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The summation package Sigma simplifies harmonic sum expressions

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Warmup example (bonus problem 6.69 in "Concrete Mathematics")

FIND a closed form for

$$\sum_{k=1}^{n} k^2 \mathsf{H}_{n+k} = ?,$$

where $\mathsf{H}_n := \sum_{k=1}^{n} \frac{1}{k}$.

Telescoping

GIVEN $f(k) = k^2 H_{n+k}$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

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Sigma computes

$$g(k) = \frac{1}{36}(k(-4k^2 + (6n+3)k - 12n(n+1) + 1) \\ + 6(2k^3 - 3k^2 + k + n(n+1)(2n+1))H_{k+n}).$$

Telescoping

GIVEN $f(k) = k^2 H_{n+k}$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Summing this equation over k from 1 to n gives

$$\boxed{\sum_{k=1}^{n} k^2 \mathsf{H}_{n+k}} = \boxed{g(n+1) - g(1)}$$

$$= -\frac{1}{36} n(n+1)\left(10n + 6(2n+1)H_n - 12(2n+1)H_{2n} - 1\right).$$

A problem from quadratic Padé approximation of $log(x)$

For all $n \geq 0$,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\left[3(H_{n-k}-H_k)^2+H_{n-k}^{(2)}+H_k^{(2)}\right]=0$$

where

$$H_k=\sum_{i=1}^{k}\frac{1}{i},\quad H_k^{(2)}=\sum_{i=1}^{k}\frac{1}{i^2}.$$

A problem from quadratic Padé approximation of $log(x)$

Theorem (Sigma; 2002)

For all $n \geq 0$,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\left[3(H_{n-k}-H_k)^2+H_{n-k}^{(2)}+H_k^{(2)}\right]=0$$

where

$$H_k=\sum_{i=1}^{k}\frac{1}{i},\quad H_k^{(2)}=\sum_{i=1}^{k}\frac{1}{i^2}.$$

Proof.

Sigma

□

Details

Proving $\xrightarrow{\text{Sigma}}$ Finding

$$\sum_{k=0}^{2n}\binom{2n}{k}^3(-1)^k H_k H_{2n-k} = \frac{(3n)!(-1)^n}{n!n!n!}\frac{1}{12}\big(3H_n^2 - 6H_nH_{3n} + 3H_{3n}^2 + H_n^{(2)} + 12H_{2n}(H_{2n}+H_n-H_{3n}) + 4H_{2n}^{(2)} - 3H_{3n}^{(2)}\big)$$

$$\sum_{k=0}^{2n}\binom{2n}{k}^3(-1)^k H_k^2 = \frac{(3n)!(-1)^n}{n!n!n!}\frac{1}{12}\big(3H_n^2 - 6H_nH_{3n} + 3H_{3n}^2 - H_n^{(2)} + 12H_{2n}(H_{2n}+H_n-H_{3n}) + 2H_{2n}^{(2)} - 3H_{3n}^{(2)}\big)$$

$$\sum_{k=0}^{2n}\binom{2n}{k}^3(-1)^k H_k^{(2)} = \frac{1}{2}\frac{(3n)!(-1)^n}{n!n!n!}\big(H_n^{(2)} + H_{2n}^{(2)}\big)$$

Proving $\xrightarrow{\text{Sigma}}$ Finding

$$\sum_{k=0}^{2n}\binom{2n}{k}^3(-1)^k H_k H_{2n-k} = \frac{(3n)!(-1)^n}{n!n!n!}\frac{1}{12}\Big(3H_n^2 - 6H_nH_{3n} + 3H_{3n}^2 + H_n^{(2)} + 12H_{2n}(H_{2n}+H_n-H_{3n}) + 4H_{2n}^{(2)} - 3H_{3n}^{(2)}\Big)$$

$$\sum_{k=0}^{2n}\binom{2n}{k}^3(-1)^k H_k^2 = \frac{(3n)!(-1)^n}{n!n!n!}\frac{1}{12}\Big(3H_n^2 - 6H_nH_{3n} + 3H_{3n}^2 - H_n^{(2)} + 12H_{2n}(H_{2n}+H_n-H_{3n}) + 2H_{2n}^{(2)} - 3H_{3n}^{(2)}\Big)$$

$$\sum_{k=0}^{2n}\binom{2n}{k}^3(-1)^k H_k^{(2)} = \frac{1}{2}\frac{(3n)!(-1)^n}{n!n!n!}\big(H_n^{(2)} + H_{2n}^{(2)}\big)$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3\Big[3(H_{2n-k}-H_k)^2 + H_{2n-k}^{(2)} + H_k^{(2)}\Big] = 0$$

