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A Collection of Denominator Bounds to Solve Parameterized Linear Difference Equations in $\Pi\Sigma$-Extensions *

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Abstract. An important application of solving parameterized linear difference equations in $\Pi\Sigma$-fields, a very general class of difference fields, is simplifying of multi-sum expressions and proving of multi-sum identities. This article provides essential algorithmic building blocks that allow one to search for all solutions of such difference equations. More precisely, these algorithms enable one to exploit a denominator elimination strategy which amounts to look for solutions in a polynomial ring instead of searching for rational function solutions.

Key words: Difference fields, Denominator Bounds, Symbolic Summation

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1. Introduction The summation package Sigma (Schneider, 2001, Schneider, 2004a), based on the computer algebra system Mathematica, allows one to deal with indefinite and definite summation over summands being hypergeometric terms (Petkovšek et al., 1996), hypergeometric multisum expressions (Wegschaider, 1997), or frequently used ∂-finite expressions (Chyzak, 2000, Schneider, 2005c).
All the summation tools in Sigma are based on the following problem for a given difference field $(\mathbb{F},\sigma)$, i.e., a field[2] $\mathbb{F}$ together with a field

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[2]Throughout this paper all fields will have characteristic 0.

automorphism $\sigma : \mathbb{F} \to \mathbb{F}$, with constant field $\mathbb{K}$, i.e., $\mathbb{K} = \text{const}_\sigma \mathbb{F} := \{k \in \mathbb{F} \mid \sigma(k) = k\}$.

PLDE: Solving **P**arameterized **L**inear **D**ifference **E**quations.

- **Given** $(a_1 \dots, a_m) \in \mathbb{F}^m$ with $a_1\, a_m \neq 0$ and $(f_1, \dots, f_n) \in \mathbb{F}^n$.
- **Find all** $g \in \mathbb{F}$ and $(c_1, \dots, c_n) \in \mathbb{K}^n$ with

$$a_1\, \sigma^{m-1}(g) + \dots + a_m\, g = c_1\, f_1 + \dots + c_n\, f_n. \tag{1}$$

For instance, take the field of rational functions $\mathbb{F} = \mathbb{K}(k)$ with the shift $\sigma(k) = k + 1$ and consider the case $m = 2$ with $a_1 = 1$ and $a_2 = -1$. Then for $n = 1$ one obtains telescoping for a rational function $f_1 = f'(k) \in \mathbb{K}(k)$. Moreover, specializing to $\mathbb{K} = \mathbb{K}'(\nu)$ and $f_i = f'(\nu + i - 1, k) \in \mathbb{K}'(\nu)(k)$ for $1 \leqslant i \leqslant n$, one formulates the creative telescoping problem (Zeilberger, 1990) of order $n - 1$ for definite rational sums. Furthermore, setting $n = 1$ in problem *PLDE* is nothing else than solving a linear recurrence. Finally, solving the general problem *PLDE* allows one to sum over ∂-finite summand expressions as described in (Chyzak, 2000).

Slight variations of the algorithms in (Abramov, 1989b, Abramov, 1995, Abramov, 1989a) allow one to solve problem *PLDE* for the difference field $(\mathbb{K}(k), \sigma)$ from above. More generally, with the algorithms in (Schneider, 2001, Schneider, 2005e) one can attack problem *PLDE* in $\Pi\Sigma$-fields, introduced in (Karr, 1981, Karr, 1985). Loosely spoken, this are difference fields $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$ where $\mathbb{F} := \mathbb{K}(t_1) \dots (t_e)$ is a rational function field and the application of σ on the t_i's is recursively defined over $1 \leqslant i \leqslant e$ with $\sigma(t_i) = \alpha_i\, t_i + \beta_i$ for $\alpha_i, \beta_i \in \mathbb{K}(t_1) \dots (t_{i-1})$; we omitted some technical conditions given in Section 2.

Example 1. *As illustrative example consider the following problem. Given the left hand side of*

$$\prod_{k=1}^{n-1} \frac{(2+k)(3+k)^2\,(k+k!)\,(-k+k^2+k!)}{k^5\,(1+k)^5\,(1+k!)\,(1+(1+k)\,k!)} = \frac{n^5(n+1)^3(n+2)^2}{9n!^4(n!+1)(n+n!)^2(n^2-n+n!)}, \tag{2}$$

find the closed form on the right hand side. In order to accomplish this task, we look for a sequence $g'(k)$ in terms of k and $k!$ such that

$$\frac{g'(k+1)}{g'(k)} = \frac{(2+k)(3+k)^2\,(k+k!)\,(-k+k^2+k!)}{k^5\,(1+k)^5\,(1+k!)\,(1+(1+k)\,k!)} \tag{3}$$

holds. More precisely, we take the $\Pi\Sigma$*-field* $(\mathbb{Q}(k)(t),\sigma)$ *with the rational function field* $\mathbb{Q}(k)(t)$ *where* $\sigma(k)=k+1$ *and* $\sigma(t)=(k+1)t$*; note that the shift of* $(k+1)!=(k+1)k!$ *is reflected by* $\sigma(t)=(k+1)t$*. Then problem* (3) *can be rephrased in* $(\mathbb{Q}(k)(t),\sigma)$ *by looking for a solution* $g\in\mathbb{Q}(k)(t)^*$ *with*

$$a_1\,\sigma(g)+a_2\,g=0 \tag{4}$$

where $a_1=k^5\,(1+k)^5\,(1+t)\,(1+(1+k)\,t)$ *and* $a_2=-(2+k)(3+k)^2\,(k+t)\,(-k+k^2+t)$*. Note that this problem is nothing else then PLDE with* $m=2$*,* $n=1$ *and* $f_1=0$*. We get the solutions*

$$(c_1,g)\in\{x(0,\frac{k^5\,(1+k)^3\,(2+k)^2}{t^4\,(t+1)\,(t+k)^2\,(t-k+k^2)})+y(1,0)\,|\,x,y\in\mathbb{Q}\} \tag{5}$$

for (1)*; see Example 2. This implies that* $g'(k)=\frac{k^5\,(1+k)^3\,(2+k)^2}{k!^4\,(k!+1)\,(k!+k)^2\,(k!-k+k^2)}$ *is a solution of* (3)*. "Producting"* (3) *over* k *from* 1 *to* $n-1$ *produces* (2)*.*

This article delivers one of the key steps, used in (Schneider, 2005e), in order to solve problem *PLDE* in ΠΣ-fields. More precisely, we develop algorithms, implemented in the summation package Sigma, that try to solve problem

DB: **D**enominator **B**ounding.

- **Given** a difference field $(\mathbb{F}(t),\sigma)$ where $\mathbb{F}(t)$ is a rational function field, $\sigma(t)=\alpha\,t+\beta$ with $\alpha,\beta\in\mathbb{F}$, and $\mathbb{K}:=\text{const}_\sigma\mathbb{F}(t)=\text{const}_\sigma\mathbb{F}$; furthermore $(a_1,\dots,a_m)\in\mathbb{F}[t]^m$ with $a_1\,a_m\neq 0$ and $(f_1,\dots,f_n)\in\mathbb{F}[t]^n$.
- **Find** a *denominator bound* of (1), i.e., a polynomial $d\in\mathbb{F}[t]^*$ such that for all $(c_1,\dots,c_n,g)\in\mathbb{K}^n\times\mathbb{F}(t)$ with (1) we have $d\,g\in\mathbb{F}[t]$.

Given such a denominator bound $d\in\mathbb{F}[t]$ it follows that (1) holds if and only if

$$\frac{a_1}{\sigma^{m-1}(d)}\,\sigma^{m-1}(g')+\dots+\frac{a_{m-1}}{\sigma(d)}\,\sigma(g')+\frac{a_m}{d}\,g=c_1\,f_1+\dots+c_n\,f_n \tag{6}$$

with $g'=g\,d\in\mathbb{F}[t]$. In (Schneider, 2005e), using (Karr, 1981, Schneider, 2005a), algorithms are developed that allow one to find all those solutions $(c_1,\dots,c_n,g')\in\mathbb{K}^n\times\mathbb{F}[t]$ with (6). Hence one can reconstruct all the solutions $(c_1,\dots,c_n,\frac{g'}{d})$ for the original *PLDE*-problem.

