] +
    2049 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 3]^2 -
    1081 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 3]^3 +
    266 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 3]^4 -
    2052 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 3]^5 +
    562 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 3]^6 +
    120 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
```

```
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 3]^7 +
    40 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 3]^8 +
    8 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 3]^10),
 x → Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
    20 #1^9 - 16 #1^10 +
    4 #1^11 &, 3]},
{z → 1/2111 (Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 -
      20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 4]
    (-2100 + 5808 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
          41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 4] -
      747 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
           16 #1^10 + 4 #1^11 &, 4]^2 + 4783 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
           15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 4]^3 -
      14296 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
           20 #1^9 - 16 #1^10 + 4 #1^11 &, 4]^4 + 2924 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
           15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 4]^5 -
      3920 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
           20 #1^9 - 16 #1^10 + 4 #1^11 &, 4]^6 + 5804 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
           15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 4]^7 -
      880 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
           16 #1^10 + 4 #1^11 &, 4]^8 - 176 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
           15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 4]^10)),
 y → 1/2111 (-1056 - 264 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
        41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 4] +
    2049 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 4]^2 -
    1081 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 4]^3 +
    266 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 4]^4 -
    2052 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 4]^5 +
    562 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 4]^6 +
    120 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 4]^7 +
    40 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 4]^8 +
    8 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 4]^10),
 x → Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
    20 #1^9 -
    16 #1^10 +
    4 #1^11 &, 4]},
{z → 1/2111 (Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 -
      20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 5]
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(-2100 + 5808 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
      41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 5] -
   747 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
        16 #1^10 + 4 #1^11 &, 5]^2 + 4783 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
        15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 5]^3 -
   14296 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 5]^4 + 2924 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
        15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 5]^5 -
   3920 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 5]^6 + 5804 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
        15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 5]^7 -
   880 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
        16 #1^10 + 4 #1^11 &, 5]^8 - 176 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
        15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 5]^10)),
y -> 1/2111 (-1056 - 264 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
      41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 5] +
   2049 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 5]^2 -
   1081 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 5]^3 +
   266 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 5]^4 -
   2052 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 5]^5 +
   562 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 5]^6 +
   120 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 5]^7 +
   40 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 5]^8 +
   8 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 5]^10),
x -> Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
   20 #1^9 -
   16 #1^10 +
   4 #1^11 &, 5]},
{z -> 1/2111 (Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 -
      20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]
   (-2100 + 5808 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
      41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 6] -
   747 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
        16 #1^10 + 4 #1^11 &, 6]^2 + 4783 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
        15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]^3 -
   14296 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]^4 + 2924 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
        15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]^5 -
   3920 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]^6 + 5804 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
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            15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]^7 -
        880 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
            16 #1^10 + 4 #1^11 &, 6]^8 - 176 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
            15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]^10)),
  y -> 1/2111 (-1056 - 264 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
          41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 6] +
      2049 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]^2 -
      1081 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]^3 +
      266 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]^4 -
      2052 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]^5 +
      562 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]^6 +
      120 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]^7 +
      40 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]^8 +
      8 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]^10),
  x -> Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
      20 #1^9 -
      16 #1^10 +
      4 #1^11 &, 6]},
 {z -> 1/2111 (Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 -
         20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]
      (-2100 + 5808 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
            41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 7] -
        747 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
            16 #1^10 + 4 #1^11 &, 7]^2 + 4783 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
            15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^3 -
        14296 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
            20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^4 + 2924 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
            15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^5 -
        3920 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
            20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^6 + 5804 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
            15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^7 -
        880 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
            16 #1^10 + 4 #1^11 &, 7]^8 - 176 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
            15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^10)),
  y -> 1/2111 (-1056 - 264 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
          41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 7] +
      2049 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^2 -
