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SIMPLIFYING SUMS IN ΠΣ*-EXTENSIONS

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We present telescoping algorithms which compute optimal sum representations of indefinite nested sums. More precisely, given a rational summand expression in terms of nested sums and products, the algorithm splits the summand into a summable part, which can be summed by telescoping, and into a non-summable part, which is degree-optimal with respect to one of the most nested sums or products. If possible, all the most nested sums and products can be eliminated in the non-summable part. We illustrate our summation algorithms by various concrete examples.

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1. Introduction

Indefinite summation can be described by the following telescoping problem: *Given* f where f belongs to some domain of sequences $\mathbb{E}$, *find* $g \in \mathbb{E}$ such that

$$f(k) = g(k+1) - g(k). \tag{1.1}$$

Then given such a solution g, we get the closed form evaluation

$$\sum_{k=a}^{b} f(k) = g(b+1) - g(a),$$

i.e., the sum $\sum_{k=a}^{b} f(k)$ can be simplified in terms of the sequences given in $\mathbb{E}$. E.g., there are algorithms for the rational case, see [2], for hypergeometric terms, see [10,20], for q-hypergeometric terms, see [16,18], or more generally, for $\Pi\Sigma^*$-terms, see [13,24]. Here indefinite nested sums and products are represented in the difference field setting of $\Pi\Sigma^*$-fields. Typical examples of such sums and products are d'Alembertian solutions [3,22], a subclass of Liouvillian solutions [12] of linear recurrences.

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We consider the following refined problem: *Given* $f \in \mathbb{E}$, *find* $(f', g) \in \mathbb{E}^2$ such that

$$f(k) = g(k+1) - g(k) + f'(k) \tag{1.2}$$

where f' is as "small" as possible; $f' = 0$ is the optimal choice. Then given such a solution (f', g), we obtain

$$\sum_{k=a}^{b} f(k) = g(b+1) - g(a) + \sum_{k=a}^{b} f'(k),$$

i.e., the sum $\sum_{k=a}^{b} f(k)$ can be simplified in terms of the sequences given in $\mathbb{E}$ and by the sum $\sum_{k=a}^{b} f'(k)$. In a nutshell, one tries to solve the classical telescoping problem in $\mathbb{E}$ ($f' = 0$), and if this is not possible, tries to keep the non-summable part $f'(k)$ as small as possible.

For the rational case this refined telescoping approach has been considered in [4]; here the minimality of f' is defined by the degree of the denominator polynomial. Theoretical inside and additional algorithmic approaches have been derived in [17].

For the $\Pi\Sigma^*$-field case the following variation has been considered in [25,27]: find a summand $f'(k)$ where the depth of the nested sums and products is optimal.

Based on the algorithmic theory given in [13] we shall develop a framework which combines both versions: choose one of the most nested sums or products in $f(k)$ and find $f'(k)$ such that the degrees of its polynomial and fractional part are optimal w.r.t. to the selected sum or product. Applying this strategy recursively, we can eliminate, if possible, all such most nested sums and products in $f'(k)$. Typical examples are

$$\sum_{k=2}^{n} \frac{-kH_k^5 + H_k^4 - kH_k + 2}{H_k - kH_k^2} = \frac{1}{2}\sum_{k=2}^{n} \frac{2k^2 + H_k}{k^2 H_k} + (n+1)H_n^3 - (2n+1)\big(\tfrac{3}{2}H_n^2 - 3H_n\big) - \tfrac{3}{2}(4n+1) + \frac{1}{H_n}, \tag{1.3}$$

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{x}{i}\right)^2 = (x-n)\binom{x}{n}\sum_{i=0}^{n}\binom{x}{i} - \tfrac{x-2n-2}{2}\left(\sum_{i=0}^{n}\binom{x}{i}\right)^2 - \frac{x}{2}\sum_{k=0}^{n}\binom{x}{k}^2, \tag{1.4}$$

$$\sum_{k=1}^{n-1} H_k^2 H_k^{(2)} = -\tfrac{H_n^3}{3} + \big(nH_n^{(2)} + 1\big)H_n^2 + (2n+1)H_n^{(2)}\big(1 - H_n\big) - 2H_n + \tfrac{H_n^{(3)}}{3} \tag{1.5}$$

where $H_k = \sum_{j=1}^{k} 1/j$ denote the harmonic numbers and $H_k^{(r)} = \sum_{j=1}^{k} 1/j^r$, $r > 1$, are its generalized versions. In (1.3) we simplify the sum on the left-hand side by finding $f'(k) = \frac{k^2 + H_k}{k^2 H_k}$ where the degrees w.r.t H_k are optimal. Moreover, in (1.4), which is a generalized version from [6, Page 9], we compute $f'(k) = \binom{x}{k}^2$ which is free of $\sum_{i=0}^{k}\binom{x}{i}$. In (1.5) we simplify the sum on the left-hand side by finding $f'(k) = \frac{1}{k^3}$ which is free of H_k and $H_k^{(2)}$.

The algorithms under consideration are illustrated by various concrete examples; some of them pop up in [32,19,24,9]. All these ideas are implemented in the summation package Sigma [24].

The general structure is as follows. In Section 2 we formulate the refined telescoping problem *RT* in difference fields and supplement it by examples. In Section 3 we split problem *RT* in the two subproblems *PP* and *RP* which we solve in Sections 4 and 5. Using these results, we get an algorithm which can eliminate, if possible, all the extensions which are most nested, see Section 6. In Section 7 we show how problem (1.2) is related to the theory of $\Pi\Sigma^*$-extensions.

2. The problem in $\Pi\Sigma^*$-extensions

We describe the domain of sequences $\mathbb{E}$ in problem (1.2) by *difference fields*, i.e., by a field $\mathbb{E}$ and a field automorphism $\sigma : \mathbb{E} \to \mathbb{E}$; in short we write $(\mathbb{E}, \sigma)$. All fields in this article are understood as having characteristic 0. The *constant field* of $(\mathbb{E}, \sigma)$ is defined by $\mathbb{K} := \{c \in \mathbb{E} \,|\, \sigma(c) = c\}$. It is easy to see that $\mathbb{K}$ is a subfield of $\mathbb{E}$; this implies that $\mathbb{Q} \subseteq \mathbb{K}$. Then problem (1.2) can be formulated as follows. Given $f \in \mathbb{E}$, find $(f', g) \in \mathbb{E}^2$ such that

$$\sigma(g) - g + f' = f \tag{2.1}$$

where f' is as simple as possible. We call $(f', g) \in \mathbb{E}$ a Σ*-pair* for f if it fulfills (2.1).

Subsequently, we restrict to difference fields which can be obtained by certain *difference field extensions* called $\Pi\Sigma^*$-extensions. A difference field $(\mathbb{E}, \sigma)$ is called a *difference field extension* of $(\mathbb{F}, \sigma')$, if $\mathbb{F}$ is a sub-field of $\mathbb{E}$ and $\sigma|_{\mathbb{F}} = \sigma'$ (since σ and σ' agree on $\mathbb{F}$, we do not distinguish them anymore). A difference field extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ is called a $\Pi\Sigma^*$*-extension*, if $\mathbb{E} = \mathbb{F}(t)$ is a rational function field extension, the field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ is extended to $\sigma : \mathbb{F}(t) \to \mathbb{F}(t)$ by $\sigma(t) = a\,t$ or $\sigma(t) = t + a$ for some $a \in \mathbb{F}^*$, and the constant field remains unchanged, i.e., $\mathrm{const}_\sigma\mathbb{F}(t) = \mathrm{const}_\sigma\mathbb{F} = \mathbb{K}$. If $\sigma(t) = a\,t$, we call the extension also a Π*-extension*; if $\sigma(t) = t + a$, we call it a Σ^**-extension*.

Remark 2.1. Note that there are decision procedures which enable one to test if a given extension is a $\Pi\Sigma^*$-extension. For Σ^*-extensions we refer to Section 7. For general Π-extensions we refer to [13]. Here we mention only that a hypergeometric term, like $\binom{n}{k}$ or $k!$, can be always represented by a Π-extension; only objects like $(-1)^k$ cannot be handled, see [28].

For such a $\Pi\Sigma^*$-extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ we are interested in the following problem:

RT: Refined Telescoping with optimal degree

Given $f \in \mathbb{F}(t)$; **find** a Σ-pair (f', g) for f where among the possible f' with

$$f' = p + \frac{q}{d} \quad \text{where} \quad p, q, d \in \mathbb{F}[t], \text{ and } \deg(q) < \deg(d) \tag{2.2}$$

the degree of d and the degree of p are minimal; we set $\deg(0) = -\infty$.

Remark. **(1)** The constraint that $\deg(p)$ is minimal does not restrict the constraint that $\deg(d)$ is minimal and vice versa. For further explanations we refer to Section 3. **(2)** In [28] we consider the analogue problem for products: given $f \in \mathbb{F}(t)$, find $(f', g) \in \mathbb{F}(t)^2$ with $\frac{\sigma(g)}{g} f' = f$ where the degrees of the numerator and denominator of f' are minimal. For the rational case see [1].

In this article we develop algorithms for problem RT where $\mathbb{F}$ is a $\Pi\Sigma^*$-field. This means that we start with the constant field $\mathbb{K}$ and adjoin step by step either a Π- or a Σ^*-extension t_i on top. Following [13] we call such a tower of $\Pi\Sigma^*$-extensions $\mathbb{K}(t_1)\dots(t_e)$ a $\Pi\Sigma^*$*-field*.
Usually, one chooses for t in RT a sum or product which is most nested.

We illustrate problem RT by various concrete examples. In Examples 2.1–2.5 we focus on the problem to obtain non-summable parts where the degree of p is reduced. In Examples 2.6–2.9 we compute non-summable parts where the degree of d reduced. In Example 2.10 (see identity (1.3)) we compute a non-summable part where the degrees in p and d are optimal.

Example 2.1. Consider the rational case, i.e., take the difference field $(\mathbb{K}(k), \sigma)$ with $\sigma(k) = k+1$ and $\text{const}_\sigma\mathbb{K}(k) = \mathbb{K}$; note that this is a Σ^*-extension of $(\mathbb{K}(k), \sigma)$. Then for any $f \in \mathbb{K}[k]$ we can compute a $g \in \mathbb{K}[k]$ with $\sigma(g) - g = f$; see e.g. [11, (6.10),(6.11),(2.45)]. For the q-rational case we have a similar result: Take the constant field $\mathbb{K}(q)$ with a parameter q and consider the $\Pi\Sigma^*$-extension $(\mathbb{K}(q)(P), \sigma)$ with $\sigma(P) = q\,P$. Since $\sigma(\frac{P^i}{q^i-1}) - \frac{P^i}{q^i-1} = P^i$ for $i > 0$, we can find for any $f = \sum_{i=1}^n f_i P^i \in \mathbb{K}(q)[P]$ a $g \in \mathbb{K}(q)[P]$ with $\sigma(g) - g = f$.