Example 2. *In order to find the solutions of* (4)*, i.e., of* (1) *with* $m=2,n=1$ *and* $f_1=0$*, we first compute the denominator bound* $d:=t^4\,(t+$

1) $(t+k)^2\,(t-k+k^2) \in \mathbb{Q}(k)[t]$ *for problem DB; see Exp. 6. Then we can apply other algorithms (Karr, 1981, Schneider, 2005e) and compute with* Sigma *the solution set* $\mathbb{V}' = \{x(0, k^5\,(1+k)^3\,(2+k)^2) + y(1,0) | x, y \in \mathbb{Q}\}$ *for* (6) *with* $(c_1, g') \in \mathbb{V}'$. *This gives the solution set* (5) *for* (1).

The successful application of this approach is illustrated in (Paule and Schneider, 2003, Driver et al., 2005a, Andrews et al., 2005).

After introducing $\Pi\Sigma$-fields in Section 2 we develop algorithms that try to solve problem *DB* in a $\Pi\Sigma$-field $(\mathbb{F}(t), \sigma)$. More precisely, we shall divide problem *DB* into two subproblems; see Section 2. First, we bring together the work of (Karr, 1981) and (Bronstein, 2000) in Section 3 which allows us to compute a denominator bound $d \in \mathbb{F}[t]^*$ of (1) if $\frac{\sigma(t)}{t} \notin \mathbb{F}$. Otherwise, if $\frac{\sigma(t)}{t} \in \mathbb{F}$, we have to search in addition for a factor t^b in order to obtain a denominator bound $t^b\,d$. Using ideas from (Karr, 1981), in Section 4 we show how such a $b \geqslant 0$ can be determined for first order linear difference equations. Moreover, we generalize these algorithms for a certain class of higher order linear difference equations. In Section 5 we show some important properties of these algorithms which are the key step in order to obtain refined summation algorithms (Schneider, 2004b, Schneider, 2005b).

2. The Denominator Bound Problem in $\Pi\Sigma$-Extensions First we give a precise definition of $\Pi\Sigma$-extensions and $\Pi\Sigma$-fields in terms of difference field extensions. Namely, a difference field $(\mathbb{E}, \sigma')$ is a *difference field extension* of $(\mathbb{F}, \sigma)$ if $\mathbb{F}$ is a subfield of $\mathbb{E}$ and $\sigma'(g) = \sigma(g)$ for $g \in \mathbb{F}$; note that from now on σ and σ' are not distinguished anymore.

- A difference field extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ is a Π-*extension* if $\mathbb{F}(t)$ is a rational function field, $\sigma(t) = \alpha\,t$ for some $\alpha \in \mathbb{F}^*$ and $\text{const}_\sigma\mathbb{F}(t) = \text{const}_\sigma\mathbb{F}$. There is the following equivalent description of Π-extensions; see (Karr, 1985; Theorem 2.2) or (Schneider, 2001; Theorem 2.2.2) by using the *homogenous group* $\mathrm{H}_{(\mathbb{F},\sigma)} := \{\sigma(g)/g \,|\, g \in \mathbb{F}^*\}$; note that $\mathrm{H}_{(\mathbb{F},\sigma)}$ forms a multiplicative group.

Theorem 1. *A difference field extension* $(\mathbb{F}(t), \sigma)$ *of* $(\mathbb{F}, \sigma)$ *is a* Π-*ext. iff* $\sigma(t) = \alpha\,t$, $t \neq 0$, $\alpha \in \mathbb{F}^*$ *and there is no* $n > 0$ *with* $\alpha^n \in \mathrm{H}_{(\mathbb{F},\sigma)}$.

- A difference field extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ is a Σ-*extension* if $\mathbb{F}(t)$ is a rational function field, $\sigma(t) = \alpha\,t + \beta$ for some $\alpha, \beta \in \mathbb{F}^*$, $\text{const}_\sigma\mathbb{F}(t) = \text{const}_\sigma\mathbb{F}$, and the following holds: (1) there is no $g \in \mathbb{F}$ with $\sigma(g) - \alpha g = \beta$, and (2) if there is an $n \neq 0$ with $\alpha^n \in \mathrm{H}_{(\mathbb{F},\sigma)}$ then $\alpha \in \mathrm{H}_{(\mathbb{F},\sigma)}$; for more details see (Karr, 1981, Karr, 1985, Bronstein, 2000, Schneider, 2001).

• $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ is a *(nested)* $\Pi\Sigma$*-extension* of $(\mathbb{F},\sigma)$ if for all $1\leqslant i\leqslant n$ $(\mathbb{F}(t_1,\dots,t_{i-1})(t_i),\sigma)$ is a Π-extension or Σ-extension of $(\mathbb{F}(t_1,\dots,t_{i-1}),\sigma)$ (for $i=0$ we define $\mathbb{F}(t_1)\dots(t_{i-1})=\mathbb{F}$).
• A difference field $(\mathbb{F},\sigma)$ is a $\Pi\Sigma$*-field over* $\mathbb{K}$ if $\mathbb{F}=\mathbb{K}(t_1)\dots(t_e)$, $(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-extension of $(\mathbb{K},\sigma)$ and $\text{const}_\sigma\mathbb{K}=\mathbb{K}$. For instance, the difference field $(\mathbb{Q}(k)(t),\sigma)$ in Example 2 is a $\Pi\Sigma$-field over $\mathbb{Q}$.

Subsequently, let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$. In order to solve problem *DB* for $(\mathbb{F}(t),\sigma)$, we will focus on two different subproblems that can be introduced by using the spread function. The *spread* of $f\in\mathbb{F}[t]^*$ and $g\in\mathbb{F}[t]^*$ is defined as

$$\text{spread}_{(\mathbb{F},\sigma)}(f,g)=\{m\geqslant 0\,|\,\deg(\gcd(f,\sigma^m(g)))>0\}.$$

Then we try to find a denominator bound $d_0\,d_1$ of (1) with $d_0,d_1\in\mathbb{F}[t]$ where $\text{spread}_{(\mathbb{F},\sigma)}(d_0,d_0)$ is finite and $\text{spread}_{(\mathbb{F},\sigma)}(d_1,d_1)$ is infinite. We call d_0 also the *finite part* and d_1 the *infinite part*.
The next proposition states when the set $\text{spread}_{(\mathbb{F},\sigma)}(f,g)$ for $f,g\in\mathbb{F}[t]^*$ is finite.

Proposition 1. *Let* $(\mathbb{F}(t),\sigma)$ *be a* $\Pi\Sigma$*-extension of* $(\mathbb{F},\sigma)$*.*

1. *Then for* $f\in\mathbb{F}(t)\setminus\mathbb{F}$ *the following holds: there is* $k>0$ *with* $\frac{\sigma^k(f)}{f}\in\mathbb{F}$ *iff* $\frac{\sigma(t)}{t}\in\mathbb{F}$ *and* $f=u\,t^i$ *for some* $u\in\mathbb{F}^*$ *and* $i\in\mathbb{Z}^*$*.*
2. *Then for* $f,g\in\mathbb{F}[t]^*$ *the following holds:* $\text{spread}_{(\mathbb{F},\sigma)}(f,g)$ *is finite iff* $\frac{\sigma(t)}{t}\notin\mathbb{F}$ *or* $t\nmid\gcd(f,g)$*.*

Proof. The first statement holds by (Karr, 1981; Theorem 4) or (Karr, 1985; Lemma 3.2)[3]. By (Bronstein, 2000; Theorem 6) it follows that $\text{spread}_{(\mathbb{F},\sigma)}(f,g)$ is an infinite set if and only if g has a nontrivial factor $p\in\mathbb{F}[t]\setminus\mathbb{F}$ with $\sigma^k(p)/p\in\mathbb{F}$ for some $k>0$ such that $\sigma^n(p)\mid f$ for some $n\geqslant 0$. Hence the second statement follows by the first statement.

Summarizing, if $\frac{\sigma(t)}{t}\notin\mathbb{F}$ (a Σ-extension), the denominator bound consists only of the finite part, i.e., $d=d_0$. Otherwise, if $\frac{\sigma(t)}{t}\in\mathbb{F}$ (a Π-extension), one has to look besides the finite part d_0 also for the infinite part that is of the form $d_1=t^b$ for some $b\geqslant 0$.

In Section 3 we will introduce algorithms that compute the finite part under the assumption that one can solve problem

[3]Compact proofs can be found in (Bronstein, 2000; Corollary 1 and 2); see also (Schneider, 2001; Theorem 2.2.4).