      1081 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
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          20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^3 +
      266 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^4 -
      2052 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^5 +
      562 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^6 +
      120 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^7 +
      40 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^8 +
      8 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^10),
   x -> Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
      20 #1^9 -
      16 #1^10 +
      4 #1^11 &, 7]},
  {z -> 1/2111 (Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 -
         20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]
      (-2100 + 5808 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
            41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 8] -
        747 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
            16 #1^10 + 4 #1^11 &, 8]^2 + 4783 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
            15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^3 -
        14296 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
            20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^4 + 2924 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
            15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^5 -
        3920 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
            20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^6 + 5804 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
            15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^7 -
        880 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
            16 #1^10 + 4 #1^11 &, 8]^8 - 176 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
            15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^10)),
   y -> 1/2111 (-1056 - 264 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
         41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 8] +
      2049 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^2 -
      1081 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^3 +
      266 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^4 -
      2052 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^5 +
      562 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^6 +
      120 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^7 +
      40 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
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        20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^8 +
    8 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^10),
  x → Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
    20 #1^9 -
    16 #1^10 +
    4 #1^11 &, 8]},
 {z → 1/2111 (Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 -
      20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]
    (-2100 + 5808 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
        41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 9] -
     747 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
         16 #1^10 + 4 #1^11 &, 9]^2 + 4783 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
         15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^3 -
     14296 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
         20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^4 + 2924 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
         15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^5 -
     3920 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
         20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^6 + 5804 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
         15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^7 -
     880 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
         16 #1^10 + 4 #1^11 &, 9]^8 - 176 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
         15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^10)),
  y → 1/2111 (-1056 - 264 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
       41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 9] +
    2049 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^2 -
    1081 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^3 +
    266 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^4 -
    2052 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^5 +
    562 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^6 +
    120 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^7 +
    40 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^8 +
    8 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^10),
  x → Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
    20 #1^9 -
    16 #1^10 +
    4 #1^11 &, 9]},
 {z → 1/2111 (Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 -
      20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]
    (-2100 + 5808 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 -
```

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          20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 10] -
        747 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
            16 #1^10 + 4 #1^11 &, 10]^2 + 4783 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
            15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^3 -
        14296 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
            20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^4 + 2924 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
            15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^5 -
        3920 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
            20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^6 + 5804 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
            15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^7 -
        880 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
            16 #1^10 + 4 #1^11 &, 10]^8 - 176 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
            15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^10)),
  y -> 1/2111 (-1056 - 264 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
         41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 10] +
      2049 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^2 -
      1081 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^3 +
      266 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^4 -
      2052 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^5 +
      562 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^6 +
      120 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^7 +
      40 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^8 +
      8 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^10),
  x -> Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
      20 #1^9 -
      16 #1^10 +
      4 #1^11 &, 10]},
 {z -> 1/2111 (Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 -
         20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]
      (-2100 + 5808 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 -
          20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 11] -
        747 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
            16 #1^10 + 4 #1^11 &, 11]^2 + 4783 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
            15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^3 -
        14296 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
            20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^4 + 2924 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
            15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^5 -
        3920 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
            20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^6 + 5804 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
            15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^7 -
```