Example 2.2. Given $\sum_{k=1}^n H_k^4$, we derive the identity

$$\sum_{k=1}^n H_k^4 = H_n^2\big((n+1)H_n^2 - 2(2n+1)H_n + 12n\big) + \sum_{k=1}^n \frac{12k^2-8k-1-2kH_k\big(12k^2-6k-1\big)}{k^3} \tag{2.3}$$

as follows. Take the difference field $(\mathbb{Q}(k), \sigma)$ with $\sigma(k) = k+1$, and extend it with the Σ^*-extension $(\mathbb{Q}(k)(H), \sigma)$ where $\sigma(H) = H + \frac{1}{k+1}$. Note that the shift of H_k in k is reflected by the automorphism σ acting on H. Then we compute the Σ-pair $(f', g) = (\frac{12k^2-8k-1-2kH(12k^2-6k-1)}{k^3}, \frac{(Hk-1)^2((H^2-4H+12)k^2-8k-1)}{k^3})$ for $f = H^4$; for further details see Example 4.2. This delivers (1.2) with $f(k) = H_k^4$, $f'(k) = \frac{12k^2-8k-1-2kH_k(12k^2-6k-1)}{k^3}$ and $g(k) = \frac{(H_k k-1)^2((H_k^2-4H_k+12)k^2-8k-1)}{k^3}$. Summing (1.2) over k gives (2.3). Note that $\sum_{k=1}^n f'(k) = 12\sum_{k=1}^n \frac{H_k}{k} + 2\sum_{k=1}^n \frac{H_k}{k^2} - 24\sum_{k=1}^n H_k + 12H_n - 8H_n^{(2)} - H_n^{(3)}$. With the identities $\sum_{k=1}^n H_k =$

$H_n(n+1)-n$ and $\sum_{k=1}^{n}\frac{H_k}{k}=\frac{1}{2}(H_n^2+H_n^{(2)})$, which we can also find with our machinery, we get

$$\sum_{k=1}^{n}H_k^4=(n{+}1)H_n^4-(2n{+}1)\big(2H_n^3-6H_n^2+12H_n\big)+24n-H_n^{(3)}-2H_n^{(2)}+2\sum_{k=1}^{n}\frac{H_k}{k^2}. \tag{2.4}$$

Example 2.3. Given $\sum_{k=1}^{n}H_k^3$, we take $(\mathbb{Q}(k)(H),\sigma)$ from Example 2.2 and compute the Σ-pair $(f',g)=\big(-\frac{12k^2-6k-1}{2k^2},\frac{(Hk-1)\left(2H^2k^2-6Hk^2+12k^2-Hk-6k-1\right)}{2k^2}\big)$ for H^3; see Example 4.6. Summing the result over k and using $\sum_{k=1}^{n}-\frac{12k^2-6k-1}{2k^2}=\frac{1}{2}(-12n+6H_n+H_n^{(2)})$ gives

$$\sum_{k=1}^{n}H_k^3=\frac{1}{2}\left(2(n+1)H_n^3-3(2n+1)H_n^2+6(2n+1)H_n-12n+H_n^{(2)}\right). \tag{2.5}$$

Example 2.4. We find (1.4), a generalization given in [6, Page 9], as follows. Take the difference field $(\mathbb{Q}(x)(k),\sigma)$ with constant field $\mathbb{Q}(x)$ and $\sigma(k)=k+1$, and extend it with the Π-extension $(\mathbb{Q}(x)(k)(B),\sigma)$ with $\sigma(B)=\frac{x-k}{k+1}B$. Afterwards, extend it with the Σ^*-extension $(\mathbb{Q}(x)(k)(B)(S),\sigma)$ with $\sigma(S)=S+\sigma(B)$; note that the shift of $\binom{x}{k}$ and $\sum_{i=0}^{k}\binom{x}{i}$ in k is reflected by the automorphism σ acting on B and S. Then we compute the Σ-pair $(f',g)=(-\frac{x}{2}B^2,-\frac{1}{2}(B-S)(xB+(2k-x)S))$ for $f=S^2$. This gives $f'(k)=-\frac{x}{2}\binom{x}{k}^2$ and $g(k)=-\frac{1}{2}(\binom{x}{k}-\sum_{i=0}^{k}\binom{x}{i})(x\binom{x}{k}+(2k-x)\sum_{i=0}^{k}\binom{x}{i})$ for (1.2). Summing (1.2) over k gives (1.4).
With the same mechanism we find the identities

$$\sum_{k=0}^{n}(-1)^k\Big(\sum_{j=0}^{k}\tbinom{x}{j}\Big)^2=\frac{1}{2x}\Big(-\sum_{k=0}^{n}(x-2k)\tbinom{x}{k}^2(-1)^k$$
$$+(-1)^n\Big(2(x-n)\tbinom{x}{n}\sum_{j=0}^{n}\tbinom{x}{j}+x\Big(\sum_{j=0}^{n}\tbinom{x}{j}\Big)^2\Big)\Big),$$

$$\sum_{k=1}^{n}\tfrac{p(k)}{(1-3k)^2(2-3k)^2(1-2k)^2k^2}\sum_{j=1}^{k}\tfrac{108j^3-153j^2+68j-10}{j(2j-1)(3j-2)(3j-1)}=2\Big(\sum_{j=1}^{n}\tfrac{108j^3-153j^2+68j-10}{j(2j-1)(3j-2)(3j-1)}\Big)^2+$$
$$-\sum_{k=1}^{n}\tfrac{289656k^7-842886k^6+1001583k^5-622368k^4+213418k^3-38207k^2+2720k+20}{k^2(2k-1)^2(3k-2)^2(3k-1)^2}$$

where $p(k)=\big(-289656k^7+819558k^6-935487k^5+546174k^4-167482k^3+22839k^2+4\left(1944k^6-5670k^5+6759k^4-4221k^3+1460k^2-266k+20\right)k-220\big)$. The first identity is a generalization given in [32]. Note that in this identity $(-1)^k$ occurs which cannot be expressed in $\Pi\Sigma^*$-extensions; see Remark 2.1 – nevertheless the machinery under consideration can be adapted for this case, see Section 8. The second identity has been used in [9].

Example 2.5. For (1.5) we take $(\mathbb{Q}(k)(H^{(2)})(H),\sigma)$ with $\sigma(k)=k+1$, $\sigma(H^{(2)})=H^{(2)}+\frac{1}{(k+1)^2}$ and $\sigma(H)=H+\frac{1}{k+1}$, and compute the Σ-pair $(-\frac{6k^2-3k-1}{3k^3},-\frac{H^3}{3}+$

$(H^{(2)}k{+}1)H^2{-}H^{(2)}(2k{+}1)H{+}\frac{6H^{(2)}k^4-6k^2+3k+1}{3k^3})$ for $f = H^2H^{(2)}$; see Example 6.1. This gives (1.5).

Example 2.6. In [24, Page 381] we needed the simplification

$$\sum_{k=1}^{n}\frac{k+1}{k(k+2)} = -\frac{n(3n+5)}{4(n+1)(n+2)} + \sum_{k=1}^{n}\frac{1}{k}. \tag{2.6}$$

Given $(\mathbb{Q}(k), \sigma)$ with $\sigma(k) = k+1$, we can use any of the algorithms from [4,17] to compute the Σ-pair $(f', g) = (\frac{1}{k}, \frac{2k+1}{2k(k+1)})$ for $f = \frac{k+1}{k(k+2)}$; in Example 5.5 we will apply our generalized method. Then summing (1.2) over k yields (2.6).

Example 2.7. In order to find the identity $\sum_{j=0}^{n} jH_j\binom{n}{j} = -\frac{1}{2} + 2^{n-1}(1 + nH_n - n\sum_{j=1}^{n}\frac{1}{j2^j})$ in [19, Page 370] we needed the identity

$$\sum_{k=2}^{n}\frac{1}{k(k-1)2^k} = -\frac{1}{n2^{n+1}} + \frac{1}{4} - \frac{1}{2}\sum_{k=2}^{n}\frac{1}{k2^k}. \tag{2.7}$$

Extend $(\mathbb{Q}(k), \sigma)$ with the Π-extension $(\mathbb{Q}(k)(P), \sigma)$ where $\sigma(P) = 2\,P$, and compute the Σ-pair $(-\frac{1}{2kP}, \frac{-1}{(k-1)P})$ for $\frac{1}{(k-1)kP}$; see Example 5.6. This produces (1.2) with $f(k) = \frac{1}{(k-1)kP}$, $f'(k) = -\frac{1}{2k2^k}$ and $g(k) = \frac{-1}{(k-1)2^k}$. Summing (1.2) over k gives (2.7).

Example 2.8. We find the right-hand side of

$$\sum_{k=1}^{n}\frac{k!\left(k^2+k+k!\left(k(k+1)^2+k!\left(k(k+1)^2+\left(2k^2-1\right)k!-3\right)-2\right)+1\right)+1}{(k!)^3(k!+1)((k+1)k!+1)}$$
$$= \frac{3(n+1)(n!)^3+(3-2n)(n!)^2-2(n+2)n!-2}{2(n!)^2((n+1)n!+1)} + \sum_{k=1}^{n}\frac{k(k!)^3+k!+1}{(k!)^3(k!+1)} \tag{2.8}$$

as follows. Take the Π-extension $(\mathbb{Q}(k)(F), \sigma)$ with $\sigma(F) = (k+1)F$ and represent the summand with $f = \frac{F\left(k^2+k+F\left(k(k+1)^2+F\left(k(k+1)^2+F\left(2k^2-1\right)-3\right)-2\right)+1\right)+1}{F^3(F+1)(kF+F+1)}$. Then we compute the Σ-pair $(f', g) = (\frac{kF^3+F+1}{F^3(F+1)}, -\frac{kF^2-F^2+k^2F+kF+k^2}{F^2(F+1)})$ for f; the details can be found in Examples 5.1, 5.2, 5.3, 5.4, and 5.8. Reinterpreting (f', g) in terms of $k!$ gives the closed form (2.8).

Example 2.9. Starting with the left-hand side of

$$\sum_{k=2}^{n}\frac{(k+1)\left(k(k+1)^2(k+2)H_k^3+k\left(3k^2+8k+5\right)H_k^2-(k+2)H_k-k-2\right)}{H_k\left(k(k+1)^2(k+2)H_k^3+2\left(k^3+2k^2-1\right)H_k^2-\left(k^2+5k+5\right)H_k-2k-3\right)}$$
$$= \frac{-6(n+1)(n+2)H_n^2-6(2n+3)H_n+11(n+1)(n+2)}{11H_n\left(2n+(n+1)(n+2)H_n+3\right)} + \sum_{k=2}^{n}\frac{k(k+1)}{kH_k-1} \tag{2.9}$$

we take the difference field $(\mathbb{Q}(k)(H), \sigma)$ from Example 2.2 and compute for $f = \frac{(k+1)\left(k(k+1)^2(k+2)H^3+k\left(3k^2+8k+5\right)H^2-(k+2)H-k-2\right)}{H(k(k+1)^2(k+2)H^3+2(k^3+2k^2-1)H^2-(k^2+5k+5)H-2k-3)}$ the Σ-pair $(f', g) = (\frac{k(k+1)}{Hk-1}, \frac{k(k+1)}{(Hk-1)(kH+H+1)})$; see Example 5.9. This gives the right hand side of (2.9).

Example 2.10. We derive identity (1.3) as follows. Take $(\mathbb{Q}(k)(H),\sigma)$ from Example 2.2 and compute the Σ-pair $(f',g)=\big(-\frac{12Hk^2-2k^2-6Hk-H}{2Hk^2},kH^3-\frac{3}{2}(2k+1)H^2+6kH+\frac{3}{k}+\frac{k}{Hk-1}+\frac{1}{2k^2}-6\big)$ for $f=\frac{-kH^5+H^4-kH+2}{H-H^2k}$; see Example 3.1. This produces (1.3).

The following simple facts are heavily used throughout this article.

Lemma 2.1. *Let $(\mathbb{F},\sigma)$ be a difference field.*

(1) If $(f'_i,g_i)\in\mathbb{F}^2$ are Σ-pairs for $f_i\in\mathbb{F}$, $(f'_0+f'_1,g_0+g_1)$ is a Σ-pair for f_0+f_1.
(2) If $(f',g)\in\mathbb{F}^2$ is a Σ-pair for f and $(\phi,\gamma)\in\mathbb{F}^2$ is a Σ-pair for f', then $(\phi,\gamma+g)$ is a Σ-pair for f.
(3) Let $i\in\mathbb{Z}$ and $f\in\mathbb{F}$. Then (f,g) is a Σ-pair for $\sigma^i(f)$ where $g=\sum_{j=0}^{i-1}\sigma^j(f)$ if $i\geq 0$, and $g=-\sum_{j=0}^{-i-1}\sigma^{j+i}(f)$ if $i<0$.