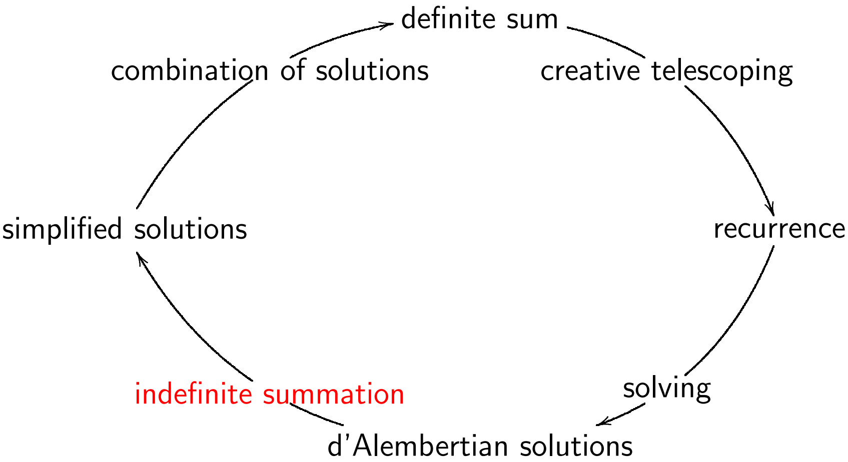
Proving $\xrightarrow{\text{Sigma}}$ Finding

$$\boxed{\sum_{k=0}^{2n}\binom{2n}{k}^3(-1)^k H_k H_{2n-k}} = \text{FIND}$$

$$\sum_{k=0}^{2n}\binom{2n}{k}^3(-1)^k H_k^2 = \frac{(3n)!(-1)^n}{n!n!n!}\frac{1}{12}\Big(3H_n^2 - 6H_nH_{3n} + 3H_{3n}^2 - H_n^{(2)} + 12H_{2n}(H_{2n}+H_n-H_{3n}) + 2H_{2n}^{(2)} - 3H_{3n}^{(2)}\Big)$$

$$\sum_{k=0}^{2n}\binom{2n}{k}^3(-1)^k H_k^{(2)} = \frac{1}{2}\frac{(3n)!(-1)^n}{n!n!n!}(H_n^{(2)} + H_{2n}^{(2)})$$

The Sigma-summation spiral:



telescoping

- GIVEN

$$\sum_{k=0}^{n} f(k).$$

- FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

telescoping

- GIVEN

$$\sum_{k=0}^{n} f(k).$$

- FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

- THEN (with some mild extra conditions)

$$\sum_{k=0}^{n} f(k) = g(n+1) - g(0)$$

Refined telescoping

- GIVEN

$$\sum_{k=0}^{n} f(k).$$

- FIND $g(k)$ and $f^*(k)$:

$$\boxed{f(k) = g(k+1) - g(k) + f^*(k)}$$

where $f^*(k)$ is simpler than $f(k)$.

- THEN (with some mild extra conditions)

$$\sum_{k=0}^{n} f(k) = g(n+1) - g(0) + \sum_{k=0}^{n} f^*(k).$$

Degree optimal w.r.t the top extension

$$\sum_{k=2}^{n} \frac{2 - kH_k + H_k^4 - kH_k^5}{H_k - kH_k^2} =$$

Sigma

$$\sum_{k=1}^{n} H_k^4 =$$

Degree optimal w.r.t the top extension

$$\sum_{k=2}^{n} \frac{2 - kH_k + H_k^4 - kH_k^5}{H_k - kH_k^2} = \sum_{k=2}^{n} \frac{k^2 + H_k}{k^2 H_k} + (n+1)H_n^3 - (2n+1)\left(\tfrac{3}{2}H_n^2 - 3H_n + \tfrac{3}{2}\right) + \tfrac{1}{H_n}$$

$$\sum_{k=1}^{n} H_k^4 = -H_n^{(3)} - 2H_n^{(2)} + 2\sum_{k=1}^{n} \frac{H_k}{k^2} + (n+1)H_n^4 - 2(2n+1)H_n^3 + 6(2n+1)H_n^2 - 12(2n+1)H_n + 24n.$$

Further examples

$$\sum_{k=1}^{n}\frac{k+1}{k(k+2)}=-\frac{n(3n+5)}{4(n+1)(n+2)}+\sum_{k=1}^{n}\frac{1}{k}$$

$$\sum_{k=2}^{n}\frac{1}{k(k-1)2^k}=\frac{-1}{n2^{n+1}}+\frac{1}{4}-\frac{1}{2}\sum_{k=2}^{n}\frac{1}{k2^k}$$

$$\begin{aligned}\sum_{k=1}^{n}&\frac{k!\left(k^2+k+k!\left(k(k+1)^2+k!\left(k(k+1)^2+\left(2k^2-1\right)k!-3\right)-2\right)+1\right)+1}{(k!)^3(k!+1)((k+1)k!+1)}\\&=\frac{3(n+1)(n!)^3+(3-2n)(n!)^2-2(n+2)n!-2}{2(n!)^2((n+1)n!+1)}+\sum_{k=1}^{n}\frac{k(k!)^3+k!+1}{(k!)^3(k!+1)}\end{aligned}$$

$$\begin{aligned}\sum_{k=2}^{n}&\frac{(k+1)\left(k(k+1)^2(k+2)H_k^3+k\left(3k^2+8k+5\right)H_k^2-(k+2)H_k-k-2\right)}{H_k\left(k(k+1)^2(k+2)H_k^3+2\left(k^3+2k^2-1\right)H_k^2-\left(k^2+5k+5\right)H_k-2k-3\right)}\\&=\frac{-6(n+1)(n+2)H_n^2-6(2n+3)H_n+11(n+1)(n+2)}{11H_n\left(2n+(n+1)(n+2)H_n+3\right)}+\sum_{k=2}^{n}\frac{k(k+1)}{kH_k-1}\end{aligned}$$

(CS. Simplifying sums in $\Pi\Sigma^*$-extensions. To appear in J. Algebra Appl.)