SE: **S**hift **E**quivalence in a $\Pi\Sigma$-extension.

- **Given** a $\Pi\Sigma$-extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ and $f,g\in\mathbb{F}(t)^*$;
- **find**, if possible, an $n\in\mathbb{Z}$ with $\frac{\sigma^n(f)}{g}\in\mathbb{F}$.

In Section 4 we show how one can compute in Π-extensions the infinite part for first order linear difference equations. In order to accomplish this task, we assume that one can solve problem

HG: **H**omogeneous **G**roup.

- **Given** $(\mathbb{F},\sigma)$ and $a,b\in\mathbb{F}^*$;
- **decide** if there exists a $d\in\mathbb{Z}$ with $a\,b^d\in \mathrm{H}_{(\mathbb{F},\sigma)}$. If yes, **compute** such a d.

Generalizations of this result will allow us to find the infinite denominator bound part for a certain class of linear difference equations of higher order.

The crucial point is that these subproblems *SE* and *HG*, and hence the denominator bound problems from above, can be solved in a $\Pi\Sigma$-field $(\mathbb{F}(t),\sigma)$ over a constant field that is *σ-computable*. This means that **(1)** for any $k\in\mathbb{K}$ one can decide if $k\in\mathbb{Z}$, **(2)** polynomials in $\mathbb{K}[t_1,\dots,t_n]$ can be factored over $\mathbb{K}$, and **(3)** one knows how to compute for $(c_1,\dots,c_k)\in\mathbb{K}^k$ a basis of $\{(n_1,\dots,n_k)\in\mathbb{Z}^k\,|\,c_1^{n_1}\dots c_k^{n_k}=1\}$ which is a submodule of $\mathbb{Z}^k$ over $\mathbb{Z}$. For instance, any rational function field $\mathbb{A}=\mathbb{K}(x_1,\dots,x_r)$ over an algebraic number field $\mathbb{A}$ is σ-computable; see (Schneider, 2005d).

In the end, we introduce some further definitions, notation and properties for difference fields. In a difference field $(\mathbb{F},\sigma)$ we define the *σ-factorial* $f_{(k)}$ for $f\in\mathbb{F}^*$ and $k\in\mathbb{Z}$ by $\prod_{i=0}^{k-1}\sigma^i(f)$, if $k\geqslant 0$, and by $\prod_{i=1}^{-k}\sigma^{-i}(1/f)$, if $k<0$. Given $\boldsymbol{a}=(a_1,\dots,a_m)\in\mathbb{F}^m$, we write $\sigma_{\boldsymbol{a}}g=a_1\,\sigma^{m-1}(g)+\dots+a_m\,g$.
Let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$. Then $\sigma:\mathbb{F}[t]\to\mathbb{F}[t]$ is a ring automorphism. For all $f,g\in\mathbb{F}[t]$ we have $\deg(\sigma(f))=\deg(f)$ and $\gcd(\sigma(f),\sigma(g))=\sigma(\gcd(f,g))$. If $f\in\mathbb{F}[t]$ is irreducible then also $\sigma(f)\in\mathbb{F}[t]$. Moreover, if $\alpha:=\sigma(t)/t\in\mathbb{F}$, we have $\sigma^k(t)=\alpha_{(k)}\,t$ for all $k\in\mathbb{Z}$.
Let $\mathbb{F}[t]$ be a polynomial ring and $\mathbb{F}(t)$ its quotient field. By $\deg(f)$ we denote the degree of $f\in\mathbb{F}[t]$; by convention we set $\deg(0):=-\infty$. Furthermore, if $f=\sum_{i=0}^n f_i\,t_i\in\mathbb{F}[t]$, the i-th *coefficient* f_i of f is denoted by $[f]_i$, i.e. $[f]_i=f_i$. We define the *order* of $f\in\mathbb{F}[t]^*$, $\mathrm{ord}(f)$, as the maximal $m\geqslant 0$ such that $t^m\mid q$; we set $\mathrm{ord}(0):=-1$. We define the denominator of $f\in\mathbb{F}(t)$ by $\mathrm{den}(f)=q\in\mathbb{F}[t]^*$ where q is monic, $f=\frac{p}{q}$ for some $q\in\mathbb{F}[t]$, and $\gcd(p,q)=1$.

3. The Finite Part of the Denominator Bound In the sequel we apply results from (Bronstein, 2000) in order to find the finite part of the denominator bound in a $\Pi\Sigma$-extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$.

Let $f,g\in\mathbb{F}[t]^*$ with the property that $t\nmid\gcd(f,g)$ if $\frac{\sigma(t)}{t}\in\mathbb{F}$. Then by Proposition 1 we may write

$$\text{spread}_{(\mathbb{F},\sigma)}(f,g)=\{m_1>m_2>\dots>m_s\}, \tag{7}$$

i.e., $\text{spread}_{(\mathbb{F},\sigma)}(f,g)=\{m_1,\dots,m_s\}$ with $m_1>m_2>\dots>m_s$. For such f,g we call $\langle(p_i,q_i,u_i)\,|\,1\leqslant i\leqslant s+1\rangle$ the *bounding sequence* of f and g if
1. $p_1:=f$, $q_1:=g$, $u_1:=1$ and
2. for $1\leqslant i\leqslant s$ we have

$$p_{i+1}:=\frac{p_i}{d_i},\quad q_{i+1}:=\frac{q_i}{\sigma^{-m_i}(d_i)}\qquad\text{and}\qquad u_{i+1}:=u_i\prod_{j=0}^{m_i}\sigma^{-j}(d_i)$$

with $d_i:=\gcd(p_i,\sigma^{m_i}(q_i))$.

The following result, a generalization of (Abramov, 1995), tells us how we can derive the finite part of the denominator part.

Theorem 2. *Let* $(\mathbb{F}(t),\sigma)$ *be a* $\Pi\Sigma$*-ext. of* $(\mathbb{F},\sigma)$*,* $\boldsymbol{a}=(a_1,\dots,a_m)\in\mathbb{F}[t]^m$ *with* $a_1\neq0\neq a_m$ *and* $h\in\mathbb{F}(t)$ *with* $\sigma_{\boldsymbol{a}}h\in\mathbb{F}[t]$*. Consider the bounding sequence* $\langle(p_i,q_i,u_i)\,|\,1\leqslant i\leqslant s+1\rangle$ *of* $f:=\sigma^{m-1}(a_1)$ *and*

$$g:=\begin{cases}\frac{a_m}{t^{\text{ord}(a_m)}} & \text{if } \frac{\sigma(t)}{t}\in\mathbb{F},\\ a_m & \text{otherwise.}\end{cases} \tag{8}$$

Then $\text{spread}_{(\mathbb{F},\sigma)}(u_{s+1},u_{s+1})$ *is finite and* $\text{den}(h)\mid u_{s+1}t^b$ *for some* $b\geqslant0$*. If* $\frac{\sigma(t)}{t}\notin\mathbb{F}$ *then* $\text{den}(h)\mid u_{s+1}$*.*

Proof. It is an easy exercise to show that $\text{spread}_{(\mathbb{F},\sigma)}(u_{s+1},u_{s+1})$ is finite; see (Schneider, 2002; Proposition 6.3). The remaining part follows by (Bronstein, 2000; Theorems 8 and 10).

Note that $\text{spread}_{(\mathbb{F},\sigma)}(f,g)=\emptyset$ for $f,g\in\mathbb{F}^*$. This gives the following result, see (Bronstein, 2000; Cor. 3), needed in the theory of d'Alembertian solutions (Schneider, 2001; Chapt. 4).

Corollary 1. *Let* $(\mathbb{F}(t),\sigma)$ *be a* $\Pi\Sigma$*-ext. of* $(\mathbb{F},\sigma)$*,* $\boldsymbol{a}=(a_1,\dots,a_m)\in\mathbb{F}^m$ *with* $a_1\,a_m\neq0$ *and* $h\in\mathbb{F}(t)$ *with* $\sigma_{\boldsymbol{a}}h\in\mathbb{F}[t]$*. Then* $h\,t^b\in\mathbb{F}[t]$ *for some* $b\geqslant0$*. If* $\frac{\sigma(t)}{t}\notin\mathbb{F}$ *then* $h\in\mathbb{F}[t]$*.*

Proof. Let $h \in \mathbb{F}(t)$ with $\sigma_{\boldsymbol{a}} h \in \mathbb{F}[t]$ and define f, g as in Theorem 2. It follows that $f, g \in \mathbb{F}^*$, thus $\operatorname{spread}_{(\mathbb{F},\sigma)}(f,g) = \emptyset$, and hence the bounding sequence of f and g is $\langle (f, g, 1) \rangle$. By Theorem 2 the corollary follows.

