A Computer Proof of Moll's Log-Concavity Conjecture

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Abstract

In his study [7] on quartic integrals, Moll met a specialized family of Jacobi polynomials. Moll conjectured that the corresponding coefficient sequences are log-concave. In this paper we settle Moll's conjecture by a non-trivial usage of computer algebra.

1 Introduction

Victor Moll begins his "personal story" [7] with the remark,

> (...) It was even more of a surprise to discover that new things can still be said today about the mundane subject of integration of rational functions of a single variable and that this subject has connections with branches of contemporary mathematics as diverse as combinatorics, special functions, elliptic curves, and dynamical systems.

In this article we want to add another essential ingredient to his story, namely *computer algebra.* In Section 2 we will show how recently developed procedures can be used to retrieve observations which in Moll's original approach were derived with classical methods. In Section 3 we demonstrate that computer algebra can do even more, namely, by proving his log-concavity conjecture with a combination of various algorithms. In the Conclusion we briefly comment on a much stronger version of his conjecture.

2 Setting the Stage

Moll's starting point is the following double sum representation of a quartic integral.

Theorem 1 *Let $a > -1$ and let m be a natural number. Then*

$$\int_0^\infty \frac{1}{(x^4+2ax^2+1)^{m+1}}dx = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}}P_m(a)$$

where

$$P_m(a) = \sum_{j,k} \binom{2m+1}{2j}\binom{m-j}{k}\binom{2k+2j}{k+j}\frac{(a+1)^j(a-1)^k}{2^{3(k+j)}} \tag{1}$$

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The proof is elementary and employs Wallis' integral formula; details are given in [1, Thm. 7.2.1]. Obviously, Theorem 1 gives a description of the quartic which for specific m can be computed in a finite number of steps. The remainder of Moll's article [7] discusses how further information about the family of polynomials $P_m(a)$ can be obtained. For instance, expanding $P_m(a)$ as

$$P_m(a) = \sum_{l=0}^{m} d_l(m) a^l,$$

inspection quickly leads to the following positivity conjecture on the coefficients.

Proposition 1 *For $0 \leq l \leq m$, we have $d_l(m) > 0$.*

2.1 $P_m(a)$ and Ramanujan's Master Theorem

Moll found a remarkable proof of Prop. 1. Namely, he was able to reformulate the problem in such a way that Ramanujan's Master Theorem could be invoked to prove the following alternative representation for the $P_m(a)$.

Theorem 2 *Let m be a natural number. Then*

$$P_m(a) = 2^{-2m} \sum_{k} 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} (a+1)^k. \tag{2}$$

For further details, see [1, Thm. 7.9.1]. Obviously, Theorem 2 immediately implies positivity of the coefficients, i.e., the correctness of Proposition 1. In particular, after applying the binomial theorem one obtains the following single sum representation for the $d_l(m)$.

Proposition 2 *For $l, m \geq 0$,*

$$d_l(m) = 2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} \binom{k}{l}. \tag{3}$$

In addition, a reformulation of the sum representation (2) allows to identify the $P_m(a)$ within the mathematical knowledge base. Namely, rewriting it into standard hypergeometric format, $P_m(a) = 2^{-2m}\binom{2m}{m} {}_2F_1(-m, m+1; \frac{1}{2} - m; \frac{a+1}{2})$, reveals that the $P_m(a)$ are nothing but special instances of Jacobi polynomials; i.e.,

$$P_m(a) = P_m^{(\alpha,\beta)}(a) \qquad \text{with } \alpha = m + \tfrac{1}{2} \text{ and } \beta = -\left(m + \tfrac{1}{2}\right). \tag{4}$$

2.2 Positivity using computer algebra

In this subsection we demonstrate that computer algebra allows to prove Proposition 1 in a completely routine fashion.

To this end we represent $d_l(m)$ as a triple sum,

$$d_l(m) = \sum_{j,s,k} \frac{(-1)^{k+j-l}}{2^{3(k+3)}} \binom{2m+1}{2s} \binom{m-s}{k} \binom{2(k+s)}{k+s} \binom{s}{j} \binom{k}{l-j}, \tag{5}$$

which is immediate from (1); see Theorem 7.3.1 in [1] after replacing s by $s-j$ and then k by $k+s$. Note that despite summing over all integers j, s, k the sum is finite owing to the finite supports of the binomials involved.