The analogue problem for Π-extensions

$$\prod_{k=1}^{n}\frac{(-k-1)(k+7)}{(k+4)^2}=\frac{4}{35}\,\frac{(n+5)(n+6)(n+7)}{(n+2)(n+3)(n+4)}\,(-1)^n,$$

$$\prod_{k=1}^{n}\frac{(k+3)(H_k(k+1)+1)^2(H_k(k+2)(k+1)+2k+3)}{(k+1)^2H_k(H_k(k+3)(k+2)(k+1)+3(k+4)k+11)}$$
$$=\frac{11}{6}\,\frac{(n+3)(n+2)(H_n(n+1)+1)^2}{(n+1)(H_n(n+3)(n+2)(n+1)+3(n+4)n+11)}\prod_{k=1}^{n}H_k,$$

$$\prod_{k=1}^{n}\frac{k!(H_k(k+2)(k+1)+2k+3)(H_k(k+1)+1)}{H_k(k+3)(k+2)(k+1)+3(k+4)k+11}$$
$$=\frac{11(H_n(n+1)+1)}{H_n(n+3)(n+2)(n+1)+3(n+4)n+11}\prod_{k=1}^{n}k!H_k,$$

$$\prod_{k=1}^{n}\frac{(q^{k+2}+(k+1)!)(q^{k+1}+k!)(k+2)(k+1)}{(q^{k+3}+(k+2)!)(k+3)}$$
$$=\frac{3(q^3+2)}{q+1}\,\frac{(q^{n+1}(n+1)+(n+1)!)}{(q^{n+3}+(n+2)!)(n+3)}\prod_{k=1}^{n}(kq^k+k!)$$

(CS. Product Representations in ΠΣ-Fields. 2005.)

Simpler w.r.t. the depth

$$\sum_{k=1}^{n} H_k^2 H_k^{(2)} =$$

$$\sum_{k=0}^{a} \left(\sum_{i=0}^{k} \binom{n}{i} \right)^2 =$$

Sigma

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{H_j}{j^2}}{k^3} =$$

Simpler w.r.t. the depth

$$\sum_{k=1}^{n} H_k^2 H_k^{(2)} = \frac{1}{3} H_n^{(3)} - \frac{1}{3} H_n^3 + \left((n+1)H_n^{(2)} + 1\right) H_n^2 + (2n+1)\left(1 - H_n\right) H_n^{(2)} - 2H_n$$

$$\sum_{k=0}^{a} \left(\sum_{i=0}^{k} \binom{n}{i} \right)^2 = (n-a)\binom{n}{a} \sum_{i=0}^{a} \binom{n}{i} - \frac{n-2a-2}{2} \left(\sum_{i=0}^{a} \binom{n}{i} \right)^2 - \frac{n}{2} \sum_{i=1}^{a} \binom{n}{i}^2$$

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{H_j}{j^2}}{k^3} =$$

$$= H_n^{(3)} \sum_{j=1}^{n} \frac{H_j}{j^2} - \sum_{j=1}^{n} \frac{H_j H_j^{(3)}}{j^2} + \sum_{j=1}^{n} \frac{H_j}{j^5}$$

Simpler w.r.t. the depth

$$\sum_{k=1}^{n} H_k^2 H_k^{(2)} = \tfrac{1}{3} H_n^{(3)} - \tfrac{1}{3} H_n^3 + \left((n+1) H_n^{(2)} + 1\right) H_n^2 + (2n+1)\,(1 - H_n)\, H_n^{(2)} - 2H_n$$

$$\sum_{k=0}^{a}\left(\sum_{i=0}^{k}\binom{n}{i}\right)^2 = (n-a)\binom{n}{a}\sum_{i=0}^{a}\binom{n}{i} - \frac{n-2a-2}{2}\left(\sum_{i=0}^{a}\binom{n}{i}\right)^2 - \frac{n}{2}\sum_{i=1}^{a}\binom{n}{i}^2$$

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{H_j}{j^2}}{k^3} = S(3,2,1,N) \qquad \text{(Harmonic sum)}$$

$$= H_n^{(3)}\sum_{j=1}^{n}\frac{H_j}{j^2} - \sum_{j=1}^{n}\frac{H_j H_j^{(3)}}{j^2} + \sum_{j=1}^{n}\frac{H_j}{j^5} \qquad \text{(Euler sums)}$$