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      880 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
          16 #1^10 + 4 #1^11 &, 11]^8 - 176 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
          15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^10)),
 y -> 1/2111 (-1056 - 264 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
        41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 11] +
    2049 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^2 -
    1081 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^3 +
    266 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^4 -
    2052 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^5 +
    562 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^6 +
    120 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^7 +
    40 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^8 +
    8 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^10),
 x -> Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
    20 #1^9 -
    16 #1^10 +
    4 #1^11 &, 11]}}
```

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S /.
 {Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &,
     1] -> α1,
   Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &,
     2] -> α2,
   Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &,
     3] -> α3,
   Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &,
     4] -> α4,
   Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &,
     5] -> α5}
```

```
{{z -> 0, y -> 0, x -> 0}, {z -> 0, y -> 0, x -> 0},
 {z -> 1/2111 (α1 (-2100 + 5808 α1 - 747 α1^2 + 4783 α1^3 -
        14296 α1^4 + 2924 α1^5 - 3920 α1^6 + 5804 α1^7 - 880 α1^8 - 176 α1^10)),
  y -> 1/2111 (-1056 - 264 α1 + 2049 α1^2 - 1081 α1^3 + 266 α1^4 - 2052 α1^5 +
      562 α1^6 + 120 α1^7 + 40 α1^8 + 8 α1^10), x -> α1},
 {z -> 1/2111 (α2 (-2100 + 5808 α2 - 747 α2^2 + 4783 α2^3 - 14296 α2^4 +
        2924 α2^5 - 3920 α2^6 + 5804 α2^7 - 880 α2^8 - 176 α2^10)),
  y -> 1/2111 (-1056 - 264 α2 + 2049 α2^2 - 1081 α2^3 + 266 α2^4 - 2052 α2^5 +
      562 α2^6 + 120 α2^7 + 40 α2^8 + 8 α2^10), x -> α2},
```

```
{z → 1/2111 (α3 (-2100 + 5808 α3 - 747 α3^2 + 4783 α3^3 - 14296 α3^4 +
      2924 α3^5 - 3920 α3^6 + 5804 α3^7 - 880 α3^8 - 176 α3^10)),
  y → 1/2111 (-1056 - 264 α3 + 2049 α3^2 - 1081 α3^3 + 266 α3^4 - 2052 α3^5 +
     562 α3^6 + 120 α3^7 + 40 α3^8 + 8 α3^10), x → α3},
{z → 1/2111 (α4 (-2100 + 5808 α4 - 747 α4^2 + 4783 α4^3 - 14296 α4^4 +
      2924 α4^5 - 3920 α4^6 + 5804 α4^7 - 880 α4^8 - 176 α4^10)),
  y → 1/2111 (-1056 - 264 α4 + 2049 α4^2 - 1081 α4^3 + 266 α4^4 - 2052 α4^5 +
     562 α4^6 + 120 α4^7 + 40 α4^8 + 8 α4^10), x → α4},
{z → 1/2111 (α5 (-2100 + 5808 α5 - 747 α5^2 + 4783 α5^3 - 14296 α5^4 +
      2924 α5^5 - 3920 α5^6 + 5804 α5^7 - 880 α5^8 - 176 α5^10)),
  y → 1/2111 (-1056 - 264 α5 + 2049 α5^2 - 1081 α5^3 + 266 α5^4 - 2052 α5^5 + 562 α5^6 +
     120 α5^7 + 40 α5^8 + 8 α5^10), x → α5}, {z → 1/2111
   (Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &,
      6] (-2100 + 5808 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
            20 #1^9 - 16 #1^10 + 4 #1^11 &, 6] - 747 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 -
             39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]^2 + 4783 Root[1 - 8 #1^2 +
             20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]^3 -
        14296 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
             16 #1^10 + 4 #1^11 &, 6]^4 + 2924 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
             41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]^5 - 3920 Root[1 - 8 #1^2 + 20 #1^3 -
             17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]^6 +
        5804 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
             16 #1^10 + 4 #1^11 &, 6]^7 - 880 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
             41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]^8 - 176 Root[1 - 8 #1^2 + 20 #1^3 -
             17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]^10)),
  y → 1/2111 (-1056 - 264 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
          41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 6] +
     2049 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
           16 #1^10 + 4 #1^11 &, 6]^2 - 1081 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
           15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]^3 +
     266 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
           16 #1^10 + 4 #1^11 &, 6]^4 - 2052 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
           15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]^5 +
     562 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
           16 #1^10 + 4 #1^11 &, 6]^6 + 120 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
           15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]^7 +
     40 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
           16 #1^10 + 4 #1^11 &, 6]^8 + 8 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 -
           39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]^10),
  x → Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
     20 #1^9 - 16 #1^10 + 4 #1^11 &, 6]},
{z → 1/2111 (Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 -
        20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]
     (-2100 + 5808 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
```

```
20 #1^9 - 16 #1^10 + 4 #1^11 &, 7] - 747 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
    15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^2 +
 4783 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
    20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^3 - 14296 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
    15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^4 +
 2924 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
    20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^5 - 3920 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
    15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^6 +
 5804 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
    16 #1^10 + 4 #1^11 &, 7]^7 - 880 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
    41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^8 - 176 Root[1 - 8 #1^2 + 20 #1^3 -
    17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^10)),
y -> 1/2111 (-1056 - 264 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
    41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 7] +
  2049 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
     16 #1^10 + 4 #1^11 &, 7]^2 - 1081 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
     15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^3 +
  266 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
     16 #1^10 + 4 #1^11 &, 7]^4 - 2052 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
     15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^5 +
  562 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
     16 #1^10 + 4 #1^11 &, 7]^6 + 120 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
     15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^7 +
  40 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
     16 #1^10 + 4 #1^11 &, 7]^8 + 8 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 -
     39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]^10),
x -> Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
  20 #1^9 - 16 #1^10 + 4 #1^11 &, 7]},
{z -> 1/2111 (Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 -
     20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]
  (-2100 + 5808 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
     20 #1^9 - 16 #1^10 + 4 #1^11 &, 8] - 747 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
     15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^2 +
   4783 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
     20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^3 - 14296 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
     15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^4 +
   2924 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
     20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^5 - 3920 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
     15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^6 +
   5804 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
     16 #1^10 + 4 #1^11 &, 8]^7 - 880 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
     41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^8 - 176 Root[1 - 8 #1^2 + 20 #1^3 -
     17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^10)),
y -> 1/2111 (-1056 - 264 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
    41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 8] +
  2049 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
```