The first step is to derive with the RISC package MultiSum [8] a suitable recurrence for the $d_l(m)$. MultiSum is an implementation of Wegschaider's algorithm which is an extension of multivariate WZ summation [9]. After loading the package into Mathematica by

In[1]:= **<< MultiSum.m**
MultiSum Package by Kurt Wegschaider (enhanced by Axel Riese and Burkhard Zimmermann) – © RISC Linz – V2.02β (02/21/05)

we input the summand of the triple sum (5):

In[2]:= $\boldsymbol{f = (-1)^{k+j-1}/2^{3(k+s)}}$ **Binomial[**$\boldsymbol{2m+1, 2s}$**]Binomial[**$\boldsymbol{m-s, k}$**]Binomial[**$\boldsymbol{2(k+s), k+s}$**]**
Binomial[$\boldsymbol{s, j}$**]Binomial[**$\boldsymbol{k, l-j}$**];**

With the next command we determine candidate sets of shifts possibly arising in recurrences for $f =: F_{l,m,k,s,i}$:

In[3]:= **regions = FindStructureSet[**$\boldsymbol{f, \{l, m\}, \{1, 0\}, \{k, s, j\}, \{0, 1, 0\}, 1}$**];**
In[4]:= **Length[regions]**

Out[4]= 9

This means, we obtain nine such candidate sets. For the second one in this list we compute a summand recurrence as follows:

In[5]:= **FindRecurrence[**$\boldsymbol{f, \{l, m\}, \{k, s, j\}}$**, regions[[2]], 1, WZ → True]**

Out[5]=
$$\begin{aligned}&\{-4(l+m)F_{l-1,m,k-1,s-1,j-1} - 2(2l+4m+3)F_{l,m,k-1,s-1,j-1} + 4(m+1)F_{l,m+1,k-1,s-1,j-1}\\ &= \Delta_j\big((2k+6s-7)F_{l-1,m,k-1,s-1,j-1} + 2(2l+4m+3)F_{l,m,k-1,s-1,j-1} - 4(m+1)F_{l,m+1,k-1,s-1,j-1}\big)\\ &\quad +\Delta_k\big((-2k+4l+4m-6s+7)F_{l-1,m,k-1,s-1,j-1} - 4(k-l+s)F_{l-1,m,k,s-1,j-1}\\ &\qquad +(-2k+4l+4m-2s+7)F_{l,m,k-1,s-1,j} + 4(k+m+2s)F_{l,m,k-1,s,j}\\ &\qquad -4(k-l)F_{l,m,k,s-1,j} - 4(m+1)F_{l,m+1,k-1,s,j}\big)\\ &\quad +\Delta_s\big((2k+6s-7)F_{l-1,m,k-1,s-1,j} + (2k+4m+2s-1)F_{l,m,k-1,s-1,j} - 4(m+1)F_{l,m+1,k-1,s-1,j}\big)\}\end{aligned}$$

In the output, Δ_i denotes the forward difference operator, e.g.,

$$\Delta_k\big((k-l+s)F_{l,m,k,s,j}\big) = (k+1-l+s)F_{l,m,k+1,s,j} - (k-l+s)F_{l,m,k,s,j}.$$

Note: Since $F_{l,m,k,s,j}$ is hypergeometric w.r.t. each of the variables, any shift of $F_{l,m,k,s,j}$ is a rational function multiple of $F_{l,m,k,s,j}$. Consequently, after dividing both sides by $F_{l,m,k,s,j}$ the correctness check of Out[5] reduces to checking an equality of rational functions. – Since $f = F_{l,m,k,s,j}$ has finite support w.r.t. the variables j, k, and s, we can sum both sides of Out[5] to obtain a recurrence for the sum $d_l(m) =: \mathrm{SUM}(l,m)$. This is done automatically by

In[6]:= **SumCertificate[%]**

Out[6]= $\{-2(l+m)\mathrm{SUM}(l-1,m) + (-2l-4m-3)\mathrm{SUM}(l,m) + 2(m+1)\mathrm{SUM}(l,m+1) = 0\}$

In other words, we have that for $0 \le l \le m+1$,

$$-2(l+m)d_{l-1}(m) - (2l+4m+3)d_l(m) + 2(m+1)d_l(m+1) = 0. \tag{6}$$

This recurrence implies positivity of the $d_l(m)$.