Further examples

$$\sum_{k=1}^{n} H_k^3 = \frac{1}{2}\left(2(n+1)H_n^3 - 3(2n+1)H_n^2 + 6(2n+1)H_n - 12n - 1 + H_n^{(2)}\right)$$

$$\sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k} \frac{1}{j} \sum_{i=1}^{j} \frac{1}{i} = \frac{1}{6}\left[H_n^3 + 3H_n\, H_n^{(2)} + 2\, H_n^{(3)}\right]$$

$$\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\frac{H_j^{(2)}}{j^3}\right)^2 = -\left({H_n^{(2)}}^2 + H_n^{(4)}\right)\sum_{j=1}^{n}\frac{H_j^{(2)}}{j^3} + (n+1)\left(\sum_{j=1}^{n}\frac{H_j^{(2)}}{j^3}\right)^2$$

$$+\sum_{j=1}^{n}\frac{{H_j^{(2)}}^3}{j^3} - \sum_{j=1}^{n}\frac{{H_j^{(2)}}^2}{j^5} + \sum_{j=1}^{n}\frac{H_j^{(2)} H_j^{(4)}}{j^3}.$$

(CS. Symbolic summation with single-nested sum extensions. 2004.
CS. Finding telescopers with minimal depth for indefinite nested sum and product expressions. 2005.)

Example: Summation with unspecified sequences X_k

A double sum

$$\sum_{k=1}^{n}\sum_{j=1}^{k} X_j =$$ Sigma

A double sum

$$\sum_{k=1}^{n}\sum_{j=1}^{k}X_j=(n{+}1)\sum_{k=1}^{n}X_k-\sum_{k=1}^{n}kX_k$$

- $X_j := \frac{1}{j^2}$:

$$\sum_{k=1}^{n}H_k^{(2)}=(n+1)H_n^{(2)}-H_k.$$

A double sum

$$\sum_{k=1}^{n}\sum_{j=1}^{k} X_j = (n+1)\sum_{k=1}^{n} X_k - \sum_{k=1}^{n} k X_k$$

- $X_j = \binom{m}{j}$:

$$\sum_{k=0}^{n}\sum_{j=0}^{k} \binom{m}{j} = (n+1)\sum_{k=0}^{n} \binom{m}{k} - \sum_{k=0}^{n} k \binom{m}{k}$$

A double sum

$$\sum_{k=1}^{n}\sum_{j=1}^{k}X_j=(n+1)\sum_{k=1}^{n}X_k-\sum_{k=1}^{n}kX_k$$

- $X_j=\binom{m}{j}$:

$$\begin{aligned}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{m}{j}&=(n+1)\sum_{k=0}^{n}\binom{m}{k}-\sum_{k=0}^{n}k\binom{m}{k}\\&=\frac{1}{2}(m-n)\binom{m}{n}+(2n-m+2)\sum_{i=0}^{n}\binom{m}{i}\end{aligned}$$

A double sum

$$\sum_{k=1}^{n}\sum_{j=1}^{k} X_j = (n+1)\sum_{k=1}^{n} X_k - \sum_{k=1}^{n} k X_k$$

- $X_j = \binom{m}{j}$:

$$\begin{aligned}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{m}{j} &= (n+1)\sum_{k=0}^{n}\binom{m}{k} - \sum_{k=0}^{n} k\binom{m}{k}\\ &= \frac{1}{2}(m-n)\binom{m}{n} + (2n-m+2)\sum_{i=0}^{n}\binom{m}{i}\\ &\overset{(m=n)}{=} \frac{m+2}{2}2^m\end{aligned}$$

(G.E. Andrews, P. Paule. MacMahon's Partition Analysis IV: Hypergeometric Multisums. 1999.)

A variation

$$\sum_{k=1}^{n} k^2 \sum_{j=1}^{k} X_j =$$ Sigma

A variation

$$\sum_{k=1}^{n} k^2 \sum_{j=1}^{k} X_j = \tfrac{1}{6}\Big(n(n+1)(2n+1)\sum_{k=1}^{n} X_k - \sum_{k=1}^{n} k X_k + 3\sum_{k=1}^{n} k^2 X_k - 2\sum_{k=1}^{n} k^3 X_k\Big)$$

A variation

$$\sum_{k=1}^{n} k^2 \sum_{j=1}^{k} X_j = \tfrac{1}{6}\Big(n(n{+}1)(2n{+}1)\sum_{k=1}^{n} X_k - \sum_{k=1}^{n} k X_k + 3\sum_{k=1}^{n} k^2 X_k - 2\sum_{k=1}^{n} k^3 X_k\Big)$$

$$\downarrow X_k = \frac{1}{n+k}$$

$$\sum_{k=1}^{n} k^2 \sum_{j=1}^{k} \frac{1}{n+j} = \frac{1}{36} n(n+1)(1-10n+12(2n+1)\sum_{k=1}^{n} \frac{1}{n+k})$$