Proof. (1) and (2) are obvious. Take f,f',g from (3). If $i\geq 0$, $\sigma(g)-g=\sum_{j=1}^{i}\sigma^j(f)-\sum_{j=0}^{i-1}\sigma^j(f)=\sigma^i(f)-f$. If $i<0$, $\sigma(g)-g=\sum_{j=0}^{-i-1}\sigma^{j+i}(f)-\sum_{j=1}^{-i}\sigma^{j+i}(f)=\sigma^i(f)-f$. □

3. Problem reduction

Subsequently, let $(\mathbb{F}(t),\sigma)$ be a ΠΣ*-extension of $(\mathbb{F},\sigma)$, $\mathbb{K}=\text{const}_\sigma\mathbb{F}$, and $f\in\mathbb{F}(t)$. By polynomial division we get $f=f_0+f_1$ with $f_0\in\mathbb{F}[t]$ and $f_1\in\mathbb{F}(t)_{(r)}$ where

$$\mathbb{F}(t)_{(r)}:=\{\frac{a}{b}\,|\,a,b\in\mathbb{F}[t]\text{ and }\deg(a)<\deg(b)\}.$$

In short we write $f=f_0+f_1\in\mathbb{F}[t]\oplus\mathbb{F}(t)_{(r)}$ and say that f_0 is the *polynomial part* and f_1 is the *fractional part*. The following lemma tells us how we can continue.

Lemma 3.1. *Let $(\mathbb{F}(t),\sigma)$ be a ΠΣ*-extension of $(\mathbb{F},\sigma)$. Let $f,f',g\in\mathbb{F}(t)$ and write $f=f_0+f_1\in\mathbb{F}[t]\oplus\mathbb{F}(t)_{(r)}$, $f'=f'_0+f'_1\in\mathbb{F}[t]\oplus\mathbb{F}(t)_{(r)}$ and $g=g_0+g_1\in\mathbb{F}[t]\oplus\mathbb{F}(t)_{(r)}$. Then (f',g) is a Σ-pair for f iff (f'_i,g_i) are Σ-pairs of f_i for $i=0,1$.*

Proof. The direction from right to left follows by Lemma 2.1.1. Suppose that (f',g) is a Σ-pair for f. Then $\big[\sigma(g_0)-g_0+f'_0-f_0\big]+\big[\sigma(g_1)-g_1+f'_1-f_1\big]=0$. Since $\sigma(g_0)-g_0+f'_0-f_0\in\mathbb{F}[t]$, $\sigma(g_1)-g_1+f'_1-f_1\in\mathbb{F}(t)_{(r)}$ and $\mathbb{F}(t)=\mathbb{F}[t]\oplus\mathbb{F}(t)_{(r)}$ is a direct sum ($\mathbb{F}[t],\mathbb{F}(t)_{(r)}$ are considered as subspaces of $\mathbb{F}(t)$ over $\mathbb{F}$), we have $\sigma(g_i)-g_i+f'_i-f_i=0$ for $i\in\{0,1\}$. □

This motivates us to consider the following problems separately.

PP: **P**olynomial **P**roblem

Given $f\in\mathbb{F}[t]$; **find** from all Σ-pairs $(f',g)\in\mathbb{F}[t]^2$ for f a pair where $\deg(f')$ is minimal.

RP: **R**ational **P**roblem

Given $f\in\mathbb{F}(t)_{(r)}$; **find** from all Σ-pairs $(f',g)\in\mathbb{F}(t)_{(r)}^2$ for f a pair where the degree of the denominator of f' is minimal.

This explains, why we can impose simultaneously optimal degrees of p and d in problem *RT*.

Example 3.1. (Cont. Example 2.10) Given f from Example 2.10 we compute the polynomial part $f_0 = H^3$ and the fractional part $f_1 = \frac{Hk-2}{H(Hk-1)}$ with $f = f_0 + f_1$. Denote with (f_0', g_0) the computed Σ-pair from Example 2.3. Next, we compute a solution of problem *RP*, namely the Σ-pair $(f_1', g_1) = (\frac{1}{H}, \frac{k}{kH-1})$ for f_1, see Example 5.7 (as byproduct we get $\sum_{k=2}^{n} \frac{kH_k-2}{H_k(kH_k-1)} = \frac{1}{H_n} - 1 + \sum_{k=2}^{n} \frac{1}{H_k}$). Combining the Σ-pairs, see Lemma 2.1.1, we get the Σ-pair $(f', g) = (f_0'+f_1', g_0+g_1)$ for f which we used in Example 2.10.

Based on the previous considerations we propose the following algorithm.

Algorithm 1. `RefinedTelescoping`$((\mathbb{F}(t), \sigma), f)$

In: A $\Pi\Sigma^*$-extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ and algorithms for *PP* and *RP*; $f \in \mathbb{F}[t]$.
Out: A solution of problem *RT*.
(1) Split $f = f_0 + f_1$ with $f_0 \in \mathbb{F}[t]$ and $f_1 \in \mathbb{F}(t)_{(r)}$ by polynomial division.
(2) Let $(f_0', g_0) \in \mathbb{F}[t]^2$ be a solution of problem *PP* for f_0.
(3) Let $(f_1', g_1) \in \mathbb{F}(t)_{(r)}^2$ be a solution of problem *RP* for f_1.
(4) RETURN $(f_0' + f_1', g_0 + g_1)$.

In Sections 4 and 5 we will solve problems *PP* and *RP* under the assumption that the two subproblems *PLDE* and *SEF* can be solved. Namely, we suppose that we can deal with

Problem *PLDE*: Solving First order-**P**arameter **L**inear **D**ifference **E**quations

Given $a_1, a_2 \in \mathbb{F}^*$ and $f, \phi \in \mathbb{F}$; **find** $g \in \mathbb{F}$ and $c \in \mathbb{K}$ with $a_1\,\sigma(g) + a_2\,g = f + c\,\phi$.

Moreover, we must be able to factorize a polynomial $f \in \mathbb{F}[t]$ into its irreducible factors. Furthermore, we must be able to solve problem *SEF*; here we need the following definition: we say that $f, g \in \mathbb{F}[t]^*$ are *σ-prime*, in short, $f \perp_\sigma g$, if $\gcd(f, \sigma^k(g)) = 1$ for all $k \in \mathbb{Z}$.

Problem *SEF*: **S**eparate **E**quivalent **F**actors

Given $q \in \mathbb{F}[t]^*$ and an irreducible $h \in \mathbb{F}[t]$; **find** $m_i \geq 0$ and $c \in \mathbb{F}[t]$ with

$$q = c \prod_i \sigma^i(h^{m_i}), \quad c \perp_\sigma h. \tag{3.1}$$

The following remarks are in place: If $f \in \mathbb{F}[t]^*$ is irreducible and $m \in \mathbb{Z}$, then $\sigma^m(f) \in \mathbb{F}[t]$ is irreducible. Hence, on the set of all irreducible polynomials from $\mathbb{F}[t]$ we get an equivalence relation $f \sim g$ (the shift-equivalence) iff $f \not\perp_\sigma g$. Thus, solving problem *SEF* means to separate the irreducible polynomials in q into the factors which are all shift-equivalent with h, i.e., $\prod_i \sigma^i(h^{m_i})$, and into the factors which are not shift-equivalent to h, i.e., c. Expanding this refined factorization on c gives Karr's σ-factorization introduced in [13].

Summarizing, we will obtain the following results.

Corollary 3.1. *Let* $(\mathbb{F}(t), \sigma)$ *be a* $\Pi\Sigma^*$*-extension of* $(\mathbb{F}, \sigma)$ *where one can solve problems PLDE and SEF; let* $f \in \mathbb{F}[t]$*. Then Algorithm 1 is applicable and the output* (f', g) *is a solution of RT. Moreover, we have:* ***(1)*** *If there is a* Σ*-pair* $(\phi', \gamma) \in \mathbb{F} \times \mathbb{F}(t)$ *for* f*, then* $f' \in \mathbb{F}$*.*
(2) *If there is a* $\gamma \in \mathbb{F}(t)$ *with* $\sigma(\gamma) - \gamma = f$*, then* $f' = 0$*.*

To this end, we emphasize that there are algorithms for problems *PLDE* and *SEF* if $\mathbb{F}$ is a $\Pi\Sigma^*$-field. For problem *PLDE* see [13, Section 3]; a simplified version is given in [29, Thm. 4.7] which uses results from [8,23,26]. For problem *SEF* see [13, Thm. 9]. Hence Algorithm 1 can be applied for any $\Pi\Sigma^*$-field $(\mathbb{F}(t), \sigma)$.

4. The Polynomial Problem

We reduce problem *PP* to problem *PLDE*. Here we consider two cases.

4.1. *The* Π*-extension case*

The solution of problem *PP* is immediate with Lemma 4.1; the proof follows by coefficient comparison.

Lemma 4.1. *Let* $(\mathbb{F}(t), \sigma)$ *be a* Π*-extension of* $(\mathbb{F}, \sigma)$ *with* $\sigma(t) = \alpha\, t$*, and suppose that* $f, f', g \in \mathbb{F}[t]$ *with* $f = \sum_{i=0}^{n} f_i t^i$*,* $f' = \sum_{i=0}^{n} f'_i t^i$*, and* $g = \sum_{i=0}^{n} g_i t^i \in \mathbb{F}[t]$*. Then* (f', g) *is a* Σ*-pair for* f *iff* $(f'_i t^i, g_i t^i)$ *are* Σ*-pairs for* $f_i t^i$ *for all* $0 \leq i \leq n$*.*

Start with the Σ-pair (f', g) given by $f' := f$ and $g := 0$. Then we can eliminate a monomial $f_i t^r \neq 0$ in f' iff there is a $g_r \in \mathbb{F}$ with $\sigma(g_r t^r) - g_r t^r = f_r t^r$ or equivalently if

$$\alpha^r \sigma(g_r) - g_r = f_r. \tag{4.1}$$

Consequently, if we find such a solution g_r with (4.1), we can adapt the Σ-pair (f', g) with $f' := f - f_r t^r$ and $g := g + g_r t^r$. In this way we can eliminate all terms of highest degree in f' and get a Σ-pair (f', g) where $\deg(f')$ is minimal. Summarizing, we get

Algorithm 2. `OptimalPolyΠExtension`$((\mathbb{F}(t), \sigma), f)$
In: A Π-extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ and $f = \sum_i f_i t^i \in \mathbb{F}[t]$; an algorithm for problem *PLDE*.
Out: A solution of problem *PP*.
(1) Set $g := 0$, $f' := f$, $r := \deg(f)$. DO
(2) If $f_r \neq 0$ and if there is no g_r with (4.1), STOP and RETURN (f', g).
(3) Otherwise, take such a g_r and set $g := g + g_r t^r$ and $f' := f - f_r t^r$. Set $r := r - 1$.
(4) UNTIL $r = -1$.
(5) RETURN (f', g).

Remark. If one continues the DO-loop although one fails to find a g_r one removes all possible terms in f. In this case the number of non-zero terms in f' is minimal.

Example 4.1. Take $(\mathbb{Q}(k)(F), \sigma)$ from Example 2.8 and let $f = \big(F^3 + (kH + 1)(kH + 2H + 1)F^2 + (k^2 + k + 1)F\big)$. The subproblems are $(k+1)^i\sigma(g_i) - g_i = f_i$ with $f_0 = 0$, $f_1 = \big(k^2 + k + 1\big)$, $f_2 = (kH + 1)\,(kH + 2H + 1)$, and $f_3 = 1$. The solutions are $g_2 = H^2$, $g_1 = k$, and $g_0 = 0$; there is no such $g_3 \in \mathbb{Q}(k)(H)$. Hence $(f', g) = (F^3, H^2F^2 + kF)$ is a Σ-pair for f which solves *PP* and is optimal w.r.t. the number of terms in F.