Summarizing, if one can compute (7) in $(\mathbb{F}(t), \sigma)$, we obtain

Algorithm 1. *Compute the finite part of the denominator bound.*
d_0=`DenFinBound(`$(\mathbb{F}(t),\sigma), \boldsymbol{a}, \boldsymbol{f}$`)`
Input: *A $\Pi\Sigma$-extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$, $\boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{F}[t]^m$ with $m \geqslant 2$ and $a_1\, a_m \neq 0$, and $\boldsymbol{f} \in \mathbb{F}[t]^n$.*
Output: *The finite part $d_0 \in \mathbb{F}[t]^*$ of the den. bound. I.e., d_0 is the den. bound of (1) if $\frac{\sigma(t)}{t} \notin \mathbb{F}$; if $\frac{\sigma(t)}{t} \in \mathbb{F}$, there is a $b \geqslant 0$ s.t. $d\, t^b$ is a den. bound of (1).*
(1) Set $f := \sigma^{m-1}(a_1) \in \mathbb{F}[t]^*$ and $g \in \mathbb{F}[t]^*$ as in (8), and compute (7).
(2) Compute the bounding sequence $\langle (p_i, q_i, u_i) \,|\, 1 \leqslant i \leqslant s+1 \rangle$ of f and g.
(3) RETURN u_{s+1}.

Example 3. *Consider the $\Pi\Sigma$-field $(\mathbb{Q}(k)(t), \sigma)$ with $\boldsymbol{a} = (a_1, a_2) \in \mathbb{Q}(k)[t]^2$ from Example 2 and let $f := \sigma^{-1}(a_1) = (k-1)^5\, k^4\, (t+1)\, (k+t)$ and $g := \frac{a_2}{t^{\operatorname{ord}(a_2)}} = -(k+2)(k+3)^2(t+k)(t-k+k^2)$. We have $\operatorname{spread}_{(\mathbb{Q}(t),\sigma)}(f,g) = \{0,1,2\}$; see Example 4. Computing the bounding sequence $\langle (p_i, q_i, u_i) \,|\, 1 \leqslant i \leqslant 4 \rangle$ of f and g, we obtain $u_4 = (t+1)\,(t+k)^2\,(t-k+k^2)$. By Theorem 2 this is nothing else than the finite part of the denominator bound of (4).*

Finally, observe that $\operatorname{spread}_{(\mathbb{F},\sigma)}(f,g)$ in Algorithm 1 can be computed if one can solve problem *SE*. Namely, there is the following

Lemma 1. *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-ext. of $(\mathbb{F}, \sigma)$ and $f, g \in \mathbb{F}[t] \setminus \mathbb{F}$. Then:*
1. If $f = f_1^{m_1} \dots f_r^{m_r}$, $g = g_1^{n_1} \dots g_s^{n_s}$ are complete factorizations,

$$\operatorname{spread}_{(\mathbb{F},\sigma)}(f,g) = \{k \in \mathbb{Z} \,|\, \frac{\sigma^k(g_j)}{f_i} \in \mathbb{F}, 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s\}. \tag{9}$$

2. Suppose that $t \nmid \gcd(f,g)$ or $\frac{\sigma(t)}{t} \notin \mathbb{F}$. If $\frac{\sigma^k(f)}{g} \in \mathbb{F}$ for some $k \in \mathbb{Z}$ then k is uniquely determined.

Proof. (1) The first statement follows by showing that for any $k \in \mathbb{Z}$ the following holds: $\deg(\gcd(f, \sigma^k(g))) > 0$ iff there exist i, j with $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$ such that $\frac{\sigma^k(f_i)}{g_j} \in \mathbb{F}$. Assume there are such i, j, k with $\frac{\sigma^k(g_j)}{f_i} \in \mathbb{F}$. Hence $\sigma^k(g_j) \mid \gcd(\sigma^k(g), f) =: h$, and thus $\deg(h) > 0$. Contrary, assume that there is a $k \in \mathbb{Z}$ such that $\deg(\gcd(f, \sigma^k(g))) > 0$. Then there is an irreducible $h \in \mathbb{F}[t]$ with $\deg(h) > 0$, $h \mid f$ and $h \mid \sigma^k(g)$. Therefore one can take $c, d \in \mathbb{F}^*$ with $h = c\, f_i$ and $h = d\, \sigma^k(g_j)$ for some

$1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$. Thus $\sigma^k(g_j)/f_i \in \mathbb{F}$.
(2) Suppose there are $k, l \in \mathbb{Z}$ with $k > l$ and $\frac{\sigma^k(f)}{g}, \frac{\sigma^l(f)}{g} \in \mathbb{F}$. Then $\frac{\sigma^k(f)}{\sigma^l(f)} = \frac{\sigma^k(f)}{g}\,\frac{g}{\sigma^l(f)} \in \mathbb{F}^*$, hence $\frac{\sigma^{k-l}(f)}{f} \in \mathbb{F}^*$ with $k-l > 0$; a contradiction to Prop. 1.

In (Karr, 1981; Chapter 2) algorithms are developed that solve problem *SE* if $(\mathbb{F}, \sigma)$ is a ΠΣ-field over a σ-computable field. Note that σ-computability enables one to compute gcds in $\mathbb{F}[t]$. Hence, with Lemma 1 we get

Theorem 3. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-field over a σ-computable constant field and $f, g \in \mathbb{F}[t]^*$ with $\frac{\sigma(t)}{t} \notin \mathbb{F}$ or $t \nmid \gcd(f, g)$. Then there is an algorithm that computes the finite set* $\mathrm{spread}_{(\mathbb{F},\sigma)}(f, g)$.

Remark 1. We give a refined strategy to compute $\mathrm{spread}_{(\mathbb{F},\sigma)}(f, g)$. For this we introduce the equivalence relation $a \sim_{(\mathbb{F},\sigma)} b$ for $a, b \in \mathbb{F}(t)^*$ which holds iff $\frac{\sigma^k(a)}{b} \in \mathbb{F}$ for some $k \in \mathbb{Z}$. Then one first factorizes f and g and groups these factors respectively into equivalence classes under $\sim_{(\mathbb{F},\sigma)}$. More precisely, we can write $f = u\, C_1, \dots, C_p$, $g = v\, D_1, \dots, D_p$ with $u, v \in \mathbb{F}^*$ and $C_i = \prod_{j=0}^{r_i} \sigma^j(\alpha_i)^{x_{ij}}$, $D_i = \prod_{j=0}^{r_i} \sigma^j(\alpha_i)^{y_{ij}}$ where the $\alpha_i \in \mathbb{F}[t]$ are irreducible and pairwise not shift-equivalent w.r.t. $\sim_{(\mathbb{F},\sigma)}$. Note that also the $\sigma^j(\alpha_i)$ are irreducible. Hence, in order to compute $\mathrm{spread}_{(\mathbb{F},\sigma)}(f, g)$ with (9), it suffices to compare its multiplicities $x_{ij}, y_{ij} \geqslant 0$.

Example 4. *Consider the ΠΣ-field $(\mathbb{Q}(k)(t), \sigma)$ with $f, g \in \mathbb{Q}(k)[t]$ from Example 3. Then for $\alpha = t + k^2 - k$ we can write $f = \frac{k^4(k-1)^5}{(k+1)(k^2+3k+2)}\sigma(\alpha)\,\sigma^2(\alpha)$ and $g = -\frac{(k+2)(k+3)^2}{k+1}\alpha\,\sigma(\alpha)$ where $\sigma(\alpha) = (k+1)(t+k)$ and $\sigma^2(\alpha) = (k+1)(k+2)(t+1)$. By (9) we can read off* $\mathrm{spread}_{(\mathbb{Q}(t),\sigma)}(f, g) = \{0, 1, 2\}$.