A variation

$$\sum_{k=1}^{n}k^2\sum_{j=1}^{k}X_j=\tfrac{1}{6}\Big(n(n{+}1)(2n{+}1)\sum_{k=1}^{n}X_k-\sum_{k=1}^{n}kX_k+3\sum_{k=1}^{n}k^2X_k-2\sum_{k=1}^{n}k^3X_k\Big)$$

$$\downarrow X_k=\frac{1}{n+k}$$

$$\sum_{k=1}^{n}k^2\sum_{j=1}^{k}\frac{1}{n+j}=\frac{1}{36}n(n+1)(1-10n+12(2n+1)\sum_{k=1}^{n}\frac{1}{n+k})$$

$$\downarrow H_{n+k}=H_n+\sum_{i=1}^{k}\frac{1}{n+i}$$

$$\sum_{k=1}^{n}k^2H_{n+k}=\tfrac{1}{3}n(n+\tfrac{1}{2})(n+1)(2H_{2n}-H_n)-\tfrac{1}{36}(10n^2+9n-1)$$

(The warmup example)

Nested sums

$$\sum_{k=1}^{n} X_k \sum_{j=1}^{k} X_j \sum_{i=1}^{j} X_i \sum_{l=1}^{i} X_l =$$ Sigma

Nested sums

$$\sum_{k=1}^{n} X_k \sum_{j=1}^{k} X_j \sum_{i=1}^{j} X_i \sum_{l=1}^{i} X_l = \frac{1}{2}\left(\sum_{k=1}^{n} X_k \sum_{j=1}^{k} X_j\right)^2 - \frac{3}{2}\sum_{k=1}^{n} X_k^2 \left(\sum_{j=1}^{k} X_j\right)^2$$
$$+ \sum_{k=1}^{n} X_k^3 \sum_{j=1}^{k} X_j + \sum_{k=1}^{n} X_k \left(\sum_{j=1}^{k} X_j\right)^3$$
$$+ \left(\sum_{k=1}^{n} X_k\right)\left(\sum_{k=1}^{n} X_k^2 \sum_{j=1}^{k} X_j - \sum_{k=1}^{n} X_k \left(\sum_{j=1}^{k} X_j\right)^2\right)$$

Nested sums

$$\sum_{k=1}^{n} X_k \sum_{j=1}^{k} X_j \sum_{i=1}^{j} X_i \sum_{l=1}^{i} X_l = \frac{1}{2}\left(\sum_{k=1}^{n} X_k \sum_{j=1}^{k} X_j\right)^2 - \frac{3}{2}\sum_{k=1}^{n} X_k^2 \left(\sum_{j=1}^{k} X_j\right)^2 + \sum_{k=1}^{n} X_k^3 \sum_{j=1}^{k} X_j + \sum_{k=1}^{n} X_k \left(\sum_{j=1}^{k} X_j\right)^3 + \left(\sum_{k=1}^{n} X_k\right)\left(\sum_{k=1}^{n} X_k^2 \sum_{j=1}^{k} X_j - \sum_{k=1}^{n} X_k \left(\sum_{j=1}^{k} X_j\right)^2\right)$$

- $X_j = \frac{1}{j}$:

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{\sum_{i=1}^{j} \frac{H_i}{i}}{j}}{k} = \frac{1}{2}\left(\sum_{k=1}^{n} \frac{H_k}{k}\right)^2 - \frac{3}{2}\sum_{k=1}^{n} \frac{H_k^2}{k^2} + H_k\left(\sum_{k=1}^{n} \frac{H_k}{k^2} - \sum_{k=1}^{n} \frac{H_k^2}{k}\right) + \sum_{k=1}^{n} \frac{H_k^3}{k} + \sum_{k=1}^{n} \frac{H_k}{k^3}$$

Nested sums

$$\sum_{k=1}^{n} X_k \sum_{j=1}^{k} X_j \sum_{i=1}^{j} X_i \sum_{l=1}^{i} X_l = \frac{1}{2}\left(\sum_{k=1}^{n} X_k \sum_{j=1}^{k} X_j\right)^2 - \frac{3}{2}\sum_{k=1}^{n} X_k^2 \left(\sum_{j=1}^{k} X_j\right)^2$$
$$+ \sum_{k=1}^{n} X_k^3 \sum_{j=1}^{k} X_j + \sum_{k=1}^{n} X_k \left(\sum_{j=1}^{k} X_j\right)^3$$
$$+ \left(\sum_{k=1}^{n} X_k\right)\left(\sum_{k=1}^{n} X_k^2 \sum_{j=1}^{k} X_j - \sum_{k=1}^{n} X_k \left(\sum_{j=1}^{k} X_j\right)^2\right)$$

- $X_j = \frac{1}{j}$:

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{\sum_{i=1}^{j} \frac{H_i}{i}}{j}}{k} = \frac{1}{2}\left(\sum_{k=1}^{n} \frac{H_k}{k}\right)^2 - \frac{3}{2}\sum_{k=1}^{n} \frac{H_k^2}{k^2} + H_k\left(\sum_{k=1}^{n} \frac{H_k}{k^2} - \sum_{k=1}^{n} \frac{H_k^2}{k}\right)$$
$$+ \sum_{k=1}^{n} \frac{H_k^3}{k} + \sum_{k=1}^{n} \frac{H_k}{k^3}$$
$$= \frac{1}{24}\left(H_n^4 + 6H_n^{(2)} H_n^2 + 8H_n^{(3)} H_n + 3\left(H_n^{(2)}\right)^2 + 6H_n^{(4)}\right)$$