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          20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^2 -
      1081 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^3 +
      266 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^4 -
      2052 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^5 +
      562 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^6 +
      120 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^7 +
      40 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^8 +
      8 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]^10),
   x -> Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
      20 #1^9 - 16 #1^10 + 4 #1^11 &, 8]},
  {z -> 1/2111 (Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 -
         20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]
      (-2100 + 5808 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
           20 #1^9 - 16 #1^10 + 4 #1^11 &, 9] - 747 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
            15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^2 +
        4783 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
            20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^3 - 14296 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
            15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^4 +
        2924 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
            20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^5 - 3920 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
            15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^6 +
        5804 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
            16 #1^10 + 4 #1^11 &, 9]^7 - 880 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
            41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^8 - 176 Root[1 - 8 #1^2 + 20 #1^3 -
            17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^10)),
   y -> 1/2111 (-1056 - 264 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
          41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 9] +
      2049 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^2 -
      1081 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^3 +
      266 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^4 -
      2052 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^5 +
      562 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^6 +
      120 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^7 +
      40 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
          20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^8 +
```

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    8 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]^10),
  x -> Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
      20 #1^9 - 16 #1^10 + 4 #1^11 &, 9]},
 {z -> 1/2111 (Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 -
       20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]
     (-2100 + 5808 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 -
          20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 10] -
      747 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
           16 #1^10 + 4 #1^11 &, 10]^2 + 4783 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
           15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^3 -
      14296 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
           20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^4 + 2924 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
           15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^5 -
      3920 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
           20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^6 + 5804 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
           15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^7 -
      880 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
           16 #1^10 + 4 #1^11 &, 10]^8 - 176 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
           15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^10)),
  y -> 1/2111 (-1056 - 264 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
        41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 10] +
    2049 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^2 -
    1081 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^3 +
    266 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^4 -
    2052 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^5 +
    562 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^6 +
    120 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^7 +
    40 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^8 +
    8 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 10]^10),
  x -> Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
      20 #1^9 - 16 #1^10 +
      4 #1^11 &, 10]},
 {z -> 1/2111 (Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 -
       20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]
     (-2100 + 5808 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 -
          20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 11] -
      747 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
           16 #1^10 + 4 #1^11 &, 11]^2 + 4783 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
```

```
        15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^3 -
    14296 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^4 + 2924 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
        15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^5 -
    3920 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^6 + 5804 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
        15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^7 -
    880 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 -
        16 #1^10 + 4 #1^11 &, 11]^8 - 176 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 +
        15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^10)),
y -> 1/2111 (-1056 - 264 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 +
      41 #1^7 - 20 #1^8 + 20 #1^9 - 16 #1^10 + 4 #1^11 &, 11] +
    2049 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^2 -
    1081 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^3 +
    266 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^4 -
    2052 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^5 +
    562 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^6 +
    120 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^7 +
    40 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^8 +
    8 Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
        20 #1^9 - 16 #1^10 + 4 #1^11 &, 11]^10),
x -> Root[1 - 8 #1^2 + 20 #1^3 - 17 #1^4 + 15 #1^5 - 39 #1^6 + 41 #1^7 - 20 #1^8 +
    20 #1^9 - 16 #1^10 +
    4 #1^11 &, 11]}}
```