4.2. *The Σ^*-extension case*

We solve problem *PP* by refining Karr's algorithm. First we bound the degree of the possible solutions.

Lemma 4.2. *[13, Cor. 1] Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ and $f \in \mathbb{F}[t]^*$. If there is a $g \in \mathbb{F}[t]$ with $\sigma(g) - g = f$, then* $\deg(g) \leq \deg(f) + 1$.

Then we try to compute step by step the coefficients of the polynomial solution $g = \sum_{k=0}^{b} g_i t^i$ with $b := \deg(f) + 1$. If this fails, i.e., if there does not exist a telescoping solution, we can extract a solution of problem *PP*. The following example illustrates these ideas.

Example 4.2. (Cont. Example 2.2) Take the $\Pi\Sigma^*$-field $(\mathbb{Q}(k)(H), \sigma)$ with $\sigma(k) = k + 1$ and $\sigma(H) = H + \frac{1}{k+1}$. We look for a $g \in \mathbb{Q}(k)[H]$ such that

$$\sigma(g) - g = H^4; \tag{4.2}$$

for convenience we set $f_4 := H^4$. Since $\deg(g) \leq 5$ by Lemma 4.2, we can set up the solution as $g = \sum_{i=0}^{5} g_i H^i$ with $g_i \in \mathbb{Q}(k)$. By plugging in g into (4.2) we get by coefficient comparison the constraint $\sigma(g_5) - g_5 = 0$ for the leading coefficient g_5. It follows that $g_5 := c_4 \in \mathbb{Q}$ for a constant c_4 which is not determined yet. Using this information it remains to look for $c_4 \in \mathbb{Q}$ and $\sum_{i=0}^{4} g_i H^i$ such that

$$\sigma(\sum_{i=0}^{4} g_i H^i) - \sum_{i=0}^{4} g_i H^i = f_4 - c_4\, \psi_4$$

where $\psi_4 := \sigma(H^5) - H^5$, i.e., $\psi_4 = \frac{1+5(1+k)H+10(1+k)^2H^2+10(1+k)^3H^3+5(1+k)^4H^4}{(1+k)^5}$. Coefficient comparison gives the constraint $\sigma(g_4) - g_4 = 1 + c_4\,\frac{-5}{k+1}$ for g_4. The only possible solution is $c_4 = 0$ and $g_4 = k + c_3$ for a new parameter $c_3 \in \mathbb{Q}$. Thus, we have to find $g_i \in \mathbb{Q}(k)$ and $c_3 \in \mathbb{Q}$ with

$$\sigma(\sum_{i=0}^{3} g_i H^i) - \sum_{i=0}^{3} g_i H^i = f_3 - c_3\psi_3$$

where $f_3 := f_4 - c_4\,\psi_4 - (\sigma(kH^3) - kH^3) = -\frac{1+4(1+k)H+6(1+k)^2H^2+4(1+k)^3H^3}{(1+k)^3}$ and $\psi_3 := \sigma(H^3) - H^3 = \frac{1+4(1+k)H+6(1+k)^2H^2+4(1+k)^3H^3}{(1+k)^4}$. Coefficient comparison gives the constraint $\sigma(g_3) - g_3 = -4 + c_3\frac{-4}{k+1}$. The only possible solution for $g_3 \in \mathbb{Q}(k)$

and $c_3 \in \mathbb{Q}$ is $g_3 = -4k + c_2$ with a new parameter $c_2 \in \mathbb{Q}$ and $c_3 = 0$. Therefore, it remains to look for $g_i \in \mathbb{Q}(k)$ and $c_2 \in \mathbb{Q}$ such that

$$\sigma(\sum_{i=0}^{2} g_i H^i) - \sum_{i=0}^{2} g_i H^i = f_2 - c_2\psi_2$$

where $f_2 := f_3 - c_3\,\psi_3 - (\sigma(-4kH^3) + 4kH^3) = \frac{3+4k+4(2+5k+3k^2)H+6(1+k)^2(1+2k)H^2}{(1+k)^3}$ and $\psi_2 := \sigma(H^3) - H^3 = \frac{1+3(1+k)H+3(1+k)^2H^2}{(1+k)^3}$. We obtain the constraint $\sigma(g_2) - g_2 = 6\frac{1+2k}{k+1} + c_2\frac{-3}{k+1}$. The solution is $g_2 = 12k + c_1$ with $c_1 \in \mathbb{Q}$ and $c_2 = -2$. To this end, we have to look for $g_i \in \mathbb{Q}(k)$ and $c_1 \in \mathbb{Q}$ such that

$$\sigma(\sum_{i=0}^{1} g_i H^i) - \sum_{i=0}^{1} g_i H^i = f_1 - c_1\psi_1$$

where $f_1 = f_2 - c_2\,\psi_2 - (\sigma(12kH^2) - 12kH^2) = \frac{-7-20k-12k^2+(-10-46k-60k^2-24k^3)H}{(1+k)^3}$ and $\psi_1 = \sigma(H^2) - H^2 = \frac{1+2(1+k)H}{(1+k)^2}$. This time we obtain the constraint $\sigma(g_1) - g_1 = \frac{-2(5+18k+12k^2)}{(1+k)^3} + c_1\frac{-2}{(1+k)^2}$ which does not have any solution for $g_1 \in \mathbb{Q}(k)$ and $c_1 \in \mathbb{Q}$. Here Karr's algorithm stops with the answer: there is no $g \in \mathbb{Q}(k)[H]$ with (4.2). Note that there is the following sub-result. Define $\gamma_r := \sum_{i=r}^{5} g_i H^i$ for $1 \leq r \leq 4$ by the given $g_r \in \mathbb{Q}(k)$. Then

$$\sigma(\gamma_r) - \gamma_r = f - f_r,$$

i.e., (f_r, γ_r) is a Σ-pair for f. As it turns out (f_1, γ_1) solves problem *PP* for H^4.

In general, let $(\mathbb{F}(t), \sigma)$ be a Σ^*-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = t + \beta$ and $\mathbb{K} := \text{const}_\sigma\mathbb{F}$. Then we can solve problem *PP* for $f \in \mathbb{F}[t]$ with $s := \deg(f)$ in the following way.

We start with the trivial Σ-pair (f', g) with $f' := f$ and $g := 0$. Given (f', g), we check if f' has already minimal degree; this will be possible by Lemma 4.3.1. If yes, we are done. If no, Lemma 4.3.2 explains how we can construct a Σ-pair $(\phi, \gamma) \in \mathbb{F}[t]^2$ for f with $\deg(\phi) < \deg(f')$. Applying this degree reduction at most s times we find a Σ-pair $(\phi, \gamma) \in \mathbb{F}[t]^2$ where $\deg(\phi)$ is minimal.

Lemma 4.3. *Let* $(\mathbb{F}(t), \sigma)$ *be a* Σ^**-extension of* $(\mathbb{F}, \sigma)$ *and* $\mathbb{K} := \text{const}_\sigma\mathbb{F}(t)$. *Let* $(f', g) \in \mathbb{F}[t]^2$ *be a* Σ*-pair for* $f \in \mathbb{F}[t]$ *with* $s := \deg(f')$ *and define* $\psi := \sigma(t^{s+1}) - t^{s+1}$. *Then:*

(1) If there are no $w \in \mathbb{F}$ *and* $c \in \mathbb{K}$ *with*[a]

$$\sigma(w) - w = \mathit{coeff}(f', s) - c\,\mathit{coeff}(\psi, s), \tag{4.3}$$

then (f', g) *is a* Σ*-pair for* f *where* $\deg(f')$ *is minimal.*

[a] coeff(f, s) denotes the coefficient f_s in $\sum_i f_i t^i \in \mathbb{F}[t]$.

(2) If there are $w \in \mathbb{F}$ and $c \in \mathbb{K}$ with (4.3)*, then we get the Σ-pair (ϕ, γ) for f with*

$$\phi := \sigma(w\,t^s) - w\,t^s + c\,\psi - f' \qquad \text{and} \qquad \gamma := g + c\,t^{s+1} + w\,t^s \tag{4.4}$$

where $\deg(\psi) < \deg(f')$.

Proof. (1) Suppose there is a Σ-pair $(\phi, \gamma) \in \mathbb{F}[t]$ with $\deg(\phi) < s$. Then $\sigma(g-\gamma)-(g-\gamma) = f'-\phi$ with $\deg(f'-\phi) = s$. By Lemma 4.2 it follows that $\deg(g-\gamma) \leq s+1$. Consequently, $g-\gamma = c\,t^{s+1}+wt^s+v$ with $c \in \mathbb{K}$, $w \in \mathbb{F}$ and $v \in \mathbb{F}[t]$ with $\deg(v) < s$. Therefore

$$\sigma(w\,t^s + v) - (w\,t^s + v) = f' - \phi - c\,\psi.$$

Note that $\deg(\psi) \leq s$ (we even have equality by Lemma 4.2). By coefficient comparison of the leading coefficient we get (4.3).
(2) Conversely, suppose there are such $w \in \mathbb{F}$ and $c \in \mathbb{K}$ with (4.3). Then take $\gamma := g + c\,t^{s+1} + w\,t^s$. We have $\phi := f - (\sigma(\gamma) - \gamma) = f' - (\sigma(w\,t^s) - w\,t^s) - c\,\psi$ with $\deg(\phi) \leq s$. By (4.3), $\deg(\phi) < s$. By construction (ϕ, γ) is a Σ-pair for f. □

Corollary 4.1. *Let $(\mathbb{F}(t), \sigma)$ be a Σ^*-extension of $(\mathbb{F}, \sigma)$ and* $\mathbb{K} := \text{const}_\sigma\mathbb{F}(t)$. *Let $(f', g) \in \mathbb{F}[t]^2$ be a Σ-pair for $f \in \mathbb{F}[t]$ with $s := \deg(f')$ and define $\psi := \sigma(t^{s+1}) - t^{s+1}$. Then (f', g) is a solution of problem PP iff there are no $w \in \mathbb{F}$ and $c \in \mathbb{K}$ with* (4.3).

Summarizing, we reduce problem *PP* to problem *PLDE* as follows.

Algorithm 3. `OptimalPolyΣExtension`$((\mathbb{F}(t), \sigma), f)$
In: A Σ^*-extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ with $\mathbb{K} := \text{const}_\sigma\mathbb{F}$, $f \in \mathbb{F}[t]$; an algorithm for problem *PLDE*.
Out: A solution of problem *PP*.
(1) Set $(f', g) := (f, 0)$.
(2) WHILE $f' \neq 0$ DO
(3) Define $s := \deg(f)$, $\psi := \sigma(t^{s+1}) - t^{s+1}$. Decide if there are $w \in \mathbb{F}$, $c \in \mathbb{K}$ with (4.3).
(4) IF not, STOP and RETURN (f', g).
(5) Otherwise, take such a w and c, and define (ϕ, γ) as in (4.4). Set $(f', g) := (\phi, \gamma)$.
(6) OD
(7) RETURN (f', g)

Example 4.3. (Cont. Example 3.1) With Algorithm 3 we compute for $f = H^3$ the Σ-pairs $(H^3, 0)$, $\big(-\frac{3k^2H^2+6kH^2+3H^2+3kH+3H+1}{(k+1)^2}, H^3k\big)$, $\big(\frac{6Hk^2+9Hk+3k+3H+2}{(k+1)^2}, (H-3)H^2k\big)$ and $\big(-\frac{12k^2+18k+5}{2(k+1)^2}, \frac{1}{2}H(2kH^2 - 6kH - 3H + 12k)\big)$. Since there are no $g \in \mathbb{Q}(k)$ and $c \in \mathbb{Q}$ with $\sigma(g) - g = -\frac{12k^2+18k+5}{2(k+1)^2} + \frac{c}{k+1}$, the last Σ-pair solves problem *PP* by Corollary 4.1.