X-Collection

$$\sum_{k=1}^{a}(-1)^k\Big(\sum_{j=1}^{k}X_j-\frac{X_k}{2}\Big)^2=\frac{1}{2}(-1)^a\Big(\sum_{k=1}^{a}X_k\Big)^2-\frac{1}{4}\sum_{k=1}^{a}(-1)^kX_k^2,$$

$$\sum_{k=1}^{a}\Big(\sum_{j=1}^{k}X_j+X_k(k-1)\Big)^2=n\Big(\sum_{k=1}^{a}X_k\Big)^2-\sum_{k=1}^{a}kX_k^2+\sum_{k=1}^{a}k^2X_k^2\Big),$$

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}X_{n-i}\right)^2=2\sum_{k=0}^{n}X_k\sum_{j=0}^{k}jX_{j-1}+\sum_{k=0}^{n}X_k^2+\sum_{k=0}^{n}kX_k^2,$$

$$\sum_{k=1}^{a}(-1)^k\binom{n}{k}\sum_{j=1}^{k}X_j=\frac{1}{n}\Big[(n-a)\binom{n}{a}(-1)^a\sum_{k=1}^{a}X_k+\sum_{k=1}^{a}(-1)^kk\binom{n}{k}X_k\Big].$$

(M. Kauers, CS. Application of unspecified sequences in symbolic summation. 2006.
M. Kauers, CS. Indefinite summation with unspecified sequences. 2006.)

Sven Moch Example:
An application in particle physics (even case)

GIVEN $F(N)$ by

$$F(0) = f_0(0) + f_1(0)\epsilon + f_2(0)\epsilon^2 + f_3(0)\epsilon^3 + f_4(0)\epsilon^4 + O(\epsilon^5)$$

and

$$\mathrm{F(N)} = \frac{2(N-2\epsilon)(-1+2N)}{2(-1+2\epsilon-2N)(-1+3\epsilon-N)}\mathrm{F(N\text{-}1)} + \frac{\mathrm{X(2N)}}{2(-1+2\epsilon-2N)(-1+3\epsilon-N)}$$

where

$$X(2N) = x_0(N) + x_1(N)\epsilon + x_2(N)\epsilon^2 + x_3(N)\epsilon^3 + x_4(N)\epsilon^4 + O(\epsilon^5)$$

GIVEN $F(N)$ by

$$F(0) = f_0(0) + f_1(0)\epsilon + f_2(0)\epsilon^2 + f_3(0)\epsilon^3 + f_4(0)\epsilon^4 + O(\epsilon^5)$$

and

Sigma

$$\text{F(N)} = \frac{2(N-2\epsilon)(-1+2N)}{2(-1+2\epsilon-2N)(-1+3\epsilon-N)}\text{F(N-1)} + \frac{\text{X(2N)}}{2(-1+2\epsilon-2N)(-1+3\epsilon-N)}$$

where

$$X(2N) = x_0(N) + x_1(N)\epsilon + x_2(N)\epsilon^2 + x_3(N)\epsilon^3 + x_4(N)\epsilon^4 + O(\epsilon^5)$$

FIND the expansion

$$F(N) = f_0(N) + f_1(N)\epsilon + f_2(N)\epsilon^2 + f_3(N)\epsilon^3 + f_4(N)\epsilon^4 + O(N^5)$$

$$
\begin{aligned}
F(N) =& F(0)\prod_{i=1}^{N}\frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}\\
&+\frac{1}{2}\prod_{i=1}^{N}\frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}\sum_{i=1}^{N}\frac{\prod_{j=1}^{i}\frac{(2\epsilon-2j-1)(3\epsilon-j-1)}{(j-2\epsilon)(2j-1)}}{(2\epsilon-2i-1)(3\epsilon-i-1)}X(2i)
\end{aligned}
$$

FIND the expansion

$$F(N) = f_0(N) + f_1(N)\epsilon + f_2(N)\epsilon^2 + f_3(N)\epsilon^3 + f_4(N)\epsilon^4 + O(N^5)$$

$$F(N) = \boxed{F(0)} \cdot \boxed{\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}}$$

$$+\frac{1}{2}\boxed{\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}} \cdot \boxed{\sum_{i=1}^{N} \frac{\prod_{j=1}^{i} \frac{(2\epsilon-2j-1)(3\epsilon-j-1)}{(j-2\epsilon)(2j-1)}}{(2\epsilon-2i-1)(3\epsilon-i-1)} X(2i)}$$

1. FIND expansions for each box, e.g.,

$$\boxed{A(N)} = a_0(N) + a_1(N)\epsilon + a_2(N)\epsilon^2 + a_3(N)\epsilon^3 + a_4(N)\epsilon^4 + O(\epsilon^5)$$

$$\boxed{B(N)} = b_0(N) + b_1(N)\epsilon + b_2(N)\epsilon^2 + b_3(N)\epsilon^3 + b_4(N)\epsilon^4 + O(\epsilon^5)$$

$$F(N) = \boxed{F(0)} \cdot \boxed{\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}}$$

$$+\frac{1}{2}\boxed{\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}} \cdot \boxed{\sum_{i=1}^{N} \frac{\prod_{j=1}^{i} \frac{(2\epsilon-2j-1)(3\epsilon-j-1)}{(j-2\epsilon)(2j-1)}}{(2\epsilon-2i-1)(3\epsilon-i-1)} X(2i)}$$