2.3 Identifying $P_m(a)$ using computer algebra

In this subsection we demonstrate that computer algebra allows to identify the $P_m(a)$ in a completely routine fashion. This means, tools like Ramanujan's Master Theorem can be completely avoided. (Note: The conditions that make the Master Theorem applicable are in general nontrivial to check.)

Continuing the session above, we now compute a recurrence purely in l:

In[7]:= **SumCertificate[FindRecurrence[**$\boldsymbol{f, \{l, m\}, \{k, s, j\}}$**, regions[[1]], 1, WZ → True]]**

Out[7]= $(-l+m+2)(l+m-1)\mathrm{SUM}(l-2,m) - (l-1)(2m+1)\mathrm{SUM}(l-1,m) + (l-1)l\mathrm{SUM}(l,m) = 0$

In other words, we have that for $0 \le l \le m+1$,

$$-(-2+l-m)(-1+l+m)d_{l-2}(m)-(l-1)(2m+1)d_{l-1}(m)+(l-1)ld_l(m)=0. \qquad (7)$$

Note that in contrast to (6) this does not imply positivity of the $d_l(m)$. However, it enables to identify $P_m(a)$ as a special instance of Jacobi polynomials. To this end, using the RISC package GeneratingFunctions [6] we convert (7) to a differential equation for $P_m(a)$ which is nothing but the ordinary generating function of the $d_l(m)$ for fixed m.

After loading the package into Mathematica by

In[8]:= **<< GeneratingFunctions.m**

GeneratingFunctions Package by Christian Mallinger – © RISC Linz – V 0.68 (07/17/03)

we first input the recurrence (7)

In[9]:= $\mathbf{rec = (-(-2+l-m)(-1+l+m)d[-2+l]-(-1+l)(1+2m)d[-1+l]+(-1+l)ld[l] == 0);}$

The conversion to a differential equation for $P_m(a) =: P(a)$ is done automatically as follows:

In[10]:= $\mathbf{RE2DE[rec, d[l], P[a]]}$

Out[10]= $$(m+m^2)P(a)-(1+2a+2m)P'(a)-(a^2-1)P''(a)=0$$

Finally we recall the differential equation for Jacobi polynomials $y(a) := P_m^{(\alpha,\beta)}(a)$,

$$m(m+\alpha+\beta+1)y(a)-(\alpha-\beta+(\alpha+\beta+2)a)y'(a)-(a^2-1)y''(a)=0,$$

and compare the parameters,

In[11]:= $\mathbf{Solve[\{\alpha+\beta == 0, \alpha-\beta == 1+2m, \alpha+\beta+2 == 2\}, \{\alpha, \beta\}]}$

Out[11]= $$\left\{\left\{\alpha \to \tfrac{1}{2}(2m+1), \beta \to -\tfrac{1}{2}-m\right\}\right\}$$

Checking that $P_m(0), P_m'(0)$ and the initial values $y(0), y'(0)$ of $y(a) = P_m^{(\alpha,\beta)}(a)$ with $\alpha = m+\frac{1}{2}$ and $\beta = -(m+\frac{1}{2})$ coincide, we can conclude (4).

Remarks

1. Owing to (5) the values $P_m(0)$ and $P_m'(0)$ are hypergeometric double sums and thus can be evaluated with the package MultiSum.

2. Taking the standard 3-term recurrence for Jacobi polynomials would fail since this one usually is derived under the assumption that the parameters α and β are independent of m.

2.4 Further recurrences for $d_l(m)$

In this subsection we compute further recurrences for $d_l(m)$ which will be needed in the next section. The MultiSum package would be powerful enough to continue to work with (5), but for the sake of simplicity we will use the representation (3).

In[12]:= $\mathbf{g = 2^{k-2m}Binomial[2m-2k, m-k]Binomial[m+k, k]Binomial[k, l];}$

In[13]:= $\mathbf{FindRecurrence[g, \{m\}, \{k\}, 1, NumericalCheck \to True][[3]]}$

Out[13]= $$\begin{gathered}-4(l-m)(m-1)m^2F_{m,k-2}+(l+m-1)(4m-5)(4m-3)mF_{m-2,k-2}\\ -2(m-1)(8m^2-4l^2-8m+3)mF_{m-1,k-2}\\ =\Delta_k\big(4m^2(l-m)(m-1)F_{m,k-2}-m(l+m-1)(4m-5)(4m-3)F_{m-2,k-2}\\ +2m(m-1)(-4l^2+12m^2-14m+5)F_{m-1,k-2}+4m(l-m)(m-1)(2m-1)F_{m,k-1}\\ -2(m-1)(-4m^3+4m^2+4l^2m+2km-3m-2kl+l)F_{m-1,k-1}\big)\end{gathered}$$