Example 4.4. (Cont. Example 4.2) The computed Σ-pairs (f_r, γ_r) for H^4 from Example 4.2 are the (f', g) in each iteration step. By Corollary 4.1 the output (f_1, γ_1) is a solution of *PP*.

Remark 4.1. Let (ϕ,γ) be a Σ-pair for f with $s := \deg(\phi)$ minimal. The following remarks are in place. **(1)** The coefficients of the monomials t^i with $i > s+1$ in γ are uniquely determined. Namely, take any other Σ-pair $(\phi',\gamma') \in \mathbb{F}[t]^2$ for f with $\deg(\phi') = s$. Then $\sigma(\gamma-\gamma') - (\gamma-\gamma') = \phi' - \phi$, and therefore by Lemma 4.3 it follows that $\gamma - \gamma' \leq \deg(\phi'-\phi)+1 \leq s+1$. Hence all coefficients of the monomials t^i with $i > s+1$ in γ and γ' must be equal.
(2) From Remark 4.1.1 we get the following additional consequence. If $(\phi',\gamma') \in \mathbb{F}[t]^2$ is a Σ-pair for f with $\deg(\phi') = s$, then there is a $w \in \mathbb{F}[t]$ with $\deg(w) \leq s+1$ such that

$$\sigma(w) - w + \phi' = \phi. \tag{4.5}$$

Hence for all degree optimal ϕ, ϕ' we have (4.5) for some $w \in \mathbb{F}[t]$ with $\deg(w) \leq s+1$.

E.g., with Lemma 2.1.3 in combination with Lemma 2.1.2 we obtain a rather simple transformation: We can shift the non-summable part in positive or negative direction.

Example 4.5. (Cont. Example 4.4) Take for $f = H^4$ the already computed Σ-pair $(f',g) = (-\frac{12k^2+20k+2H(12k^3+30k^2+23k+5)+7}{(k+1)^3}, H^2\big(kH^2 - 2(2k+1)H + 12k\big))$. By Lemma 2.1.3 we get the Σ-pair (f',f') for $\sigma(f')$. Hence $(\sigma^{-1}(f'),\sigma^{-1}(f'))$ is a Σ-pair for f'. With Lemma 2.1.1 we get the Σ-pair $(\sigma^{-1}(f'),\sigma^{-1}(f')+g)$ for H^4 which we used in Example 2.2.

Example 4.6. (Cont. Example 4.3) Let $(f',g) = (-\frac{12k^2+18k+5}{2(k+1)^2}, \frac{1}{2}H(2kH^2 - 6kH - 3H + 12k))$ be the Σ-pair for H^3 from Example 4.3. Like in Example 4.5 we get the Σ-pair $(\sigma^{-1}(f'),\sigma^{-1}(f')+g)$ for H^3 which we used in Example 3.1.

5. The Rational Problem

Under the assumption that we can solve *SEF* and *PLDE* we reduce problem *RP* to problem *SFP* given below. The corresponding algorithms generalize the results in [4,17].

To accomplish this task, we proceed as follows. Write[b] $f = \frac{p}{q} \in \mathbb{F}(t)_{(r)} \setminus \{0\}$ and let $h \in \mathbb{F}[t]$ be an irreducible factor of q. Then solve *SEF* and compute $m_i \geq 0$ and $c \in \mathbb{F}[t]^*$ with (3.1); note that not all m_i are zero.

Example 5.1. (Cont. Example 2.8) Given $f = \frac{p}{q}$ from Example 2.8 with $q = F^3(F+1)(kF+F+1)$, we choose $h = F$ and get $q = F^3\,c$ with $c = (F+1)(kF+F+1)$ and $c \perp_\sigma F$.

[b]If not stated differently, we suppose that $p, q \in \mathbb{F}[t]$, $q \neq 0$, and $\gcd(p,q) = 1$, whenever we write $f = \frac{p}{q}$; we define $\mathrm{den}(f) = q$ (up to a unit in $\mathbb{F}$).

Since c and $\prod_i \sigma^i(h^{m_i})$ are σ-prime, in particular coprime, we can compute by the extended Euclidean algorithm, see [31, Corollary, p. 53], polynomials $a, b, c \in \mathbb{F}[t]$ such that

$$f = \frac{p}{\prod_i \sigma^i(h^{m_i})c} = \frac{a}{\prod_i \sigma^i(h^{m_i})} + \frac{b}{c}$$

where $\frac{a}{\prod_i \sigma^i(h^{m_i})}, \frac{b}{c} \in \mathbb{F}(t)_{(r)}$.

Example 5.2. (Cont. Example 5.1) We get $f = f_0 + f_1$ with $f_0 := \frac{kF^2-F^2+k^2F-F+1}{F^3}$ and $f_1 = \frac{b}{c} := \frac{k(Fk+1)}{(F+1)(kF+F+1)}$.

Given this representation of f the following lemma tells us how to proceed.

Lemma 5.1. *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$, let $f, f', g \in \mathbb{F}(t)_{(r)}$, and let $h \in \mathbb{F}[t]^*$ be irreducible. Write $f = f_0 + f_1$, $f' = f'_0 + f'_1$ and $g = g_0 + g_1$ with $f_i, f'_i, g_i \in \mathbb{F}(t)_{(r)}$ and*

$$f_0 = \frac{a}{\prod_i \sigma^i(h^{m_i})}, \quad f_1 = \frac{b}{c} \quad \text{for some } a, b, c \in \mathbb{F}[t] \text{ with } h \perp_\sigma c,\ m_i \geq 0, \tag{5.1}$$

$$f'_0 = \frac{a'}{\prod_i \sigma^i(h^{m'_i})} \quad f'_1 = \frac{b'}{c'} \quad \text{for some } a', b', c' \in \mathbb{F}[t] \text{ with } h \perp_\sigma c',\ m'_i \geq 0, \tag{5.2}$$

$$g_0 = \frac{\alpha}{\prod_i \sigma^i(h^{\mu_i})} \quad g_1 = \frac{\beta}{\gamma} \quad \text{for some } \alpha, \beta, \gamma \in \mathbb{F}[t] \text{ with } h \perp_\sigma \gamma,\ \mu_i \geq 0. \tag{5.3}$$

(i) Then (f', g) is a Σ-pair for f iff (f'_i, g_i) are Σ-pairs for f_i with $i \in \{0, 1\}$.
(ii) Let (f', g) be a Σ-pair for f, and (f'_i, g_i) be Σ-pairs for f_i with $i \in \{0, 1\}$ where the f_i, f'_i and g_i are as above. Then (f', g) is a solution of problem RP for f iff (f'_1, g_1) is a solution of problem RP for f_1 and $\deg(\mathrm{den}(f'_0))$ is minimal w.r.t. all Σ-pairs in $\mathbb{F}(t)^2_{(r)}$ where the denominators are of the form $\prod_i \sigma^i(h^{\nu_i})$ for some $\nu_i \geq 0$ (see problem SFP).

Proof. (i) The direction from left to right follows by Lemma 2.1.1. Now suppose that (f', g) is a Σ-pair for f. Then $0 = \sigma(g) - g + f' - f = [\sigma(g_0) - g_0 + f'_0 - f_0] + [\sigma(g_1) - g_1 + f'_1 - f_1] = h_0 + h_1$ with $h_i := \sigma(g_i) - g_i + f'_i - f_i$. We have $h_0 = \frac{A}{\prod_i \sigma^i(h^{\nu_i})}$ and $h_1 = \frac{B}{C}$ for some $\nu_i \geq 0$, and $A, B, C \in \mathbb{F}[t]$ with $h \perp_\sigma C$. Suppose that $h_0, h_1 \neq 0$. Since $h_0 = -h_1$, $C = u \prod_i \sigma^i(h^{\nu_i})$ with $u \in \mathbb{F}^*$ and $\deg(C) > 0$. A contradiction that $h \perp_\sigma C$. Hence $h_0 = 0 = h_1$, and therefore (f'_i, g_i) are Σ-pairs for f_i with $i \in \{0, 1\}$.
(ii) Suppose that $\deg(\mathrm{den}(f'_i))$ is not minimal for some $i \in \{0, 1\}$ as stated in the lemma. Let $j \in \{0, 1\} \setminus \{i\}$ and take $\psi, \gamma \in \mathbb{F}(t)_{(r)}$ such that $\sigma(\gamma) - \gamma + \psi = f_i$ and $\deg(\mathrm{den}(\psi)) < \deg(\mathrm{den}(f'_i))$. Then $(\psi + f'_j, \gamma + g_j)$ is a Σ-pair for f by Lemma 2.1.1. We have

$$\begin{aligned}\deg(\mathrm{den}(\psi + f'_j)) &\leq \deg(den(\psi)) + \deg(den(f'_j)) \\ &< \deg(den(f'_0)) + \deg(den(f'_1)) = \deg(\mathrm{den}(f')).\end{aligned}$$

Conversely, suppose that $\deg(\mathrm{den}(f_0))$ and $\deg(\mathrm{den}(f_1))$ are minimal as stated in the lemma, but (f', g) does not solve *RP*. Take a Σ-pair $(\psi, \gamma) \in \mathbb{F}(t)^2_{(r)}$ for f with $\deg(\mathrm{den}(\psi)) < \deg(\mathrm{den}(f'))$. By (i) there are $\psi = \psi_0 + \psi_1$ and $\gamma = \gamma_0 + \gamma_1$ such that (ψ_i, γ_i) are Σ-pairs for f_i with $i \in \{0,1\}$ and where we can write $\psi_0 = \frac{A}{\prod_i \sigma^i(h^{\nu_i})}$ and $\psi_1 = \frac{B}{C}$ for some $\nu_i \geq 0$, and $A, B, C \in \mathbb{F}[t]$ with $h \perp_\sigma C$. Then it follows that

$$\begin{aligned}\deg(\mathrm{den}(\psi_0)) + \deg(\mathrm{den}(\psi_1)) &= \deg(\mathrm{den}(\psi)) \\ &< \deg(\mathrm{den}(f')) = \deg(\mathrm{den}(f_0)) + \deg(\mathrm{den}(f_1)),\end{aligned}$$

a contradiction to $\deg(\mathrm{den}(f'_i)) \leq \deg(\mathrm{den}(\psi_i))$ for $i = 0, 1$. □

This gives the following reduction. Write f in the representation $f = f_0 + f_1$ with (5.1), see above. Then find a Σ-pair (f'_0, g_0) for f_0 where the degree of the denominator of f'_0 is minimal. More precisely, solve problem *SFP* for f_0.

SFP: **S**imple **F**ractional **Part**

Given $f = \frac{a}{\prod_i \sigma^i(h^{m_i})} \in \mathbb{F}(t)_{(r)} \setminus \{0\}$ for some $a \in \mathbb{F}[t]$ and $h \in \mathbb{F}[t]$ irreducible (not all m_i are zero); **find** $f' = \frac{a'}{\prod_i \sigma^i(h^{m'_i})} \in \mathbb{F}(t)_{(r)}$ and $g = \frac{\alpha}{\prod_i \sigma^i(h^{\mu_i})} \in \mathbb{F}(t)_{(r)}$ for some $a', \alpha \in \mathbb{F}[t]$ and $m_i, \mu_i \geq 0$ with (2.1) and the following property: the degree of $\prod_i \sigma^i(h^{m'_i})$ is optimal w.r.t. all Σ-pairs in $\mathbb{F}(t)^2_{(r)}$ where the denominators are of the form $\prod_i \sigma^i(h^{\nu_i})$ for some $\nu_i \geq 0$.