1. FIND expansions for each box, e.g.,

$$\boxed{A(N)} = a_0(N) + a_1(N)\epsilon + a_2(N)\epsilon^2 + a_3(N)\epsilon^3 + a_4(N)\epsilon^4 + O(\epsilon^5)$$

$$\boxed{B(N)} = b_0(N) + b_1(N)\epsilon + b_2(N)\epsilon^2 + b_3(N)\epsilon^3 + b_4(N)\epsilon^4 + O(\epsilon^5)$$

2. COMBINE:

$$A(N) + B(N) = \dots + (a_r(N) + b_r(N))\epsilon^r + \dots$$ component wise

$$A(N) \cdot B(N) = \dots + \Big(\sum_{l=0}^{r} a_l(N) b_{r-l}(N)\Big)\epsilon^r + \dots$$ Cauchy-product

GIVEN $F(N) = \boxed{F(0)} \cdot \boxed{\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}}$

$$+\frac{1}{2}\boxed{\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}} \cdot \boxed{\sum_{i=1}^{N} \frac{\prod_{j=1}^{i} \frac{(2\epsilon-2j-1)(3\epsilon-j-1)}{(j-2\epsilon)(2j-1)}}{(2\epsilon-2i-1)(3\epsilon-i-1)} X(2i)}$$

$$\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)} =$$

Sigma

$$\text{GIVEN } F(N) = \boxed{F(0)} \cdot \boxed{\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}}$$

$$+ \frac{1}{2} \boxed{\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}} \cdot \boxed{\sum_{i=1}^{N} \frac{\prod_{j=1}^{i} \frac{(2\epsilon-2j-1)(3\epsilon-j-1)}{(j-2\epsilon)(2j-1)}}{(2\epsilon-2i-1)(3\epsilon-i-1)} X(2i)}$$

$$\frac{\prod_{j=1}^{i} \frac{(2\epsilon-2j-1)(3\epsilon-j-1)}{(j-2\epsilon)(2j-1)}}{(2\epsilon-2i-1)(3\epsilon-i-1)} X(2i) =$$

Sigma

$$\text{GIVEN } F(N) = \boxed{F(0)} \cdot \boxed{\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}}$$

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$$\frac{\prod_{j=1}^{i} \frac{(2\epsilon-2j-1)(3\epsilon-j-1)}{(j-2\epsilon)(2j-1)}}{(2\epsilon-2i-1)(3\epsilon-i-1)} X(2i) = \qquad h_0(i) + h_1(i)\epsilon + h_2(i)\epsilon^2 + \dots$$

GIVEN $F(N) = \boxed{F(0)} \cdot \boxed{\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}}$

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$$\sum_{j=1}^{N} \frac{\prod_{j=1}^{i} \frac{(2\epsilon-2j-1)(3\epsilon-j-1)}{(j-2\epsilon)(2j-1)}}{(2\epsilon-2i-1)(3\epsilon-i-1)} X(2i) = \underbrace{\sum_{i=1}^{N} h_0(i)}_{\text{sum 0}} + \epsilon \underbrace{\sum_{i=1}^{N} h_1(i)}_{\text{sum 1}} + \epsilon^2 \underbrace{\sum_{i=1}^{N} h_2(i)}_{\text{sum 2}} + \dots$$

Sigma

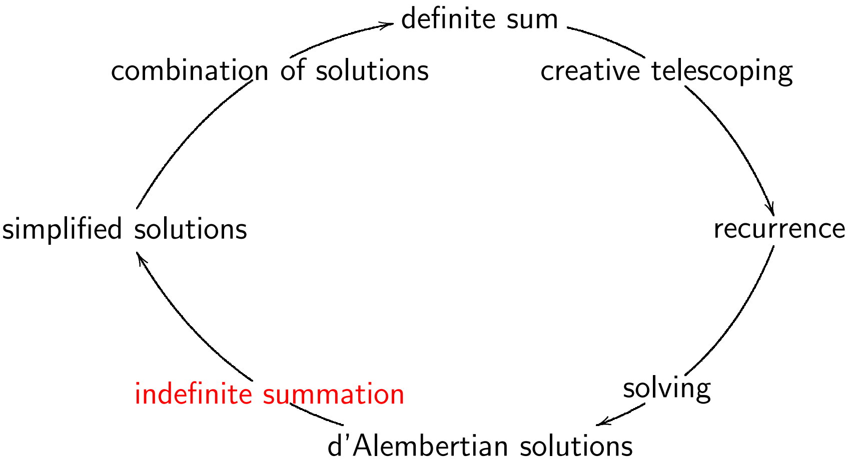
$$\text{GIVEN } F(N) = \boxed{F(0)} \cdot \boxed{\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}}$$

$$+\frac{1}{2}\boxed{\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}} \cdot \boxed{\sum_{i=1}^{N} \frac{\prod_{j=1}^{i} \frac{(2\epsilon-2j-1)(3\epsilon-j-1)}{(j-2\epsilon)(2j-1)}}{(2\epsilon-2i-1)(3\epsilon-i-1)} X(2i)}$$

$$= \boxed{f_0(N) + f_1(N)\epsilon + f_2(N)\epsilon^2 + f_3(N)\epsilon^3 + f_4(N)\epsilon^4 + O(N^5)}$$