In[14]:= **SumCertificate[%] // Simplify**

Out[14]= $$\begin{gathered}(1+l+m)(4m+3)(4m+5)\mathrm{SUM}(m)=2(m+1)\big((8m^2+24m-4l^2+19)\mathrm{SUM}(m+1)\\ +2(l-m-2)(m+2)\mathrm{SUM}(m+2)\big)\end{gathered}$$

Note that here $g = F_{m,k}$. In other words, after replacing m by $m+2$ we have for $0 \le l \le m$

$$\begin{aligned} 4(l-m-2)(m+1)(m+2)d_l(m+2) = (l+m+1)(4m+3)(4m+5)d_l(m) \\ -2(m+1)(-4l^2+8m^2+24m+19)d_l(m+1). \end{aligned} \tag{8}$$

Finally, we derive another mixed recurrence, which is a counterpart to (6):

In[15]:= **regions = FindStructureSet[g, {m, l}, {1, 2}, {k}, {1}, 1];**

In[16]:= **Length[regions]**

Out[16]= 6

This means, we obtained six candidate regions for a mixed recurrence. For the second one in the list we compute a summand recurrence as follows:

In[17]:= **FindRecurrence[g, {m, l}, {k}, regions[[2]], 1, WZ → True]**

Out[17]=

$$\begin{gathered} \{(2l-4m-5)(2l-2m-3)(l+m)F_{m,l-1,k} \\ +2(l-1)l(2l-2m-3)F_{m,l,k} \\ -2(2l-2m-3)(l-m-2)(m+1)F_{m+1,l-1,k} \\ = \Delta_k\big((2l-2m-3)(l+m)(-2l+4m+5)F_{m,l-1,k} \\ -2l(2l^2-2ml-5l-2m^2-2k-2km-2m+1)F_{m,l,k} \\ -2(m+1)(-2l^2+4ml+7l-4m^2+2k+2km-11m-8)F_{m+1,l-1,k}\big)\} \end{gathered}$$

In[18]:= **SumCertificate[%]**

Out[18]= $\{(2l-4m-5)(l+m)\mathrm{SUM}(m,l-1)+2(l-1)l\mathrm{SUM}(m,l)-2(l-m-2)(m+1)\mathrm{SUM}(m+1,l-1)=0\}$

In other words, after replacing l by $l+1$, we have

$$-2(2l-4m-3)(l+m+1)d_l(m)+4(l-m-1)(m+1)d_l(m+1)-4l(l+1)d_{l+1}(m)=0. \tag{9}$$

3 Proof of the Log-Concavity Conjecture

In addition to his positivity observation, Proposition 1, Moll observed that the sequence $d_l(m)$ is a candidate for being log-concave. The exact statement of his conjecture, proved below, is as follows.

Theorem 3 *For $0 < l < m$, we have*

$$d_{l-1}(m)d_{l+1}(m) \le d_l(m)^2. \tag{10}$$

In this section, we prove this theorem by first deriving a sufficient condition on the sequence $d_l(m)$, which implies the log-concavity of $d_l(m)$. Then, we employ again computer algebra tools for proving this sufficient condition.

3.1 Derivation of a Sufficient Condition

In a first step, we use the recurrence equations (9) and (6) in order to eliminate the shifts in l from (10) at the cost of introducing shifts in m. This gives the equivalent formulation

$$\begin{aligned} &-\left(16m^3+16lm^2+40m^2+28lm+33m+9l+9\right)d_l(m)^2 \\ &\quad -4(m+1)\left(2l^2-4m^2-7m-3\right)d_l(m+1)d_l(m) \\ &\quad +4(l-m-1)(m+1)^2d_l(m+1)^2 \le 0. \end{aligned} \tag{11}$$

Using Cylindrical Algebraic Decomposition (CAD) [3, 2], we compute in the second step an equivalent condition for the negation of (11), interpreting m, l, $d_l(m)$, and $d_l(m+1)$ as independent real variables M, L, D_0, D_1, respectively, and taking into account the knowledge $0 < l < m$ and $d_l(m) > 0$ (Prop. 1).