■ Algebraic Relations

Can

$$x_1^7 x_2 - x_1 x_2^7$$

be expressed as a polynomial in

$$x_1^2 + x_2^2$$
$$x_1^2 x_2^2$$
$$x_1^3 x_2 - x_1 x_2^3$$

?

Note: The above polynomials forms a system of fundamental invariants for $\mathbb{Z}_4$, i.e. a set of generators for the ring

```
{f ∈ ℂ[x1, x2] | f (x1, x2) = f (-x2, x1)},
```

i.e.

```
{x1^2 + x2^2, x1^2 x2^2, x1^3 x2 - x1 x2^3} /. {x1 → -x2, x2 → x1}
```

$\{x_1^2+x_2^2,\ x_1^2 x_2^2,\ x_1^3 x_2 - x_1 x_2^3\}$

and all invariants can be expressed as polynomials in these invariants.

```
{time, GB} = GroebnerBasis[
   {-i1 + x1^2 + x2^2, -i2 + x1^2 x2^2, -i3 + x1^3 x2 - x1 x2^3}, {x2, x1, i3, i2, i1}] // Timing
```

$\{0.06\ \text{Second},\ \{-i_1^2 i_2 + 4 i_2^2 + i_3^2,\ -i_2 + i_1 x_1^2 - x_1^4,\ i_1^2 i_3 x_1 - 2 i_2 i_3 x_1 - i_1 i_3 x_1^3 + i_1^2 i_2 x_2 - 4 i_2^2 x_2,\ i_1^2 x_1 - 2 i_2 x_1 - i_1 x_1^3 + i_3 x_2,\ -i_1 i_3 + 2 i_3 x_1^2 - i_1^2 x_1 x_2 + 4 i_2 x_1 x_2,\ -i_3 x_1 - 2 i_2 x_2 + i_1 x_1^2 x_2,\ -i_3 - i_1 x_1 x_2 + 2 x_1^3 x_2,\ -i_1 + x_1^2 + x_2^2\}\}$

```
PolynomialReduce[x1^7 x2 - x1 x2^7, GB,
 {x2, x1, i3, i2, i1}, MonomialOrder → Lexicographic]
```

$\left\{\left\{0,\ -i_3 - \frac{1}{2} i_1 x_1 x_2 - x_1^3 x_2,\ 0,\ \frac{3 i_1 x_2}{4} - \frac{1}{2} x_1^2 x_2 + \frac{x_2^3}{2},\ i_1 - \frac{x_1^2}{2} + \frac{3 x_2^2}{4},\ \frac{3 i_1 x_1}{2} + x_1 x_2^2,\ \frac{x_2^4}{2},\ -\frac{1}{4} i_1^2 x_1 x_2 - \frac{1}{2} i_1 x_1 x_2^3 - x_1 x_2^5\right\},\ i_1^2 i_3 - i_2 i_3\right\}$

```
i1^2 i3 - i2 i3 /. {i1 → x1^2 + x2^2, i2 → x1^2 x2^2, i3 → x1^3 x2 - x1 x2^3} // Expand
```

$x_1^7 x_2 - x_1 x_2^7$

```
R = PolynomialReduce[x1^6 x2 - x1 x2^6,
  GB, {x2, x1, i3, i2, i1}, MonomialOrder → Lexicographic]
```

$\left\{\left\{0,\ \frac{i_1 x_1}{2} - i_1 x_2 - x_1^2 x_2,\ 0,\ \frac{3 i_1}{4} - \frac{x_1^2}{2} + \frac{x_2^2}{2},\ -\frac{x_1}{4} + \frac{3 x_2}{4},\ \frac{3 i_1}{4} + x_1 x_2,\ \frac{x_2^3}{2},\ -\frac{1}{4} i_1^2 x_1 - \frac{1}{2} i_1 x_1 x_2^2 - x_1 x_2^4\right\},\ -i_1^3 x_1 + 2 i_1 i_2 x_1 + \frac{1}{2} i_1 i_3 x_1 + i_1^2 x_1^3 - i_2 x_1^3 + \frac{1}{2} i_3 x_1^3 + \frac{1}{2} i_1 i_2 x_2\right\}$

```
R[[1]]
```

$\left\{0,\ \frac{i_1 x_1}{2} - i_1 x_2 - x_1^2 x_2,\ 0,\ \frac{3 i_1}{4} - \frac{x_1^2}{2} + \frac{x_2^2}{2},\ -\frac{x_1}{4} + \frac{3 x_2}{4},\ \frac{3 i_1}{4} + x_1 x_2,\ \frac{x_2^3}{2},\ -\frac{1}{4} i_1^2 x_1 - \frac{1}{2} i_1 x_1 x_2^2 - x_1 x_2^4\right\}$

```
GB
```