COMBINE

The Sigma-summation spiral:



Quadratic Padé approximation to $\log(x)$ at $x = 1$

Back

FIND $r_n(x) = \sum_{k=0}^{n} a_k x^k, \quad s_n(x) = \sum_{k=0}^{n} b_k x^k, \quad t_n(x) = \sum_{k=0}^{n} c_k x^k$:

$$r_n(x)\,(\log x)^2 + s_n(x)\,\log(x) + t_n(x) = O((x-1)^{3n+2}).$$

Quadratic Padé approximation to $\log(x)$ at $x=1$

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$$r_n(x)\,(\log x)^2+s_n(x)\,\log(x)+t_n(x)=O((x-1)^{3n+2}).$$

A. Weideman finds

$$\begin{aligned}
r_n(x)&=c_3A(n,x)\\
s_n(x)&=c_2A(n,x)+2c_3B(n,x)\\
t_n(x)&=c_1A(n,x)+c_2B(n,x)+c_3C(n,x)
\end{aligned}$$

where

$$A(n,x)=\sum_{k=0}^{n}\binom{n}{k}^3(-x)^k \qquad B(n,x)=\sum_{k=0}^{n}\Big[\frac{d}{dk}\binom{n}{k}^3\Big](-x)^k$$

$$C(n,x)=\sum_{k=0}^{n}\Big[\frac{d^2}{dk^2}\binom{n}{k}^3\Big](-x)^k.$$

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Back

FIND $r_n(x) = \sum_{k=0}^{n} a_k x^k, \quad s_n(x) = \sum_{k=0}^{n} b_k x^k, \quad t_n(x) = \sum_{k=0}^{n} c_k x^k$:

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A. Weideman finds

$$\begin{aligned} r_n(x) &= c_3 A(n,x) \\ s_n(x) &= c_2 A(n,x) + 2c_3 B(n,x) \\ t_n(x) &= c_1 A(n,x) + c_2 B(n,x) + c_3 C(n,x) \end{aligned}$$

where

$$A(n,x) = \sum_{k=0}^{n} \binom{n}{k}^3 (-x)^k \qquad B(n,x) = \sum_{k=0}^{n} \left[\frac{d}{dk}\binom{n}{k}^3\right](-x)^k$$

$$C(n,x) = \sum_{k=0}^{n} \left[\frac{d^2}{dk^2}\binom{n}{k}^3\right](-x)^k.$$

Tests at $x = 1$:
$c_1 = \pi^2, \quad c_2 = 0, \quad c_3 = 1$

Z's creative telescoping paradigm

GIVEN

$$\mathrm{SUM}(n) := \sum_{k=0}^{n} \underbrace{(-1)^k \binom{n}{k}^3 \left[3(H_{n-k} - H_k)^2 + H_{n-k}^{(2)} + H_k^{(2)}\right]}_{=: f(n,k)}.$$

FIND $c_0(n)$, $c_1(n)$, $c_2(n)$, and $g(n,k)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n)\, f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Back

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FIND $c_0(n)$, $c_1(n)$, $c_2(n)$, and $g(n,k)$:

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for all $0 \le k \le n$ and all $n \ge 0$.

Sigma: $c_0(n) := 3(3n+2)(3n+4)(3n+8), \quad c_1(n) := 0,$
$c_2(n) := (n+2)^2(3n+8),$

$$g(n,k) := (-1)^k \binom{n}{k}^3 \frac{p_1(k,n,H_k,H^{(2)}_k,H_{n-k},H^{(2)}_{n-k})}{(n-k+1)^5(n-k+2)^5},$$

$$g(n,k+1) := (-1)^k \binom{n}{k}^3 \frac{p_2(k,n,H_k,H^{(2)}_k,H_{n-k},H^{(2)}_{n-k})}{(n-k+1)^5}.$$

Back

Z's creative telescoping paradigm

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$$\text{SUM}(n) := \sum_{k=0}^{n} \underbrace{(-1)^k \binom{n}{k}^3 \left[3(H_{n-k} - H_k)^2 + H_{n-k}^{(2)} + H_k^{(2)}\right]}_{=:\, f(n,k)}.$$

GIVEN $c_0(n)$, $c_1(n)$, $c_2(n)$, and $g(n,k)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n)\, f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 0 to n gives:

$$\boxed{g(n,n+1) - g(n,0)}$$

$$= \boxed{\begin{array}{l} c_0(n)\,\text{SUM}(n) + \\ c_1(n)\left[\text{SUM}(n+1) - f(n+1,n+1)\right] \\ c_2(n)\left[\text{SUM}(n+2) - f(n+2,n+1) - f(n+2,n+2)\right]. \end{array}}$$

Back