In[19]:= $\mathbf{CylindricalDecomposition[\{0 < L < M, D_0 > 0, D_1 > 0, (16M^3 + 16LM^2 + 40M^2 + 28LM + 33M + 9L + 9)D_0^2 - 4(M+1)\left(2L^2 - 4M^2 - 7M - 3\right) D_1 D_0 + 4(L - M - 1)(M+1)^2 D_1^2 < 0\}, \{M, L, D_0, D_1\}]}$

Out[19]= $$M > \frac{1}{2} + \sqrt{2} \wedge \mathrm{Root}\left[4\#1^3 - 3\#1 - 4M^2 - 4M\&, 1\right] < L < M \wedge D_0 > 0 \wedge$$
$$\frac{3-2L^2+M(4M+7)-\sqrt{L(4L^3-3L-4M(M+1))}}{2(M+1)(M-L+1)} D_0 < D_1 < \frac{3-2L^2+M(4M+7)+\sqrt{L(4L^3-3L-4M(M+1))}}{2(M+1)(M-L+1)} D_0$$

In other words, we obtain that the formula

$$0 < l < m \wedge d_l(m) > 0 \wedge d_l(m+1) > 0 \wedge \neg(11) \tag{12}$$

is equivalent over the reals to the formula

$$\begin{aligned} m > \tfrac{1}{2}(1 + 2\sqrt{2}) \wedge \mathrm{Root}(-4m - 4m^2 - 3X + 4X^3, 1) < l < m \wedge d_l(m) > 0 \\ \wedge \frac{p_1(l,m) - \sqrt{p_2(l,m)}}{p_3(l,m)} d_l(m) < d_l(m+1) < \frac{p_1(l,m) + \sqrt{p_2(l,m)}}{p_3(l,m)} d_l(m), \end{aligned} \tag{13}$$

where

$$\begin{aligned} p_1(l,m) &= -2l^2 + (m+1)(4m+3), \\ p_2(l,m) &= l(4l^3 - 3l - 4m(m+1)), \\ p_3(l,m) &= 2(m+1)(m-l+1). \end{aligned}$$

The symbol $\mathrm{Root}(-4m - 4m^2 - 3X + 4X^3, 1)$ denotes the algebraic function which maps m to the smallest real root of the polynomial $-4m - 4m^2 - 3X + 4X^3$. The graph of this function is depicted in Figure 1 as a dotted curve. It divides the set of all points (m,l) under consideration into the sets

$$S_- := \{ (m,l) \in \mathbb{Z}^2 : 0 < m \le \tfrac{1}{2} + \sqrt{2} \vee 0 < l \le \mathrm{Root}(-4m - 4m^2 - 3X + 4X^3, 1) \} \quad \text{and}$$
$$S_+ := \{ (m,l) \in \mathbb{Z}^2 : m > \tfrac{1}{2} + \sqrt{2} \wedge \mathrm{Root}(-4m - 4m^2 - 3X + 4X^3, 1) < l < m \}$$

of points below and above this graph, respectively.

From the output (13), it follows that the log-concavity condition (10) cannot be violated at all for the points $(m,l) \in S_-$. And, in addition, for some point $(m,l) \in S_+$, it is violated if and only if

$$\frac{p_1(l,m) - \sqrt{p_2(l,m)}}{p_3(l,m)} d_l(m) < d_l(m+1) < \frac{p_1(l,m) + \sqrt{p_2(l,m)}}{p_3(l,m)} d_l(m) \tag{14}$$

for this point. In order to complete the proof for all $0 < l < m$, it is therefore sufficient to show

$$d_l(m+1) \ge \frac{p_1(l,m) + \sqrt{p_2(l,m)}}{p_3(l,m)} d_l(m) \tag{15}$$

for all $(m,l) \in S_+$. Because then the violating condition (14) can never become true, and thus there does not exist any counterexample to the log-concavity condition for $0 < l < m$.