$\{-i_1^2\, i_2 + 4\, i_2^2 + i_3^2,\ -i_2 + i_1\, x_1^2 - x_1^4,\ i_1^2\, i_3\, x_1 - 2\, i_2\, i_3\, x_1 - i_1\, i_3\, x_1^3 + i_1^2\, i_2\, x_2 - 4\, i_2^2\, x_2,$
$i_1^2\, x_1 - 2\, i_2\, x_1 - i_1\, x_1^3 + i_3\, x_2,\ -i_1\, i_3 + 2\, i_3\, x_1^2 - i_1^2\, x_1\, x_2 + 4\, i_2\, x_1\, x_2,$
$-i_3\, x_1 - 2\, i_2\, x_2 + i_1\, x_1^2\, x_2,\ -i_3 - i_1\, x_1\, x_2 + 2\, x_1^3\, x_2,\ -i_1 + x_1^2 + x_2^2\}$

```
R[[1]].GB // Expand
```

$i_1^3\, x_1 - 2\, i_1\, i_2\, x_1 - \frac{1}{2}\, i_1\, i_3\, x_1 - i_1^2\, x_1^3 + i_2\, x_1^3 - \frac{1}{2}\, i_3\, x_1^3 - \frac{1}{2}\, i_1\, i_2\, x_2 + x_1^6\, x_2 - x_1\, x_2^6$

```
R[[2]]
```

$-i_1^3\, x_1 + 2\, i_1\, i_2\, x_1 + \frac{1}{2}\, i_1\, i_3\, x_1 + i_1^2\, x_1^3 - i_2\, x_1^3 + \frac{1}{2}\, i_3\, x_1^3 + \frac{1}{2}\, i_1\, i_2\, x_2$

```
x1^6 x2 - x1 x2^6 /. {x1 -> -x2, x2 -> x1}
```

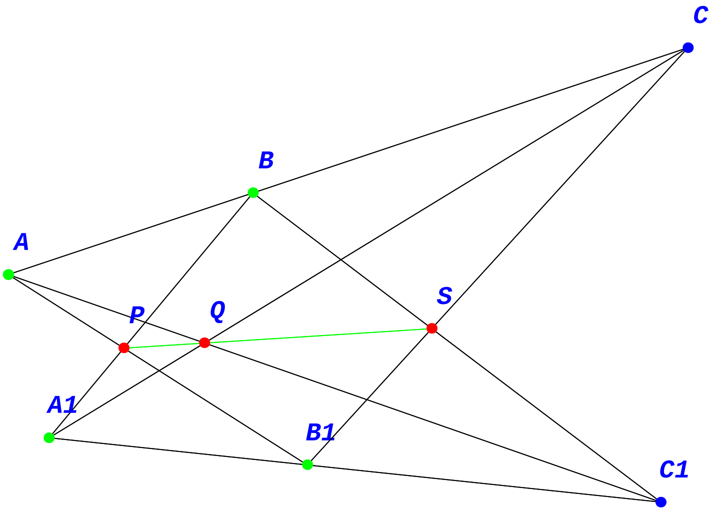
$x_1^6\, x_2 + x_1\, x_2^6$

→

■ The Groebner Bases Automated Prover for Coordinate Geometry

■ Example Pappus Theorem: Input to Theorema

We want to prove / disprove the following conjecture: Let A,B, C and A1,B1, C1 be on two lines and P = A B1 ⋂ A1B, Q = AC1 ⋂ A1C, S = BC1 ⋂ B1C. Then P, Q, and S are collinear.



For this we only have to input the conjecture into Theorema in the following form:

```
Proposition["Pappus", any[A, B, A1, B1, C, C1, P, Q, S],
 point[A, B, A1, B1] ∧ pon[C, line[A, B]] ∧ pon[C1, line[A1, B1]] ∧
   inter[P, line[A, B1], line[A1, B]] ∧ inter[Q, line[A, C1], line[A1, C]] ∧
   inter[S, line[B, C1], line[B1, C]] ⇒ collinear[P, Q, S]]
```

and then to call the Theorema Groebner bases prover by the following command:

```
Prove[Proposition["Pappus"], by → GeometryProver,
 ProverOptions → {Method -> "GroebnerProver", Refutation → True}]
```

■ Proof Generatet Automatical by Theorema

This call will generate, completely automatically, the following proof (which also contains an explanation of the proof method by Groebner bases theory):

Prove:

(Proposition (Pappus))

$$\underset{A,B,A1,B1,C,C1,P,Q,S}{\forall}(\text{point}[A, B, A1, B1] \wedge \text{pon}[C, \text{line}[A, B]] \wedge \text{pon}[C1, \text{line}[A1, B1]] \wedge \text{inter}[P, \text{line}[A, B1], \text{line}[A1, B]] \wedge \text{inter}[Q, \text{line}[A, C1], \text{line}[A1, C]] \wedge \text{inter}[S, \text{line}[B, C1], \text{line}[B1, C]] \Rightarrow \text{collinear}[P, Q, S])$$

with no assumptions.