4. The Infinite Part of the Denominator Bound What remains is to find the infinite part of the denominator bound. More precisely, given a Π-extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ with $\mathbb{K} = \mathrm{const}_\sigma\mathbb{F}$, $\mathbf{0} \neq \boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{F}[t]^m$ and $\boldsymbol{f} = (f_1, \dots, f_n) \in \mathbb{F}[t]^n$. Find $b \geqslant 0$ such for all $(c_1, \dots, c_n) \in \mathbb{K}^n$ and $g \in \mathbb{F}(t)$ with $\sigma_{\boldsymbol{a}} g = \sum_{i=1}^n c_i\, f_i$ we have $\mathrm{ord}(\mathrm{den}(g)) \leqslant b$.

We attack the problem by looking for a $b \in \mathbb{N}_0$ such that

$$\forall g \in \mathbb{F}(t):\ \sigma_{\boldsymbol{a}} g \in \mathbb{F}[t] \ \Rightarrow\ \mathrm{ord}(\mathrm{den}(g)) \leqslant b. \tag{10}$$

Remark 2. In the sequel we suppose that for

$$k := \min(\mathrm{ord}(a_1), \dots, \mathrm{ord}(a_m), \mathrm{ord}(f_1), \dots, \mathrm{ord}(f_n))$$

we have $k = 0$. If not, we may divide (1) through t^k without changing problem *PLDE*. Hence we may suppose that $k = 0$. Actually, the assumption $k = 0$ is not necessary for the following considerations. Nevertheless, this preparation step might decrease p in Situations 1, 2 and 3. This might also result in a smaller bound b for (10); see Theorems 4, 6 and 8.

Based on (Karr, 1981) we will show how such a b can be computed for first order difference equations. In addition, we extend these ideas to the higher order case; see Situation 1 and Situation 3.

4.1. Some Properties of the Order and Denominator Function The proof of the following lemma is straightforward; see (Schneider, 2002).

Lemma 2. *Let* $(\mathbb{F}(t), \sigma)$ *be a* Π*-extension of* $(\mathbb{F}, \sigma)$*. Then:*

1. *If* $f_1 \in \mathbb{F}[\frac{1}{t}] \setminus \mathbb{F}^*$, $f_2 \in \mathbb{F}(t)$ *with* $t \nmid \operatorname{den}(f_2)$ *then* $\operatorname{ord}(\operatorname{den}(f_1)) = \operatorname{ord}(\operatorname{den}(f_1 + f_2))$.
2. *Let* $d \geqslant 1$ *and* $f = \sum_{i=0}^{d} \frac{f_i}{t^i} \in \mathbb{F}[1/t]^*$*. Then* $\operatorname{ord}(\operatorname{den}(f)) = d$ *iff* $f_d \neq 0$.
3. *If* $f \in \mathbb{F}[t]$ *then* $\operatorname{ord}(f) = \operatorname{ord}(\sigma^k(f))$ *for all* $k \in \mathbb{Z}$.
4. *If* $f \in \mathbb{F}(t)$ *then* $\sigma(\operatorname{den}(f)) = u \operatorname{den}(\sigma(f))$ *for some* $u \in \mathbb{F}^*$.
5. *Let* $a \in \mathbb{F}[t]^*$ *and* $g \in \mathbb{F}(t)^*$ *with* $\operatorname{ord}(\operatorname{den}(g)) > 0$*. Then for all* $i \geqslant 0$ *we have* $\operatorname{ord}(\operatorname{den}(a\,\sigma^i(g))) = \max(0, \operatorname{ord}(\operatorname{den}(g)) - \operatorname{ord}(a))$.

Proposition 2. *Let* $(\mathbb{F}(t), \sigma)$ *be a* Π*-extension of* $(\mathbb{F}, \sigma)$*, assume* $\mathbf{0} \neq \boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{F}[t]^m$ *and take* $p := \min_i \{\operatorname{ord}(a_i) \,|\, a_i \neq 0\}$*. Let* $g \in \mathbb{F}(t)^*$ *with* $d := \operatorname{ord}(\operatorname{den}(g)) > p$ *and define* $S := \{a_i \,|\, \operatorname{ord}(a_i) = p\}$*. Then* $\operatorname{ord}(\operatorname{den}(\sigma_{\boldsymbol{a}} g)) < d - p$ *if and only if* $\operatorname{ord}(\operatorname{den}(\sum_{i \in S} a_i\, \sigma^{m-i}(g))) < d - p$.

Proof. Take $h_i := a_i\, \sigma^{m-i}(g)$ for all $1 \leqslant i \leqslant m$, set $o_i := \operatorname{ord}(\operatorname{den}(h_i))$ and write (by using partial fraction decomposition) $h_i = h_{i1} + h_{i2}$ where $h_{i1} = \sum_{j=1}^{o_i} \frac{\tilde{h}_{ij}}{t^j} \in \mathbb{F}[\frac{1}{t}] \setminus \mathbb{F}^*$ for some $\tilde{h}_{ij} \in \mathbb{F}$ and $h_{i2} \in \mathbb{F}(t)$ where $t \nmid \operatorname{den}(h_{i2})$. First we show that $0 \leqslant o_i < d - p$. If $a_i = 0$ then $h_i = 0$, and hence $o_i = \operatorname{ord}(1) = 0 < d - p$. Otherwise, if $a_i \neq 0$, by Lemma 2.5 it follows that $o_i = \max(0, d - p_i) \leqslant d - p$. Now split $\sigma_{\boldsymbol{a}} g$ via $\sigma_{\boldsymbol{a}} g = f_1 + f_2$ where $f_1 \in \mathbb{F}[1/t] \setminus \mathbb{F}^*$ and $f_2 \in \mathbb{F}(t)$ with $t \nmid \operatorname{den}(f_2)$. Then we have $f_1 = h_{11} + \dots + h_{m1} = \sum_{j=1}^{o_1} \frac{\tilde{h}_{1j}}{t^j} + \dots + \sum_{j=1}^{o_m} \frac{\tilde{h}_{mj}}{t^j}$ and hence

$$\operatorname{ord}(\operatorname{den}(f)) < d - p \overset{\text{Lemma 2.1}}{\Leftrightarrow} \operatorname{ord}(\operatorname{den}(\sum_{j=1}^{o_1} \frac{\tilde{h}_{1j}}{t^j} + \dots + \sum_{j=1}^{o_m} \frac{\tilde{h}_{mj}}{t^j})) < d - p$$

$$\overset{\text{Lemma 2.2}}{\Leftrightarrow} \sum_{i \in S} \tilde{h}_{i o_i} = 0 \overset{\text{Lemma 2.2}}{\Leftrightarrow} \operatorname{ord}(\operatorname{den}(\sum_{i \in S} \sum_{j=1}^{o_i} \frac{\tilde{h}_{ij}}{t^j})) < d - p.$$

The last inequality is equivalent to $\operatorname{ord}(\operatorname{den}(\sum_{i\in S} a_i\,\sigma^{m-i}(g))) < d-p$ by using Lemma 2.1 again.

4.2. A Simple Case The next theorem delivers a bound $b \in \mathbb{N}_0$ with (10) for the following

Situation 1. *Assume* $\mathbf{0} \neq \boldsymbol{a} = (a_1,\dots,a_m) \in \mathbb{F}[t]^m$ *with* $\operatorname{ord}(a_r) = p$ *for some* $r \in \{1,\dots,m\}$ *and* $\operatorname{ord}(a_i) > p$ *for all* $a_i \neq 0$ *with* $i \neq r$

Theorem 4. *Let* $(\mathbb{F}(t),\sigma)$ *be a* Π*-extension of* $(\mathbb{F},\sigma)$ *and* $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ *as in Situation 1. Then for all* $g \in \mathbb{F}(t)$ *with* $\sigma_{\boldsymbol{a}} g \in \mathbb{F}[t]$ *we have* $\operatorname{ord}(\operatorname{den}(g)) \leqslant p$.

Proof. Suppose that $g \in \mathbb{F}(t)$ with $\sigma_{\boldsymbol{a}} g \in \mathbb{F}[t]$ and $\operatorname{ord}(\operatorname{den}(g)) > p$. Then $\operatorname{ord}(\operatorname{den}(\sigma_{\boldsymbol{a}} g)) = \operatorname{ord}(1) = 0$ and thus $\operatorname{ord}(\operatorname{den}(a_r\,\sigma^{m-r}(g))) < \operatorname{ord}(\operatorname{den}(g)) - p$ by Proposition 2. But by Lemma 2.5 this contradicts, since $\operatorname{ord}(\operatorname{den}(a_r\,\sigma^{m-r}(g))) = \max(0, \operatorname{ord}(\operatorname{den}(g)) - p) = \operatorname{ord}(\operatorname{den}(g)) - p$.