For reasons to be explained in Section 3.2, we are not pleased with the square root expression $\sqrt{p_2(l,m)}$ in (15). It would be preferable to have a condition of the form $d_l(m+1) \ge \mathrm{rat}(l,m) d_l(m)$ for some rational function $\mathrm{rat}(l,m)$. In order to achieve this form, observe that it is sufficient to prove the modified condition

$$d_l(m+1) \ge \frac{p_1(l,m) + \sqrt{p_2(l,m) + u(l,m)}}{p_3(l,m)} d_l(m) \tag{16}$$

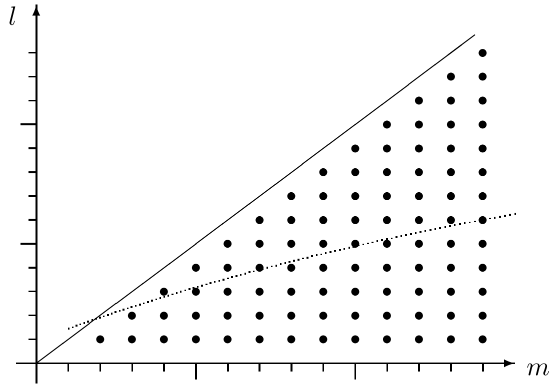

Figure 1: The points (m, l) under consideration.

for a polynomial $u(l, m)$ which is nonnegative for $0 < l < m$.

The idea is now to choose $u(l, m)$ in such a way that $p_2(l, m) + u(l, m)$ becomes a square. It turns out that $u(l, m) = 4l^2 + 4l^3 + 4lm(m + 1)$ is a good choice, because $u(l, m) \geq 0$ and

$$p_2(l, m) + u(l, m) = l^2(2l + 1)^2.$$

Hence, (16) turns into the stronger condition

$$\boxed{d_l(m+1) \geq \frac{4m^2 + 7m + l + 1}{2(m + 1 - l)(m + 1)} d_l(m)}, \tag{17}$$

which is stronger than (15) because $p_3(l, m) > 0$ for the l, m in question.

Summarizing, if (17) is true for $(m, l) \in S_+$, then also (15) is true, and then (14) can never become true. But (14) is true for every possible counterexample to (11), so we are save to conclude that no counterexample to (11) exists. Hence, since (11) is equivalent to (10), the log-concavity of $d_l(m)$ is established as soon as we have shown the boxed condition (17) above.

3.2 Proof of the Sufficient Condition

We now prove that (17) is true for all $0 < l < m$. Let us denote this condition by $\Phi(l, m)$. Again, we employ symbolic computation. The condition (17) is in a form to which a procedure by Gerhold and Kauers for proving special function inequalities [4] can be applied. An implementation of this method, based on the Cylindrical Algebraic Decomposition implementation in Mathematica, is contained in the RISC-package SumCracker [5]. This package can be used to prove automatically that the *induction step formula*

$$\forall\, m \in \mathbb{N}\ \forall\, l \in \mathbb{N} : \big(0 < l < m \wedge \Phi(l, m)\big) \Rightarrow \Phi(l, m + 1)$$

is a consequence of the three term recurrence (8) and the positivity $d_l(m) > 0$ $(0 < l < m)$:

In[20]:= $\mathbf{ProveInequality}[\boldsymbol{d}[\mathbf{1}+\boldsymbol{m}] \geq (\mathbf{4}\boldsymbol{m}^{\mathbf{2}} + \mathbf{7}\boldsymbol{m} + \boldsymbol{l} + \mathbf{3})\boldsymbol{d}[\boldsymbol{m}]/(\mathbf{2}(\boldsymbol{m} + \mathbf{1} - \boldsymbol{l})(\mathbf{1} + \boldsymbol{m})),$
$\quad\mathbf{Where} \to \{\boldsymbol{d}[\mathbf{2}+\boldsymbol{m}] == ((\mathbf{1}+\boldsymbol{l}+\boldsymbol{m})(\mathbf{3}+\mathbf{4}\boldsymbol{m})(\mathbf{5}+\mathbf{4}\boldsymbol{m})\boldsymbol{d}[\boldsymbol{m}] -$
$\quad\quad\mathbf{2}(\mathbf{1}+\boldsymbol{m})(\mathbf{19} - \mathbf{4}\boldsymbol{l}^{\mathbf{2}} + \mathbf{24}\boldsymbol{m} + \mathbf{8}\boldsymbol{m}^{\mathbf{2}})\boldsymbol{d}[\mathbf{1}+\boldsymbol{m}])/(\mathbf{4}(-\mathbf{2}+\boldsymbol{l}-\boldsymbol{m})(\mathbf{1}+\boldsymbol{m})(\mathbf{2}+\boldsymbol{m}))\},$
$\quad\mathbf{Using} \to \{\mathbf{0} < \boldsymbol{l} < \boldsymbol{m}, \boldsymbol{d}[\boldsymbol{m}] > \mathbf{0}\},$
$\quad\mathbf{Variable} \to \boldsymbol{m},$
$\quad\mathbf{CheckIntialValues} \to \mathbf{False}$
$\quad]$

Out[20]= True

This result states that whenever $\Phi(l,m)$ is true for some bullet (m,l) in Figure 1, then it is also true for its right neighboring bullet $(m+1,l)$. The verification of this formula is the key step in the whole proof. On a contemporary personal computer, it requires about 40 minutes computation time. The result of this long computation is simply the boolean value "True", which, by the correctness of the underlying algorithms, constitutes a rigorous proof that the induction step formula holds—even though the procedure does not tell us *why* the formula holds.