Then continue to solve problem *RP* for f_1; note that the degree of the denominator of f_1 is reduced by $\deg(h) \sum_i m_i > 0$. If $f_1 = 0$, take the Σ-pair $(0, 0)$. Otherwise, apply the same reduction strategy to $f_1 \in \mathbb{F}(t)_{(r)} \setminus \{0\}$ as sketched above (for a new irreducible polynomial $h \in \mathbb{F}[t]$ in the denominator of f_1). This finally gives the solution (f'_1, g_1) of problem *RP* for f_1. By Lemma 5.1 we get the solution $(f'_0 + f'_1, g_0 + g_1)$ of problem *RP* for f.

Example 5.3. (Cont. Example 5.2) We solve problem *SFP* for f_0 and get the Σ-pair $(f'_0, g_0) = (\frac{1}{F^3}, -\frac{k^2}{F^2} - \frac{k}{F})$; see Example 5.4. As byproduct we get $\sum_{k=0}^{n} \frac{(k-1)k!^2 + (k^2-1)k! - 1}{(k!)^3} = \frac{2n!^2 - n! - 1}{n!^2} + \sum_{k=1}^{n} \frac{1}{k!}$. Next we solve problem *RP* for f_1. As result we get the Σ-pair $(f'_1, g_1) = (\frac{1}{F+1}, \frac{k}{F+1})$ for f_1; see Example 5.8. This results in $\sum_{k=1}^{n} \frac{k(k!k+1)}{(k!+1)(kk!+k!+1)} = \frac{-(n+1)n!+1}{2((n+1)n!+1)} + \sum_{k=1}^{n} \frac{k}{1+k!}$. Combining the results we get the solution $(f', g) = (f'_0 + f'_1, g_0 + g_1)$ of problem *RP* for f.

Summarizing, we can reduce problem *SEF* to problem *SFP* as follows.

Algorithm 4. `ReduceFractionalPart`$((\mathbb{F}(t), \sigma), f)$

In: A $\Pi\Sigma^*$-extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ and $f \in \mathbb{F}(t)_{(r)}$; algorithms for problems *SFP*, *SEF*.
Out: A solution of problem *RP*.
(1) Set $g := 0$ and $f' := f$. WHILE $f \neq 0$ DO
(2) Let $f = \frac{p}{q}$. Take an irreducible factor $h \in \mathbb{F}[t]^*$ of q and represent q in the form (3.1).
(3) By the extended Euclidean algorithm write $f = f_0 + f_1$ in the form (5.1).
(4) Compute a Σ-pair (f'_0, g_0) for f_0 which is a solution of problem *SFP*.

(5) Set $f := f - f_0$, $f' := f' + f'_0$ and $g := g + g_0$.
(6) OD
(7) RETURN (f', g)

To this end, we show how we can solve problem *SFP*. Here the following property is essential.

Lemma 5.2. *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ and $h \in \mathbb{F}[t]^*$ be irreducible. Suppose that $\frac{\sigma(t)}{t} \notin \mathbb{F}$ or $\frac{h}{t} \in \mathbb{F}$. Then $\gcd(\sigma^k(h), \sigma^l(h)) = 1$ for all integers k, l with $k \neq l$.*

Proof. Assume $\gcd(\sigma^k(h), \sigma^l(h)) \neq 1$. Since $\sigma^k(h), \sigma^l(h) \in \mathbb{F}[t]$ are irreducible, $\frac{\sigma^k(h)}{\sigma^l(h)} \in \mathbb{F}$. Hence $\frac{\sigma^{k-l}(h)}{h} \in \mathbb{F}$. By [13, Thm. 4] (compare [8, Cor. 1,2] or [22, Thm. 2.2.4]) it follows that $\frac{\sigma(t)}{t} \in \mathbb{F}$ and $\frac{h}{t} \in \mathbb{F}$. □

Since Lemma 5.2 cannot be applied if $h = t$ and $\frac{\sigma(t)}{t} \in \mathbb{F}$, we do a case distinction.

5.1. *A special case*

Let $(\mathbb{F}(t), \sigma)$ be a Π-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t$, let $h = t$, and let $f = \frac{a}{\prod_{i=1}^{n} \sigma^i(h^{m_i})} \neq 0$ as in problem *SFP*. Then for some $u \in \mathbb{F}^*$, $n > 0$ and $t \nmid a$ we have

$$f = \frac{a\, u}{t^n}, \quad 0 \leq \deg(a) < n.$$

Hence we can write $f = \sum_{i=1}^{n} f_i \frac{1}{t^i}$ for some $f_i \in \mathbb{F}$, i.e., $f \in \mathbb{F}[\frac{1}{t}]$. Similarly, the $f', g \in \mathbb{F}(t)_{(r)}$ in problem *SFP* are also elements from $\mathbb{F}[\frac{1}{t}]$. Thus, problem *SFP* boils down to find a Σ-pair $(f', g) \in \mathbb{F}[\frac{1}{t}]^2$ for $f \in \mathbb{F}[\frac{1}{t}]$ where in $f' = \sum_{i=0}^{n'} f'_i \frac{1}{t^i}$ the degree n' is minimal.

Now observe that the difference field $(\mathbb{F}(\frac{1}{t}), \sigma)$ with $\sigma(\frac{1}{t}) = \frac{1}{\alpha} \frac{1}{t}$ is a Π-extension of $(\mathbb{F}, \sigma)$. This is a direct consequence of [13, Thm. 2]; see also [28, Prop. 4.4]. Hence in $(\mathbb{F}(\frac{1}{t}), \sigma)$ problem *SFP* is nothing else than problem *PP* handled in Subsection 4.1. In a nutshell, we can apply Algorithm 2 with the function call `OptimalPolyΠExtension`$((\mathbb{F}(\frac{1}{t}), \sigma), f)$.

Example 5.4. (Cont. Example 5.3) Given $f = \frac{(k-1)(F)^2 + (k^2-1)F - 1}{F^3} = \frac{1}{F^3} + \frac{k^2-1}{F^2} + \frac{k-1}{F} = \sum_{i=1}^{3} f_i \frac{1}{F^i}$ we have to solve the problems $\frac{1}{(k+1)^i}\sigma(g_i) - g_i = f_i$ with $f_3 = 1$, $f_2 = k^2 - 1$ and $f_1 = k - 1$. We get the solutions $g_2 = -k^2$, $g_1 = -k$; there is no solution $g_3 \in \mathbb{Q}(k)$. Hence we obtain $(f', g) = (\frac{1}{F^3}, \frac{-k^2}{F^2} + \frac{-k}{F})$ for problem *SFP*.

5.2. *The remaining cases*

The solution of problem *SFP* can be summarized in

Theorem 5.1. *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ and let $h \in \mathbb{F}[t]$ be irreducible with $h \neq t$ or $\frac{\sigma(t)}{t} \notin \mathbb{F}$. Let $f \in \mathbb{F}(t)_{(r)} \setminus \{0\}$ with $\mathrm{den}(f) = \prod_i \sigma^i(h^{m_i})$ for some $m_i \geq 0$. Then:*

(1) A Σ*-pair* $(f', g) \in \mathbb{F}(t)^2_{(r)}$ *of* f *can be computed where the denominator of* f' *has the form*

$$u\sigma^i(h)^m \quad \text{for some } u \in \mathbb{F}^*,\ i \in \mathbb{Z} \text{ and } m \geq 0. \tag{5.4}$$

(2) A Σ*-pair* $(f', g) \in \mathbb{F}(t)^2_{(r)}$ *of* f *solves SFP iff the denominator of* f' *is of the form* (5.4)*.*

In the remaining part of the subsection we prove the theorem. Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ where either $\frac{\sigma(t)}{t} \notin \mathbb{F}$ (a Σ^*-extension) or $h \neq t$. Moreover, consider $f = \frac{a}{\prod_{i=1}^n \sigma^i(h^{m_i})}$ with $m_i \geq 0$ as in problem *SFP*.

Proof of Theorem 5.1.1. By Lemma 5.2 all the $\sigma^i(h^{m_i})$ with $m_i \neq 0$ are pairwise coprime. Thus, we can invoke the extended Euclidean algorithm and compute polynomials $s_i \in \mathbb{F}[t]$ with

$$f = \frac{a}{\prod_{i=1}^n \sigma^i(h^{m_i})} = \sum_{i=0}^n \frac{s_i}{\sigma^i(h^{m_i})} \tag{5.5}$$

where $\deg(s_i) < \deg(h)m_i$. Equivalently, we can write

$$f = \sum_i \sigma^i(f_i)$$

with $f_i := \frac{\sigma^{-i}(s_i)}{h^{m_i}}$. Then by Lemma 2.1.3 we can compute $g_i \in \mathbb{F}(t)_{(r)}$ with

$$\sigma^i(f_i) = \sigma(g_i) - g_i + f_i. \tag{5.6}$$

Therefore, with (5.5), (5.6) and Lemma 2.1.1 we get a Σ-pair (f', g) for f defined by

$$f' := \sum_{i=0}^n \frac{\sigma^{-i}(s_i)}{h^{m_i}} \text{ and } g := \sum_{i=0}^n g_i \text{ with } g_i = \begin{cases} \sum_{j=0}^{i-1} \sigma^j(\frac{\sigma^{-i}(s_i)}{h^{m_i}}) & \text{if } i \geq 0 \\ -\sum_{j=0}^{-i-1} \sigma^{j+i}(\frac{\sigma^{-i}(s_i)}{h^{m_i}}) & \text{if } i < 0; \end{cases} \tag{5.7}$$

f' and g are of the required form given in *SFP*. This proves Theorem 5.1.1. □

Example 5.5. (Cont. Example 2.6) Write $f = \frac{1+k}{k(k+2)} = \frac{1}{2k} + \frac{1}{2(k+2)}$. We apply Lemma 2.1.3 and get $\frac{1}{2(k+2)} = \frac{1}{2k} + \sigma(g) - g$ with $g = \frac{1}{2k} + \frac{1}{2(k+1)} = \frac{2k+1}{2k(k+1)}$. Hence we obtain $f = \frac{1}{2k} + \frac{1}{2k} + \sigma(g) - g$, and therefore $(\frac{1}{k}, \frac{2k+1}{2k(k+1)})$ is a Σ-pair for f.

Example 5.6. (Cont. Example 2.7) Write $f = \frac{1}{k(k-1)P} = \frac{1}{(k-1)P} - \frac{1}{kP}$. By $\frac{1}{(k-1)P} = \sigma(\frac{1}{(k-1)P}) + \sigma(g) - g$ with $g = \frac{-1}{(k-1)P}$ we get the Σ-pair $(-\frac{1}{2kP}, \frac{-1}{(k-1)P})$ for f.

Example 5.7. (Cont. Example 3.1) Write $f = \frac{Hk-2}{H(Hk-1)} = \frac{2}{H} + \frac{-1}{\sigma^{-1}(H)}$. We have $\frac{-1}{\sigma^{-1}(H)} = \frac{-1}{H} + \sigma(g) - g$ with $g = \frac{1}{\sigma^{-1}(H)} = \frac{k}{kH-1}$. Thus we get the Σ-pair $(f', g) = (\frac{1}{H}, \frac{k}{kH-1})$ for f.

Example 5.8. (Cont. Example 5.3) Write $f = \frac{k(Fk+1)}{(F+1)(kF+F+1)} = \frac{k(Fk+1)}{(F+1)\sigma(F+1)} = \frac{k-1}{F+1} + \frac{1}{\sigma(F+1)}$. Then $\sigma(\frac{1}{F+1}) = \sigma(g) - (g) + \frac{1}{F+1}$ with $g = \frac{1}{F+1}$. Hence $(f', g) = (\frac{1}{F+1}, \frac{k}{F+1})$ is a Σ-pair for f.