4.3. The First Order Case In this section we will deal with the problem to find a bound b with (10) where $\boldsymbol{a} = (a_1,a_2) \in (\mathbb{F}[t]^*)^2$. If $\operatorname{ord}(a_1) \neq \operatorname{ord}(a_2)$, Thm. 4 does the job. What remains is the case $\operatorname{ord}(a_1) = \operatorname{ord}(a_2)$. More precisely we focus on finding a b with (10) for

Situation 2. *Assume that* $\boldsymbol{a} = (a_1,a_1) \in \mathbb{F}[t]^2$ *with* $a_1 = t^p(1+r_1)$ *and* $a_2 = t^p(-c+r_2)$ *where* $c \in \mathbb{F}^*$ *and* $r_1, r_2 \in \mathbb{F}[t]$ *with* $\operatorname{ord}(r_i) > 0$.

We extend the ideas from (Karr, 1981; Theorem 18).

Theorem 5. *Let* $(\mathbb{F}(t),\sigma)$ *be a* Π*-extension of* $(\mathbb{F},\sigma)$*, and* $\boldsymbol{a} = (a_1,a_2) \in \mathbb{F}[t]^2$ *as in Situation 2. Let* $g \in \mathbb{F}(t)^*$ *with* $d := \operatorname{ord}(g)$.
1. If $d > p$ *and* $\operatorname{ord}(\operatorname{den}(a_1\,\sigma(g) - a_2\,g)) < d-p$ *then* $c\,\alpha^d \in \mathrm{H}_{(\mathbb{F},\sigma)}$.
2. If $\sigma_{\boldsymbol{a}} g = 0$ *then* $c\,\alpha^d \in \mathrm{H}_{(\mathbb{F},\sigma)}$.

Proof. Write $g = \frac{u}{v\,t^d}$ with $u, v \in \mathbb{F}[t]^*$ where $\gcd(u,v) = 1$ and $t \nmid u, v$. We have

$$\begin{aligned} a_1\,\sigma(g) - a_2\,g &= (1+r_1)\,\frac{\sigma(u)}{\sigma(v)}\,\frac{1}{\alpha^d\,t^{d-p}} - (c-r_2)\frac{u}{v}\frac{1}{t^{d-p}} \\ &= \frac{(1+r_1)\,\sigma(u)\,v - (c-r_2)\,u\,\sigma(v)\,\alpha^d}{\sigma(v)\,v}\,\frac{1}{\alpha^d\,t^{d-p}}. \end{aligned} \tag{11}$$

1. Suppose that $d > p$ and $\operatorname{ord}(\operatorname{den}(a_1\,\sigma(g) - a_2\,g)) < d-p$. Then by (11),

$$t \mid ((1+r_1)\,\sigma(u)\,v - (c-r_2)\,u\,\sigma(v)\,\alpha^d). \tag{12}$$

2. Similarly, if $a_1\sigma(g) + a_2(g) = 0$, we have (12).

Hence, in both cases, it follows $\big[(1 + r_1)\,\sigma(u)\,v - (c - r_2)\,u\,\sigma(v)\,\alpha^d\big]_0 = 0$. Let $u_0 := [u]_0 \in \mathbb{F}^*$ and $v_0 := [v]_0 \in \mathbb{F}^*$. Since $t \mid r_1$ and $t \mid r_2$, we get

$$\sigma(u_0)\,v_0 - c\,u_0\,\sigma(v_0)\,\alpha^d = 0 \Leftrightarrow \frac{\sigma(u_0)\,v_0}{u_0\,\sigma(v_0)} = c\,\alpha^d \Leftrightarrow \frac{\sigma(h)}{h} = c\,\alpha^d$$

for $h := \frac{u_0}{v_0} \in \mathbb{F}^*$ and thus $c\,\alpha^d \in \mathrm{H}_{(\mathbb{F},\sigma)}$.

So far we just required that in the difference field $(\mathbb{F}(t), \sigma)$ the extension t is transcendental and that $\sigma(t) = \alpha\,t$ for some $\alpha \in \mathbb{F}^*$. Only in the next lemma all properties of Π-extensions are really exploited.

Lemma 3. *Let $(\mathbb{F}(t), \sigma)$ be a Π-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\,t$, $\alpha \in \mathbb{F}^*$ and $c \in \mathbb{F}^*$. If there exists a $d \in \mathbb{Z}$ with $c\,\alpha^d \in \mathrm{H}_{(\mathbb{F},\sigma)}$ then d is uniquely determined.*

Proof. Assume there are $d_1, d_2 \in \mathbb{Z}$ with $d_1 < d_2$ and $c\,\alpha^{d_1} \in \mathrm{H}_{(\mathbb{F},\sigma)}$, $c\,\alpha^{d_2} \in \mathrm{H}_{(\mathbb{F},\sigma)}$, i.e., there are $g_1, g_2 \in \mathbb{F}^*$ such that $\frac{\sigma(g_1)}{g_1} = c\,\alpha^{d_1}$, $\frac{\sigma(g_2)}{g_2} = c\,\alpha^{d_2}$. Since $d_2 - d_1 > 0$, it follows that $\alpha^{d_2-d_1} = \frac{\sigma(g_2)/g_2}{\sigma(g_1)/g_1} = \frac{\sigma(g_2/g_1)}{g_2/g_1}$ and thus $\alpha^{d_2-d_1} \in \mathrm{H}_{(\mathbb{F},\sigma)}$. By Theorem 1 $(\mathbb{F}(t), \sigma)$ is not a Π-extension of $(\mathbb{F}, \sigma)$, a contradiction.

Combining Theorem 5 and Lemma 3 leads to a recipe how to compute b with (10).

Theorem 6. *Let $(\mathbb{F}(t), \sigma)$ be a Π-extension of $(\mathbb{F}, \sigma)$ and $a_1, a_2 \in \mathbb{F}[t]$ as in Situation 2. Moreover let $g \in \mathbb{F}(t)$ with $a_1\,\sigma(g) + a_2\,g \in \mathbb{F}[t]$. If there exists a $d \in \mathbb{N}_0$ such that $c\,\alpha^d \in \mathrm{H}_{(\mathbb{F},\sigma)}$ then d is uniquely determined and we have* $\mathrm{ord}(\mathrm{den}(g)) \leqslant \max(d, p)$. *If there does not exist such a d then* $\mathrm{ord}(\mathrm{den}(g)) \leqslant p$.

Proof. Let $g \in \mathbb{F}(t)$ with $f := a_1\,\sigma(g) - a_2\,g \in \mathbb{F}[t]$. We have

$$\mathrm{ord}(\mathrm{den}(f)) = \mathrm{ord}(1) = 0. \tag{13}$$

First assume there exists a $d \geqslant 0$ with $c\,\alpha^d \in \mathrm{H}_{(\mathbb{F},\sigma)}$. Then by Lemma 3 d is uniquely determined. Assume $\mathrm{ord}(\mathrm{den}(g)) > p$. Since (13), by Theorem 5.1 it follows that $\mathrm{ord}(\mathrm{den}(g)) = d$ and therefore $\mathrm{ord}(\mathrm{den}(g)) = d = \max(p, d)$. Otherwise, if $\mathrm{ord}(\mathrm{den}(g)) \leqslant p$ then we have $\mathrm{ord}(\mathrm{den}(g)) \leqslant \max(p, d)$.

Now assume that there does not exist a $d \geqslant 0$ with $c\,\alpha^d \in \mathrm{H}_{(\mathbb{F},\sigma)}$. Since (13), by Theorem 5.1 it follows that $\mathrm{ord}(\mathrm{den}(g)) \leqslant p$.

By the previous considerations one obtains the following result.

Algorithm 2. *Compute the infinite part of the denominator bound.*
b=`DenInfBound`$((\mathbb{F}(t),\sigma),\boldsymbol{a})$
Input: *A* $\Pi\Sigma$*-extension* $(\mathbb{F}(t),\sigma)$ *of* $(\mathbb{F},\sigma)$ *in which one can solve problem HG;* $\mathbf{0}\neq\boldsymbol{a}=(a_1,a_2)\in\mathbb{F}[t]^2$.
Output: *A* $b\in\mathbb{N}_0$ *that fulfills* (10).