The proof is completed by showing, as base of the induction, that $\Phi(l,m)$ is true for all the bullets (l,m) right below the diagonal, i.e., that $\Phi(m,m+1)$ is true for all $m \geq 1$. Using the sum representations of $d_l(m)$, it is easy to obtain the special values

$$d_m(m+1) = 2^{-m-2}(2m+3)\binom{2m+2}{m+1}, \quad d_m(m+2) = \frac{(m+1)(4m^2+18m+21)}{2^{m+4}(2m+3)}\binom{2m+4}{m+2}$$

(cf. Prop. 7.5.1 in [1]). Using these values, a straightforward computation now confirms that

$$\Phi(m,m+1) \iff \frac{d_m(m+2)}{d_m(m+1)} \geq \frac{2m^2+8m+7}{2(m+1)} \iff \frac{m}{(2m+3)(m+2)} \geq 0,$$

which is evident for $m \geq 1$. $\square$

4 Moll Conjectured Much More

Define the operator $\mathcal{L}$ which maps a sequence a_l to the sequence

$$\mathcal{L}a_l := a_{l-1}a_{l+1} - a_l^2.$$

Then log-concavity of a_l amounts to positivity of $\mathcal{L}a_l$. If the sequence $\mathcal{L}a_l$ is not only positive but also log-concave, then we say that a_l is 2-log-concave. In general, we say that a_l is k-log-concave if $\mathcal{L}^k a_l$ is positive, and that a_l is ∞-log-concave if $\mathcal{L}^k a_l$ is positive for every $k \geq 1$.

Moll has conjectured that the coefficients $d_l(m)$ are not only log-concave but even ∞-log-concave. This remains an open problem. Infinite log-concavity is open even for simpler sequences such as the binomial coefficient sequence $\binom{m}{l}$ (compare Project 7.9.3 in [1]). Here, we find that the k-log-concavity condition can always be brought into the form

$$\operatorname{rat}_k(m,l)\binom{m}{l}^{(2^k)} \geq 0$$

for some rational functions $\operatorname{rat}_k(m,l)$ which can be easily determined for each particular value of k. Applying again CAD, we were able to confirm that $\binom{m}{l}$ is k-log-concave for $k = 0,1,2,3,4,5$. For $k \geq 6$ it is hopeless to apply CAD, because the complexity of this algorithm is sensitive towards the degree of the polynomials appearing in the inequalities, and the degrees of numerator and denominator of $\operatorname{rat}_k(m,l)$ grow exponentially in k. For $k = 6$, the degrees of numerator and denominator are 133 and 196, respectively. As for the ∞-log-concavity, we do not see how currently available methods in symbolic computation could help in proving this conjecture.

Concerning the k-log-concavity of the original sequence $d_l(m)$, we have tried to apply the proof technique of Section 3 to establish 2-log-concavity, i.e.,

$$\big(d_l(m)^2 - d_{l-1}(m)d_{l+1}(m)\big)^2 - \big(d_{l-1}(m)^2 - d_{l-2}(m)d_l(m)\big)\big(d_{l+1}(m)^2 - d_l(m)d_{l+2}(m)\big) \geq 0.$$

The recurrences (9) and (6) can again be used for obtaining an equivalent statement involving only shifts in m but no shifts in l. This statement is polynomial in the $d_l(m+i)$ of degree 4. As a consequence, the condition corresponding to (13) is much more complicated. It involves algebraic functions of degree up to 15, and it would require more than thirty pages to print it here. Under these circumstances, we have little hope that a proof of 2-log-concavity could be completed along these lines. – Not to mention that a human reader would have hard times to digest it.

Final Remark. All RISC software packages mentioned in this paper are available online at

http://www.risc.uni-linz.ac.at/research/combinat/software/

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