To prove the above statement we shall use the Gröbner basis method. First we have to transform the problem into algebraic form.

Algebraic Form:

To transform the geometric problem into algebraic form we have to chose first an orthogonal coordinate system.

Let's have the origin in point *A*, and points { } and {*B*, *C*} on the two axes.

Using this coordinate system we have the following points:

$\{\{A, 0, 0\}, \{B, 0, u_1\}, \{A1, u_2, u_3\}, \{B1, u_4, u_5\}, \{C, 0, u_6\}, \{C1, u_7, x_1\}, \{P, x_2, x_3\}, \{Q, x_4, x_5\}, \{S, x_6, x_7\}\}$

The algebraic form of the given construction is:

(1)

$$\underset{x_1,x_2,x_3,x_4,x_5,x_6,x_7}{\forall} (u_3 u_4 + -u_2 u_5 + -u_3 u_7 + u_5 u_7 + u_2 x_1 + -u_4 x_1 == 0 \wedge u_5 x_2 + -u_4 x_3 == 0 \wedge -u_1 u_2 + u_1 x_2 + -u_3 x_2 + u_2 x_3 == 0 \wedge x_1 x_4 + -u_7 x_5 == 0 \wedge -u_2 u_6 + -u_3 x_4 + u_6 x_4 + u_2 x_5 == 0 \wedge u_1 u_7 + -u_1 x_6 + x_1 x_6 + -u_7 x_7 == 0 \wedge -u_4 u_6 + -u_5 x_6 + u_6 x_6 + u_4 x_7 == 0 \Rightarrow x_3 x_4 + -x_2 x_5 + -x_3 x_6 + x_5 x_6 + x_2 x_7 + -x_4 x_7 == 0)$$

This problem is equivalent to:

(2)

$$\neg \left(\underset{x_1,x_2,x_3,x_4,x_5,x_6,x_7}{\exists} (u_3 u_4 + -u_2 u_5 + -u_3 u_7 + u_5 u_7 + u_2 x_1 + -u_4 x_1 == 0 \wedge u_5 x_2 + -u_4 x_3 == 0 \wedge -u_1 u_2 + u_1 x_2 + -u_3 x_2 + u_2 x_3 == 0 \wedge x_1 x_4 + -u_7 x_5 == 0 \wedge -u_2 u_6 + -u_3 x_4 + u_6 x_4 + u_2 x_5 == 0 \wedge u_1 u_7 + -u_1 x_6 + x_1 x_6 + -u_7 x_7 == 0 \wedge -u_4 u_6 + -u_5 x_6 + u_6 x_6 + u_4 x_7 == 0 \wedge x_3 x_4 + -x_2 x_5 + -x_3 x_6 + x_5 x_6 + x_2 x_7 + -x_4 x_7 \neq 0)\right)$$

To remove the last inequality, we use the well-known Rabinowitsch trick. Let v_0 be a new variable. Then the problem becomes:

(3)

$$\neg \left(\underset{x_1,x_2,x_3,x_4,x_5,x_6,x_7,v_0}{\exists} (u_3 u_4 + -u_2 u_5 + -u_3 u_7 + u_5 u_7 + u_2 x_1 + -u_4 x_1 == 0 \wedge u_5 x_2 + -u_4 x_3 == 0 \wedge -u_1 u_2 + u_1 x_2 + -u_3 x_2 + u_2 x_3 == 0 \wedge x_1 x_4 + -u_7 x_5 == 0 \wedge -u_2 u_6 + -u_3 x_4 + u_6 x_4 + u_2 x_5 == 0 \wedge u_1 u_7 + -u_1 x_6 + x_1 x_6 + -u_7 x_7 == 0 \wedge -u_4 u_6 + -u_5 x_6 + u_6 x_6 + u_4 x_7 == 0 \wedge 1 + -v_0 (x_3 x_4 + -x_2 x_5 + -x_3 x_6 + x_5 x_6 + x_2 x_7 + -x_4 x_7) == 0)\right)$$

To prove this statement we have to compute the Gröbner bases of the above polynomials.