Example 5.9. (Cont. Example 2.9) Take f from Example 2.9 and split it in the form

$$f = f_0 + f_1 + f_2 + f_3 = \frac{k(k+1)^2}{(2k+1)h} + \frac{(k+1)^2(k+2)}{(2k+3)\sigma(h)} + \frac{k(k+1)(k+2)}{(2k+1)\sigma^2(h)} - \frac{(k+1)(k+2)(k+3)}{(2k+3)\sigma^3(h)}.$$

Then by Lemma 2.1.3 we obtain (f'_i, g_i) with $f'_i = f_i + \sigma(g_i) - g_i$ where

$$\begin{aligned}
(f'_1, g_1) =& \Big(\frac{k^2(k+1)}{(2k+1)h}, \frac{k^2(k+1)}{(2k+1)(Hk-1)},\\
(f'_2, g_2) =& \Big(\frac{(k-2)(k-1)k}{(2k-3)h}, \frac{(k-1)k\big(-2k+3+H\big(4k^2-8k+2\big)\big)}{H(Hk-1)(4k^2-8k+3)}\Big),\\
(f'_3, g_3) =& \Big(\frac{-(k-2)(k-1)k}{(2k-3)h},\\
& \frac{-k\big(-4k^3+8k^2-k-3-4H\big(4k^3-6k^2-k+2\big)+H^2\big(12k^5-12k^4-21k^3+12k^2+7k-2\big)\big)}{H(8k^3-12k^2-2k+3)(k(k+1)H^2-H-1)}\Big).
\end{aligned}$$

This gives $(f', g) = (f_0 + f'_1 + f'_2 + f'_3, g'_1 + g'_2 + g'_3) = (\frac{k(k+1)}{Hk-1}, \frac{k(k+1)}{(Hk-1)(kH+H+1)})$.

With Theorem 5.1.2 it follows that all Σ-pairs from the previous examples are solutions of problem *SFP*. To prove it, we need Remark 5.1 and Lemma 5.3 (compare [8, Cor. 4]).

Remark 5.1. Let (f', g) be a Σ-pair for f which we get by (5.7). Then $\deg(\text{den}(f')) \leq \deg(\text{den}(f))$. If $m_i m_j \neq 0$ in (5.5) for some $i \neq j$, then $\deg(\text{den}(f')) < \deg(\text{den}(f))$.

Lemma 5.3. *Let* $(\mathbb{F}(t), \sigma)$ *be a* $\Pi\Sigma^*$*-extension of* $(\mathbb{F}, \sigma)$*, let* $h \in \mathbb{F}[t]$ *be irreducible and suppose that* $\frac{\sigma(t)}{t} \notin \mathbb{F}$ *or* $t \nmid h$*. Then there is no* $g \in \mathbb{F}(t)$ *with* $\sigma(g) - g = \frac{c}{h^r}$ *for* $c \in \mathbb{F}[t]^*$ *and* $r > 0$*.*

Proof. Suppose that there is a solution $g = \frac{a}{b} \in \mathbb{F}(t)$. Define $d := \gcd(b, \sigma(b))$. Then $v\,h^r = \text{lcm}(b, \sigma(b))$ with $v \mid d$; see e.g. [31, Thm. 2.3.1]. Let $m \in \mathbb{Z}$ be maximal such that $\sigma^m(h) \mid b$; this is possible by Lemma 5.2. Then $\sigma^{m+1}(h) \nmid d$. Hence $\sigma^{m+1}(h) \nmid v$ and $\sigma^{m+1}(h) \mid \text{lcm}(b, \sigma(b))$. Thus $\sigma^{m+1}(h) \mid h^r$. By Lemma 5.2 we have $m = -1$. Similarly, take m' minimal with $\sigma^{m'}(h) \mid b$. Then $\sigma^{m'}(h) \nmid d$. Hence $\sigma^{m'}(h) \nmid v$ and $\sigma^{m'}(h) \mid \text{lcm}(b, \sigma(b))$. Thus $\sigma^{m'}(h) \mid h^r$, i.e., $m' = 0$; a contradiction. □

Proof of Theorem 5.1.2. "⇒" Let (f', g) be a Σ-pair of f with $\text{den}(f') = \prod_{i=1}^{n'} \sigma^i(h^{m'_i})$ where $m'_i m'_j \neq 0$ for some $i \neq j$. By Remark 5.1 there is a Σ-pair (ϕ, γ) for f' where $\deg(\text{den}(\phi)) < \deg(\text{den}(f'))$. Hence $(\phi, g + \gamma)$ is a Σ-pair for f by Lemma 2.1.2. Thus $\deg(\text{den}(f'))$ is not minimal.
"⇐" Let (f', g) be a Σ-pair for f with $f' = \frac{p}{q}$ and $q = \sigma^i(h)^m$. If $m = 0$, then $f' = 0$, i.e., nothing has to be shown. Let $m > 0$, and hence $p \neq 0$, and assume that

there is a Σ-pair $(\frac{p'}{q'}, g')$ for f where $\deg(q')$ is minimal and $\deg(q') < \deg(q)$. By the implication "⇒" and the minimality of $\deg(q')$ it follows that $q' = u\,\sigma^j(h)^{m'}$ for some $u \in \mathbb{F}^*$, $j \in \mathbb{Z}$ and $m' \geq 0$. Then by Lemma 2.1.3 there are $a \in \mathbb{F}[t]$ and $\gamma \in \mathbb{F}(t)$ with $\sigma(\gamma) - \gamma + \frac{a}{\sigma^i(h)^{m'}} = f$. Hence

$$\sigma(g-\gamma) - (g-\gamma) = \frac{a}{\sigma^i(h)^{m'}} - \frac{p}{\sigma^i(h)^m} = \frac{a\,\sigma^i(h)^{m-m'} - p}{\sigma^i(h)^m} \tag{5.8}$$

where $\sigma^i(h) \in \mathbb{F}[t]$ is irreducible. Since $m' < m$, $p \neq 0$ and $\gcd(p, \sigma^i(h))) = 1$, the right-hand side of (5.8) is non-zero; a contradiction to Lemma 5.3. This proves Theorem 5.1.2. □

Define the *dispersion* of $f \in \mathbb{F}[t]^*$ by

$$\mathrm{disp}(f) = \max\{m \geq 0 \mid \gcd(\sigma^m(f), f) \neq 1\}.$$

By Lemma 5.1.2 (Algorithm 4) and Theorem 5.1 we get

Corollary 5.1. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ*-extension of $(\mathbb{F}, \sigma)$ and $f \in \mathbb{F}(t)_{(r)}$ where t is a Σ*-extension or $t \nmid \mathrm{den}(f)$. Let $(\frac{p}{q}, g) \in \mathbb{F}(t)^2_{(r)}$ be a Σ-pair for f. Then the following is equivalent:*

(1) $(\frac{p}{q}, g)$ is a solution of problem RP.
(2) $q = u \prod_i h_i^{m_i}$ where $u \in \mathbb{F}^*$ and where the $h_i \in \mathbb{F}[t]$ are irreducible and pairwise σ-prime.
(3) disp$(q) = 0$.

Corollary 5.1 is a generalized version of the rational case given in [4] or [17, Prop. 3.3].

Remark 5.2. A solution $(\frac{p}{q}, g)$ of problem *RP* with q as in Corollary 5.1.2 is not uniquely determined. More precisely, by splitting $\frac{p}{q}$ in the form $f' = \sum_i \frac{p_i}{{h_i}^{m_i}}$ with $p_i \in \mathbb{F}[t]$ and $\deg(p_i) < \deg(h_i)m_i$ we can apply Lemma 2.1.3 and obtain all other Σ-pairs (ϕ, γ) where $\mathrm{den}(\phi)$ is of the form $\prod_i \sigma^{z_i} h_i^{m_i}$ with $z_i \in \mathbb{Z}$.

6. Eliminating several top extensions in a sum

As shown in Corollary 3.1 we can eliminate the top extension from the non-summable part, if possible; see Examples 2.3 and 2.4. More generally, we are interested to eliminate several extensions, like for identity (1.5) or

$$\sum_{k=1}^{n}\Big(\sum_{j=1}^{k}\binom{n}{j}\Big)\Big(\sum_{j=1}^{k}\binom{n}{j}^2\Big) = \frac{n+2}{2}\sum_{j=1}^{n}\binom{n}{j}\sum_{j=1}^{n}\binom{n}{j}^2 - \frac{1}{2n}\sum_{k=1}^{n}(n^2-nk+k^2)\binom{n}{k}^3.$$

Assume we are given a tower of ΠΣ*-extensions $(\mathbb{F}(t_1)\dots(t_e), \sigma)$, i.e., $(\mathbb{F}(t_1)\dots(t_i), \sigma)$ is a ΠΣ*-extension of $(\mathbb{F}(t_1)\dots(t_{i-1}), \sigma)$ for $1 \leq i \leq e$. If the shift behavior of all these extensions t_i depend only on $\mathbb{F}$, i.e., $\sigma(t_i) = a_i\,t_i + b_i$

where $a_i, b_i \in \mathbb{F}$, then we call such an extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ a $\Pi\Sigma^*$-*extension over* $\mathbb{F}$. Given such an extension we are interested in the following problem.

ET: Eliminate top extensions

Given $f \in \mathbb{F}(t_1)\dots(t_e)$; **find** a Σ-pair $(f',g) \in \mathbb{F}(t_1)\dots(t_r) \times \mathbb{F}(t_1)\dots(t_e)$ for f where r is minimal, i.e., eliminate as many extensions in f' as possible. In particular, choose $f'=0$, if possible.

In particular, we are interested in the following application: Let $\mathbb{F}$ be a $\Pi\Sigma^*$-field where all the maximal nested sums and products are the t_i's and all less nested sums and products are in $\mathbb{F}$. Then solving *ET* enables one to decide constructively if there is a Σ-pair (f',g) for f where f' is less nested than f.

Example 6.1. (Cont. Example 2.5) Given f from Example 2.5 we compute with Algorithm 1 the Σ-pair $(f_2,g_2) = \big(\frac{6H^{(2)}(k+1)^3+3k+4}{3(k+1)^3}, -\frac{1}{3}H(H^2-3(H^{(2)}k+1)H+3H^{(2)}(2k+1))\big)$ for f. Since we managed to eliminate the extension H from the non-summable part f_2, we apply Algorithm 1 to f_2 and get as result the Σ-pair $(f_1,g_1) = \big(-\frac{6k^2+9k+2}{3(k+1)^3}, 2H^{(2)}k\big)$. Finally, we apply Algorithm 1 to f_1 and get the Σ-pair $\big(-\frac{6k^2+9k+2}{3(k+1)^3}, 0\big)$, i.e., f_1 cannot be simplified further in the degree (of the numerator or denominator). Combining all the steps by using Lemma 2.1.2 we obtain the Σ-pair $(f_1, g_1+g_2) = (-\frac{6k^2+9k+2}{3(k+1)^3}, -\frac{H^3}{3}+(H^{(2)}k+1)H^2-(2kH^{(2)}+H^{(2)})H+2H^{(2)}k)$ for f. Finally, by using Lemma 2.1.3 we change the Σ-pair for f to $(\sigma^{-1}(f'), \sigma^{-1}(f')+g)$, see also Example 4.5. This result is used in Example 2.5.

As illustrated in the previous example, we try to attack problem *ET* by running Algorithm 1 recursively and using Lemma 2.1.2. More precisely, we propose the following algorithm.

Algorithm 5. `EliminateExtensions`$((\mathbb{F}(t_1)\dots(t_e),\sigma), f)$

In: A $\Pi\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ over $\mathbb{F}$ with $e \geq 1$ where we can solve problems *PLDE* and *SEF* for all extensions t_i. $f \in \mathbb{F}(t_1)\dots(t_e)$.
Out: A solution of problem *ET*.

(1) If $e=0$, decide constructively, if there is a $g \in \mathbb{F}$ with $\sigma(g)-g=f$. If yes, RETURN $(0,g)$, otherwise RETURN $(f,0)$.
(2) Decide constructively, if there is a Σ-pair $(f',g) \in \mathbb{F}(t_1)\dots(t_{e-1}) \times \mathbb{F}(t_1)\dots(t_e)$ for f.
(3) If no, THEN RETURN (f',g). Otherwise, take such an (f',g).
(4) Compute (ϕ,γ) :=`EliminateExtensions`$((\mathbb{F}(t_1)\dots(t_{e-1}),\sigma), f')$; RETURN $(\phi, g+\gamma)$.