(1) IF $\operatorname{ord}(a_1)\neq\operatorname{ord}(a_2)$ *THEN RETURN* $\min(\operatorname{ord}(a_1),\operatorname{ord}(a_2))$.
(2) Set $p:=\operatorname{ord}(a_1)$ *and define* $c:=-\frac{[a_2]_p}{[a_1]_p}$.
(3) If there exists a $d\in\mathbb{N}_0$ *such that* $c\,\alpha^d\in\mathrm{H}_{(\mathbb{F},\sigma)}$
(4) THEN take such a d *and RETURN* $\max(d,p)$ *ELSE RETURN* p.

Now suppose in addition that $(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-field over a σ-computable constant field $\mathbb{K}$. Then by (Karr, 1981; Theorem 9) there is an algorithm that can solve problem *HG*. Hence we can apply our Algorithm 2.

Example 5. *Consider the* $\Pi\Sigma$*-field* $(\mathbb{Q}(k)(t),\sigma)$ *over the* σ*-computable constant field* $\mathbb{Q}$*, and* $\boldsymbol{a}=(a_1,a_2)\in\mathbb{Q}(k)[t]^2$ *as in Example 2. We have* $c:=-\frac{[a_2]_p}{[a_1]_p}=\frac{(-1+k)\,(2+k)\,(3+k)^2}{k^3\,(1+k)^5}$*. By Karr's algorithm, see (Karr, 1981; Theorem 9), we find* $d=4$ *such that* $c\,(k+1)^d\in\mathrm{H}_{(\mathbb{Q}(k)(t),\sigma)}$*. (Actually, there is the relation* $\frac{\sigma(h)}{h}=c\,(k+1)^d$ *with* $h:=\frac{k^2\,(1+k)^3\,(2+k)^2}{-1+k}$*.) Hence* $b:=\max(d,\operatorname{ord}(a_1))=4$ *fulfills* (10)*. This gives the infinite part* t^4 *of the denominator bound of* (4).

4.4. A Generalization for Higher Order Equations Subsequently, we look for a bound $b\in\mathbb{N}_0$ with (10) for the more general Situation 3 that contains Situation 2.

Situation 3. *Assume* $\mathbf{0}\neq\boldsymbol{a}=(a_1,\dots,a_\lambda,\dots,a_\mu\,\dots,a_m)\in\mathbb{F}[t]^m$ *with* $\lambda<\mu$, $\operatorname{ord}(a_\lambda)=\operatorname{ord}(a_\mu)=p$ *and*

$$\operatorname{ord}(a_i)>\operatorname{ord}(a_\lambda)\ \text{or}\ a_i=0\ \ \forall i\neq\lambda,\mu.$$

In particular suppose that $a_\lambda=t^p+r_1$ *and* $a_\mu=-c\,t^p+r_2$ *for* $c\in\mathbb{F}^*$ *and* $r_1,r_2\in\mathbb{F}[t]$ *with* $\operatorname{ord}(r_1),\operatorname{ord}(r_2)>0$.

First we generalize Theorem 5.1.

Theorem 7. *Let* $(\mathbb{F}(t),\sigma)$ *be a* Π*-extensions of* $(\mathbb{F},\sigma)$*,* $\boldsymbol{a}\in\mathbb{F}[t]^m$ *as in Situation 3 and assume that* $(\mathbb{F}(t),\sigma^{\mu-\lambda})$ *is a* Π*-extension of* $(\mathbb{F}^{\mu-\lambda},\sigma)$*. If there exists a* $g\in\mathbb{F}(t)$ *with* $d:=\operatorname{ord}(\operatorname{den}(g))>p$ *such that* $\operatorname{ord}(\operatorname{den}(\sigma_{\boldsymbol{a}}g))<d-p$ *then* $\sigma^{\mu-m}(c)\,\alpha^d_{(\mu-\lambda)}\in\mathrm{H}_{(\mathbb{F},\sigma^{\mu-\lambda})}$.

Proof. Let $g \in \mathbb{F}(t)$ with $d := \mathrm{ord}(\mathrm{den}(g)) \geqslant p$ and assume $\mathrm{ord}(\mathrm{den}(\sigma_{\boldsymbol{a}} g)) < d-p$. Then $d-p > \mathrm{ord}(\mathrm{den}(a_\lambda\, \sigma^{m-\lambda}(g) + a_\mu\, \sigma^{m-\mu}(g)))$ by Proposition 2 and Situation 3 and thus by Lemmas 2.4 and 2.3 we have

$$\begin{aligned} d-p &> \mathrm{ord}(\sigma^{\mu-m}(\mathrm{den}(a_\lambda\, \sigma^{m-\lambda}(g) + a_\mu\, \sigma^{m-\mu}(g)))) \\ &= \mathrm{ord}(\mathrm{den}(\sigma^{\mu-m}(a_\lambda)\, \sigma^{\mu-\lambda}(g) + \sigma^{\mu-m}(a_\mu)\, g). \end{aligned}$$

By $\sigma^{\mu-m}(a_\lambda) = \alpha^p_{(\mu-m)}\, t^p + \sigma^{\mu-m}(r_1)$, $\sigma^{\mu-m}(a_\mu) = -\sigma^{\mu-m}(c)\, \alpha^p_{(\mu-m)}\, t^p + \sigma^{\mu-m}(r_2)$ it follows that $\mathrm{ord}(\mathrm{den}(b_1\, \sigma^{\mu-\lambda}(g) + b_2\, g)) < d-p$ for $b_1 := t^p + \sigma^{\mu-m}(r_1)/\alpha^p_{(\mu-m)}$ and $b_2 := -\sigma^{\mu-m}(c)\, t^p + \sigma^{\mu-m}(r_2)/\alpha^p_{(\mu-m)}$. As $(\mathbb{F}(t), \sigma^{\mu-\lambda})$ is a Π-extension of $(\mathbb{F}, \sigma^{\mu-\lambda})$, we may apply Theorem 5.1 and thus we obtain $\sigma^{\mu-m}(c)\, \alpha^d_{\mu-\lambda} \in \mathrm{H}_{(\mathbb{F},\sigma^{\mu-\lambda})}$.

Finally we obtain a degree bound method for Situation 3.

Theorem 8. *Let $(\mathbb{F}(t), \sigma)$ be a Π-extension of $(\mathbb{F}, \sigma)$, $\boldsymbol{a} \in \mathbb{F}[t]^m$ as in Situation 3 and suppose that $(\mathbb{F}(t), \sigma^{\mu-\lambda})$ is a Π-extension of $(\mathbb{F}, \sigma^{\mu-\lambda})$. Let $g \in \mathbb{F}(t)$ such that $\sigma_{\boldsymbol{a}} g = \mathbb{F}[t]$. If there exists a $d \in \mathbb{N}_0$ such that $\sigma^{(\mu-m)}(c)\, \alpha^d_{\mu-\lambda} \in \mathrm{H}_{(\mathbb{F},\sigma^{\mu-\lambda})}$ then d is uniquely determined and* $\mathrm{ord}(\mathrm{den}(g)) \leqslant \max(d,p)$. *Otherwise, if there does not exist such a d then* $\mathrm{ord}(\mathrm{den}(g)) \leqslant p$.

Proof. Let $g \in \mathbb{F}(t)$ with $f := \sigma_{\boldsymbol{a}} g \in \mathbb{F}[t]$. It follows (13). Assume there exists a $d \geqslant 0$ with $\sigma^{\mu-m}(c)\, \alpha^d_{(\mu-\lambda)} \in \mathrm{H}_{(\mathbb{F},\sigma^{\mu-\lambda})}$. Then by Lemma 3 d is uniquely determined. Suppose $\mathrm{ord}(\mathrm{den}(g)) > p$. Since (13), by Theorem 7 it follows that $\mathrm{ord}(\mathrm{den}(g)) = d = \max(p,d)$. Otherwise, if $\mathrm{ord}(\mathrm{den}(g)) \leqslant p$, we have $\mathrm{ord}(\mathrm{den}(g)) \leqslant \max(p,d)$.
Now assume there does not exist a $d \geqslant 0$ with $\sigma^{\mu-m}(c)\, \alpha^d_{(\mu-\lambda)} \in \mathrm{H}_{(\mathbb{F},\sigma^{\mu-\lambda})}$. Since (13), by Theorem 7 it follows that $\mathrm{ord}(\mathrm{den}(g)) \leqslant p$.