The polynomials of the Gröbner bases are: {1}

As the obtained Gröbner bases is {1} the statement is generically true.

Statistics:

The length of the Gröbner bases is 1.

Time needed to compute the Gröbner bases: 0.42 Seconds.

The proof illustrates that the proving algorithm proceeds by, first, transforming the proof problem into the problem of constructing a Groebner bases for a certain set of polynomials and checking whether this Groebner basis is equal to {1} or not. Since the construction of Groebner is possible by an algorithm, this decision can be done algorithmically.

■ Why are GB Called Gröbner Bases?

Professor Wolfgang Gröbner was my PhD thesis supervisor.

In my thesis (1965) and journal publication (1970), reprinted in the GB-98 Proceedings, I introduced:

* the concept of Gröbner bases and reduced Gröbner bases
* the S-polynomials
* the main theorem with proof
* the algorithm with termination and correctness proof
* the uniqueness of Gröbner bases
* first applications (computing in residue rings, Hilbert function, algebraic systems)
* the technique of base-change w.r.t. to different orderings
* a complete computer implementation
* first complexity considerations

However, in the thesis, I did not use the name "Gröbner bases". I introduced this name only in 1976, when people started to become interested in my work.

My later contributions:

* the technique of criteria for eliminating unnecessary reductions
* an abstract characterization of "Gröbner bases rings"

■ Where Do We Find Info on GB?

■ Gröbner Bases 98 Conference

B. Buchberger, F. Winkler. Gröbner *Bases: Theory and Applications.* Cambridge University Press, 1998. 560 pages.

This book contains tutorials and original papers.

This book contains also:

B. Buchberger. *Introduction to Gröbner Bases*, pp. 3-31.

B. Buchberger. *An Algorithmic Criterion for the Solvability of Systems of Algebraic Equations, pp. 540-560.* (English translation of the original paper from 1970, in which Gröbner bases were introduced.)

A continuation of this book is the special issue of the JSC on Gröbner bases edited by Q.N. Tran and F. Winkler, 2000.

■ On Your Desk

GB implementations are contained in all the current math software systems like *Mathematica* (see demo), Maple, Magma, Macsyma, Axiom, Derive, Reduce, Mupad, ...

Software systems specialized on Gröbner bases: CoCoA, Macaulay, Singular, ...

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▪ In Your Palm

Gröbner bases are now availabe on the TI-92 (implemented in Derive).

▪ Textbooks

T. Kreuzer, L. Robbiano: *Algorithmic Commutative Algebra I.* Springer, Heidelber, 2000: Contains a list of all other textbooks on GB.

W.W.Adams, P. Loustenau. *Introduction to* Gröbner *Bases*. Graduate Studies in Mathematics: Amer. Math. Soc., Providence, R.I., 1994.

T.Becker, V.Weispfenning. Gröbner *Bases: A Computational Approach to Commutative Algebra.* Springer, New York, 1993.

D.Cox, J.Little, D.O'Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra.* Springer, New York, 1992.

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▪ In the Web

Search. E.g. in the Research Index you obtain ~ 3000 citations.

▪ Original Publications

Approximately 500 papers appeared meanwhile on Gröbner bases.

J of Symbolic Computation, in particular, special issues.

ISSAC Conferences.

Mega Conferences.

...

The essential additional original ideas in the literature:

- Gröbner bases can be constructed w.r.t. arbitrary "admissible" orderings (W. Trinks 1978)
- Gröbner bases w.r.t. to "lexical" orderings have the elimination property (W. Trinks 1978)
- Gröbner bases can be used for computing syzygies and the S-polys generate the module of syzygies (G. Zacharias 1978)
- A given F, w.r.t. the *infinitely* many admissible orderings, has only *finitely* many Gröbner bases and, hence, we can construct a "universal" Gröbner bases for F (L. Robbiano, V. Weispfenning, T. Schwarz 1988)
- Starting from a Gröbner bases for F for ordering O_1 one can "walk", by changing the basis only slightly, to a basis for a "nearby" ordering O_2 and so on ... until one arrives at a Gröbner bases for a desired ordering O_k (Kalkbrener, Mall 1995).

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