Here we want to emphasize that in each recursion step, see line (4), we drop the top most extension. In a first glance this seems to be not general enough to solve problem *ET*. The crucial point is that we suppose that $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ is a $\Pi\Sigma^*$-extension over $\mathbb{F}$, i.e., for each $1 \leq i \leq e$ we have $\sigma(t_i)=t_i+a_i$ or $\sigma(t_e)=a_it_i$ for some $a_i \in \mathbb{F}$.

Namely, let $(F,G) \in \mathbb{F}(t_1)\dots(t_r) \times \mathbb{F}(t_1)\dots(t_e)$ be a Σ-pair for f where r is minimal. If $r=e$, we certainly return the correct result in step (3) by Corollary 3.1. Now suppose that $r<e$. Hence, by Corollary 3.1 we can compute a Σ-pair (f',g)

for f with $f' \in \mathbb{F}(t_1)\dots(t_{e-1})$ and $g \in \mathbb{F}(t_1)\dots(t_e)$. Thus, $\sigma(h) - (h) = f' - F$ where $h := G - g \in \mathbb{F}(t_1)\dots(t_e)$. Summarizing, we know that there is a Σ-pair (F, h) for f' where F is free of t_e, but h might depend on t_e. Hence, if we drop the extension t_e in line (4), we might fail to find such a Σ-pair for f'.

Here the following lemma enters the game; for a proof see [13, Thm. 24] or [22, Prop. 4.1.3].

Lemma 6.1. *Let* $(\mathbb{F}(t), \sigma)$ *be a* $\Pi\Sigma^*$*-extension of* $(\mathbb{F}, \sigma)$*. Let* $g \in \mathbb{F}(t)$ *with* $\sigma(g) - g \in \mathbb{F}$*. If* $\frac{\sigma(t)}{t} \in \mathbb{F}$*, then* $g \in \mathbb{F}$*. Otherwise,* $g = c\,t + w$ *for some* $c \in \text{const}_\sigma\mathbb{F}$ *and* $w \in \mathbb{F}$*.*

More precisely, if t_e is a Π-extension, it follows by Lemma 6.1 that $h \in \mathbb{F}(t_1)\dots(t_{e-1})$. Hence, $(F, h) \in \mathbb{F}(t_1)\dots(t_r) \times \mathbb{F}(t_1)\dots(t_{e-1})$. Otherwise, if t_e is a Σ^*-extension, by Lemma 6.1 it follows that $h = c\,t_e + w$ for some $w \in \mathbb{F}(t_1)\dots(t_{e-1})$ and $c \in \text{const}_\sigma\mathbb{F}$. Since $\sigma(t_e) = t_e + a_e$ for some $a_e \in \mathbb{F}$,

$$\sigma(h) - h + F = \sigma(w) - w + (F + c\,a_e) = f'.$$

With $v := F + c\,a_e$ we obtain the Σ-pair

$$(v, w) \in \mathbb{F}(t_1)\dots(t_r) \times \mathbb{F}(t_1)\dots(t_{e-1}) \tag{6.1}$$

for f'. This is the main observation of our algorithmic idea: There is a Σ-pair (v, w) for f' where v is free of t_e and w depends only on the elements $t_1, \dots, t_r$. This fact enables us to remove the top most extension in each recursion step. By induction on the number e of extensions we arrive at the following theorem; for a complete proof see [30].

Theorem 6.1. *Let* $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ *be a* $\Pi\Sigma^*$*-extension over* $\mathbb{F}$ $(e \geq 1)$ *where one can solve problems PLDE and SEF for all extensions* t_i*. Then Algorithm 5 solves problem ET.*

Concerning Algorithm 5 the following remarks are in place:

(1) Step (2) of Algorithm 5 can be accomplished by Algorithm 1, i.e., by the function call (f', g) :=`RefinedTelescoping`$((\mathbb{F}(t_1)\dots(t_e), \sigma), f)$; see Example 6.1. In particular, if one fails to eliminate the extension t_e from f', one obtains a Σ-pair (f', g) where the degrees in t_e are optimal. Hence we can combine problems *RT* and *ET*.

(2) We can improve the computation in step (2): Since we only have to eliminate the extension t_e, if possible, but we do not have to decide, if $f' = 0$ is possible, we can avoid unnecessary computations in Algorithm 1. More precisely, in Sub-algorithm 2 we can quit the do-loop when $r = 0$; in Sub-algorithm 3 we can quit the while-loop when $\deg(f') = 0$.

(3) The proposed algorithm does not work, if there is a t_i with $\sigma(t_i) - t_i \in \mathbb{F}(t_1)\dots(t_{i-1}) \setminus \mathbb{F}$. In this case we cannot guarantee that there is a Σ-pair (6.1) for f', i.e., dropping the extension t_e in line (4) makes the algorithm wrong. In [25,27] this problem can be handled properly by using a rather complicated machinery. Similarly, the following problem cannot be handled properly.

(4) We might fail to find a sum extension where the depth is optimal. E.g., starting with the left-hand side of (6.2) we find the first simplification in

$$\sum_{j=1}^{n}\sum_{k=1}^{j}\frac{H_k}{k^2}=n\sum_{k=1}^{n}\frac{H_k}{k^2}-\sum_{k=1}^{n}\frac{H_k(k-1)}{k^2}=n\sum_{k=1}^{n}\frac{H_k}{k^2}-\Big(-\sum_{k=1}^{n}\frac{H_k}{k^2}+\frac{1}{2}\big(H_n^2+H_n^{(2)}\big)\Big). \tag{6.2}$$

But our algorithm fails to find $H_n^{(2)}$ in order to simplify $\sum_{k=1}^{j}\frac{H_k(k+1)}{k^2}$ further. Here we would need in addition the sum $\sum_{k=1}^{j}\frac{H_k(k+1)}{k^2}$ which we dropped in line (4); for a solution see [25,27].

7. Simplification of Σ^*-extensions

By [13] there is the following result concerning the construction of Σ^*-extensions.

Theorem 7.1. *Let $(\mathbb{F}(t),\sigma)$ be a difference field extension of $(\mathbb{F},\sigma)$ with $\sigma(t)=t+f$ where $f\in\mathbb{F}$. Then this is a Σ^*-extension iff there is no $g\in\mathbb{F}$ with $\sigma(g)-g=f$.*

This result provides a constructive theory to represent sums, like

$$S(n)=\sum_{k=1}^{n}f(k),$$

in $\Pi\Sigma^*$-fields. More precisely, suppose that $f(k)$ can be expressed in a $\Pi\Sigma^*$-field, say $(\mathbb{F},\sigma)$ with $f\in\mathbb{F}$; for typical examples see Section 2. Two cases can occur:
(1) One finds a $g\in\mathbb{F}$ with $\sigma(g)-g=f$. Then reconstruct from g a sequence $g(k)$ with $g(k+1)-g(k)=f(k)$ and derive, with some mild extra-conditions, the closed form $S(n)=g(n+1)-g(1)$. In particular, the sum $S(n)$ can be expressed by $t:=\sigma(g)+c\in\mathbb{F}$ for some $c\in\mathbb{K}$ ($c=g(1)$) with

$$\sigma(t)=t+\sigma(f); \tag{7.1}$$

this reflects the shift behavior $S(n+1)=S(n)+f(n+1)$.
(2) One shows that there is no $g\in\mathbb{F}$ with $\sigma(g)-g=f$. Then by Theorem 7.1 one can adjoin the sum $S(n)$ formally in form of the Σ^*-extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ with (7.1).

At this point our refined summation algorithms can contribute: Suppose there is no $g\in\mathbb{F}$ with $\sigma(g)-g=f$ and let (f',g) be any Σ-pair for f. Then there is no $h\in\mathbb{F}$ with $\sigma(h)-h=f'$; otherwise, we would have $\sigma(g+h)-(g+h)=f$ for $g+h\in\mathbb{F}$. Hence, by Theorem 7.1 we can construct the Σ^*-extension $(\mathbb{F}(s),\sigma)$ with $\sigma(s)=s+\sigma(f')$. Moreover, for $T:=s+\sigma(g)+c$ with some $c\in\mathbb{K}$ we have $\sigma(T)=\sigma(s)+\sigma^2(g)+c=s+\sigma(f')+\sigma^2(g)+c=s+\sigma(g)+c+\sigma(f)=T+\sigma(f)$, i.e., $\sigma(T)=T+\sigma(f)$. Thus, we can represent $S(n)$ by $T\in\mathbb{F}(s)$.

Remark. Note that the Σ^*-extensions $(\mathbb{F}(t),\sigma)$ and $(\mathbb{F}(s),\sigma)$ from above are isomorphic by the difference field isomorphism $\tau:\mathbb{F}(t)\to\mathbb{F}(s)$ with $\tau(f)=f$ for all $f\in\mathbb{F}$ and $\tau(t)=s+\sigma(g)+d$; $d\in\mathbb{K}$ is arbitrary, but fixed.

Summarizing, if we compute a Σ-pair (f', g) with Algorithms 1 or 5 we might represent the sum $S(n)$ in a refined Σ^*-extension.

Example 7.1. (Cont. Example 2.6) Since there is no $g \in \mathbb{Q}(k)$ with $\sigma(g) - g = f$ for $f = \frac{k+1}{k(k+2)}$, we can construct the Σ^*-extension $\mathbb{Q}(k)(t)$ with $\sigma(t) = t + \frac{k+2}{(k+1)(k+3)}$ and can represent the sum $S(n) = \sum_{k=1}^{n} \frac{k+1}{k(k+2)}$ by t. Given the Σ-pair $(\frac{1}{k}, \frac{2k+1}{2k(k+1)})$ from Example 2.6, we can represent the sum $S(n)$ with $T := H + \frac{2k+3}{2(k+1)(k+2)} - \frac{3}{4}$ in the Σ^*-extension $(\mathbb{Q}(k)(H), \sigma)$ with $\sigma(H) = H + \frac{1}{k+1}$. We get the difference field isomorphism $\tau : \mathbb{Q}(k)(t) \to \mathbb{Q}(k)(H)$ given by $\tau(t) = \frac{2k+3}{2(k+1)(k+2)} - \frac{3}{4}$. This is exactly reflected by the identity (2.6).

8. Conclusion

We developed algorithms that can express a given sum in terms of a sum $\sum f'(k)$ where $f'(k)$ is degree-optimal.

Here we restricted so far to the domain of $\Pi\Sigma^*$-fields. More generally, one can apply the presented reductions also to difference rings which involve objects like $(-1)^k$, see Example 2.4. Here further investigations are necessary. E.g., one needs more general algorithms for problems *PLDE* and *SEF*; some first steps can be found in [22]. Note that our algorithms can be applied for more general difference fields described in [14,15]

Carrying over Paule's greatest factorial factorization [17] (the discrete analogue of greatest squarefree factorization) to the $\Pi\Sigma^*$-case might give further theoretical insight to our algorithmic results. Some steps in this direction can be found in [18,7].

Following [21] one might refine our algorithms further: given $f(k)$, find $f'(k)$ and $g(k)$ with (1.2) where among all the degree optimal $f'(k)$ also $g(k)$ is "optimal"; see Remarks 4.1 and 5.2. Special cases have been considered in [1,5] for the hypergeometric case.

We presented a simple algorithm in Section 6 that computes, if possible, a summand $f'(k)$ which is less nested than $f(k)$. More general, but also more complicated algorithms have been proposed in [25,27] which find depth-optimal $f'(k)$, see e.g. identity (6.2). Using the results of this article should simplify these general algorithms.

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