Note that in (Karr, 1985; Thm: page 314) it has been shown that if $(\mathbb{F}, \sigma)$ is a $\Pi\Sigma$-field, also $(\mathbb{F}, \sigma^k)$ is a $\Pi\Sigma$-field for all $k \in \mathbb{Z}^*$. Hence, if $(\mathbb{F}(t), \sigma)$ is a $\Pi\Sigma$-field over a σ-computable constant field $\mathbb{K}$, one can solve problem *HG* in $(\mathbb{F}, \sigma^k)$; see (Karr, 1981; Theorem 9). Summarizing, Theorem 8 gives an algorithm to compute a bound $b \in \mathbb{N}_0$ that fulfills (10) for the special case described in Situation 3.

5. Extension Stable Denominator Bounds For The First Order Case
Combining Algorithms 1 and 2 we obtain the following result.

Algorithm 3. *Compute a denominator bound for the first order case.*
d=`DenBound`$((\mathbb{F}(t),\sigma),\boldsymbol{a},\boldsymbol{f})$
Input: *A $\Pi\Sigma$-extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ in which one can solve problems SE and HG; $\boldsymbol{a}=(a_1,a_2)\in(\mathbb{F}[t]^*)^2$ and $\boldsymbol{f}\in\mathbb{F}[t]^n$.*
Output: *A denominator bound $d\in\mathbb{F}[t]^*$ for (1).*

(1) Let $d\in\mathbb{F}[t]^*$ be the result of `DenFinBound`$((\mathbb{F}(t),\sigma),\boldsymbol{a},\boldsymbol{f})$ *in Algorithm 1.*
(2) If $(\mathbb{F},\sigma)$ is a Σ-extension of $(\mathbb{F},\sigma)$ THEN $b:=0$
ELSE let $b\in\mathbb{N}_0$ be given by `DenInfBound`$((\mathbb{F}(t),\sigma),\boldsymbol{a})$ *of Algorithm 2.*
(3) RETURN $d\,t^b$.

Theorem 9. *Let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$ in which one can solve problems SE, HG; let $\boldsymbol{a}\in(\mathbb{F}[t]^*)^2$ and $\boldsymbol{f}\in\mathbb{F}[t]^n$. Then there is an algorithm that computes a denominator bound for (1). Problems SE, HG can be solved if $(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-field over a σ-computable $\mathbb{K}$.*

Example 6. *Looking at Examples 3 and 5 we compute the denominator bound $d:=t^4\,(1+t)\,(t+t)^2\,(-k+k^2+t)$ for Example 2.*

In (Schneider, 2004b, Schneider, 2005b) algorithms have been developed that allow one to discover identities, like

$$\sum_{k=0}^{n}H_k^3=\frac{1}{2}\Big[-12n+6(2n+1)H_n-3(2n+1)H_n^2+2(n+1)H_n^3+H_n^{(2)}\Big]\quad(14)$$

where $H_k=\sum_{i=1}^{k}\frac{1}{i}$ and $H_k^{(2)}=\sum_{i=1}^{k}\frac{1}{i^2}$. Knowing that the left hand side of (14) can be expressed in terms of H_n and $H_n^{(2)}$, one can proceed as follows. One applies our algorithms in (Schneider, 2005e, Schneider, 2005a) in combination with Algorithm 3 in order to find $g(k)=\frac{1}{2}\big[-12k+6(2k+1)H_k-3(2k+1)H_k^2+2kH_k^3+H_k^{(2)}\big]$ that satisfies

$$g(k+1)-g(k)=H_k^3.\quad(15)$$

Then by telescoping we obtain (14); for more details see (Schneider, 2004a). The difficulty of this approach is to find the sum extension $H_k^{(2)}$ in which such a telescoper $g(k)$ of (15) exists. Our refined summation algorithms given in (Schneider, 2004b, Schneider, 2005b) solve this problem. Namely, they can decide if such a telescoper $g(k)$ in terms of a sum extension exists; if yes, they can compute such a solution.

One of the important ingredients for our refined algorithms is the following result stated in Theorem 10. Namely, we need the property that Alg. 3 is extension-stable. This means that extending the underlying difference field by certain additional $\Pi\Sigma$-extensions does not change the result. In order to prove this property, we modify Alg. 3 as follows.

REMARK 3. We normalize the output $d \in \mathbb{F}[t]^*$ of Algorithm 3. For instance, we return $d \in \mathbb{F}[t]^*$ by forcing the leading coefficient of d to be 1.

Let $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$ with $\sigma(t_i)=\alpha_i\,t_i+\beta_i$ and suppose that there is a permutation $\tau:\{1,\dots,e\}\to\{1,\dots,e\}$ such that $\alpha_{\tau(i)},\beta_{\tau(i)}\in\mathbb{F}(t_{\tau(1)})\dots(t_{\tau(i-1)})$ for all $1\leqslant i\leqslant e$. Then for such a τ the generators of a $\Pi\Sigma$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ can be reordered without changing the $\Pi\Sigma$ nature of the extension. In short, we say that $(\mathbb{F}(t_{\tau(1)})\dots(t_{\tau(e)}),\sigma)$ *is equal to* $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ *up to reordering* if there exists such a permutation τ.

Theorem 10. *Let* $(\mathbb{F}(x_1)\dots(x_e)(t)(s),\sigma)$ *and* $(\mathbb{F}(s)(x_1)\dots(x_e)(t),\sigma)$ *be* $\Pi\Sigma$*-extensions of* $(\mathbb{F},\sigma)$ *which are equal up to reordering; write* $\mathbb{G}=\mathbb{F}(x_1)\dots(x_e)$ *and* $\mathbb{E}=\mathbb{F}(s)(x_1)\dots(x_e)$*. Suppose that one can solve problem HG in* $(\mathbb{G},\sigma)$ *and* $(\mathbb{E},\sigma)$*, and problem SE in* $(\mathbb{G}(t),\sigma)$ *and* $(\mathbb{E}(t),\sigma)$*. Let* $\boldsymbol{a}\in(\mathbb{G}[t]^*)^2$ *s.t. there is an* $h\in\mathbb{G}(t)^*$ *with* $\sigma_{\boldsymbol{a}}h=0$*, and let* $\boldsymbol{f}\in\mathbb{G}[t]^n$*. Then we have*

$$\texttt{DenBound}((\mathbb{G}(\texttt{t}),\sigma),\boldsymbol{a},\boldsymbol{f})=\texttt{DenBound}((\mathbb{E}(\texttt{t}),\sigma),\boldsymbol{a},\boldsymbol{f})$$

where the result of DenBound *is normalized as stated in Remark 3.*

Proof. By assumption we can apply the Algorithms 1 and 2 as stated above. First we show that

$$u\,\texttt{DenFinBound}((\mathbb{E}(t),\sigma),\boldsymbol{a},\boldsymbol{f})=\texttt{DenFinBound}((\mathbb{G}(t),\sigma),\boldsymbol{a},\boldsymbol{f}) \tag{16}$$

for a unit $u\in\mathbb{E}$. Note that for any $p,q\in\mathbb{G}[t]^*$ and any $m\in\mathbb{Z}$ we have that $\gcd_{\mathbb{G}[t]}(p,\sigma^m(q))$ and $\gcd_{\mathbb{E}[t]}(p,\sigma^m(q))$ differ only by a factor from $\mathbb{E}$. This fact implies that the computed spreads in line (1) of Algorithm 1 are the same for the inputs (16). Hence also the computed bounding sequences in line (2) can differ only by a unit in $\mathbb{E}$. This shows (16). Now suppose that $\sigma(t)=\alpha\,t$. Then,

$$\texttt{DenInfBound}((\mathbb{E}(t),\sigma),\boldsymbol{a},\boldsymbol{f})=\texttt{DenInfBound}((\mathbb{G}(t),\sigma),\boldsymbol{a},\boldsymbol{f}): \tag{17}$$

Write $\boldsymbol{a}=(a_1,a_2)$. If we have $\mathrm{ord}(a_1)\neq\mathrm{ord}(a_2)$ then in both cases the output will be the same in line (1). Otherwise assume equality. By Theorem 5.1 we find a $d\in\mathbb{Z}$ such that $c\,\alpha^d\in\mathrm{H}_{(\mathbb{G},\sigma)}$. For this d we also have $c\,\alpha^d\in\mathrm{H}_{(\mathbb{E},\sigma)}$. Since d is unique by Lemma 3, it follows that for both cases we find the same d. Summarizing, after the normalization (Remark 3) the theorem follows.

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