Axiom Workshop 2006

Sigma - A package for multi-summation

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Summation tools in difference fields

A problem in rhombus tilings

Define

$$S_n := \sum_{k=0}^{n-1} \frac{(-1)^{n+k}(n+k+4)!H_{k+1}}{(k+2)!(k+3)!(n-k-1)!}, \quad H_k = \sum_{i=1}^{k} \frac{1}{i}$$

$$T_n := \sum_{k=0}^{n-1} \frac{(-1)^k(n+k+4)!}{(k+1)(k+2)!^2(n-k-1)!}.$$

Lemma 26 (Fulmek, Krattenthaler; 2000) For $n \geq 0$,

$$S_n + \frac{(1-(-1)^n(n+2))n!}{(n+3)!}T_n = (-1)^n(n+2) - 2.$$

Proof: 4 pages involving highly non-trivial transformations.

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Proof: Sigma finds:

$$S_n = \frac{-5-3n}{(1+n)(2+n)} - 2\mathsf{H}_n + (-1)^n\Big(\frac{5+2n-2n^2-n^3}{(1+n)(2+n)} + 2(2+n)\mathsf{H}_n\Big),$$

$$T_n = 1 - 9n - 9n^2 - 2n^3 + 2(1+n)(2+n)(3+n)H_n - (-1)^n.$$

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Sigma contributed in

Ahlgren. Gaussian hypergeometric series and combinatorial congruences. 2001.

Driver, Prodinger, CS, Weideman. Padé approximations to the logarithm II: Identities, recurrences, and symbolic computation. 2006.

Driver, Prodinger, CS, Weideman. Padé approximations to the logarithm III: Alternative methods and additional results.

Kauers, CS. Application of unspecified sequences in symbolic summation. 2006.

Kuba, Prodinger, CS. Generalized Reciprocity Laws for Sums of Harmonic Numbers.

Mortenson. A supercongruence conjecture of Rodriguez-Villegas for a certain truncated hypergeometric function. 2003.

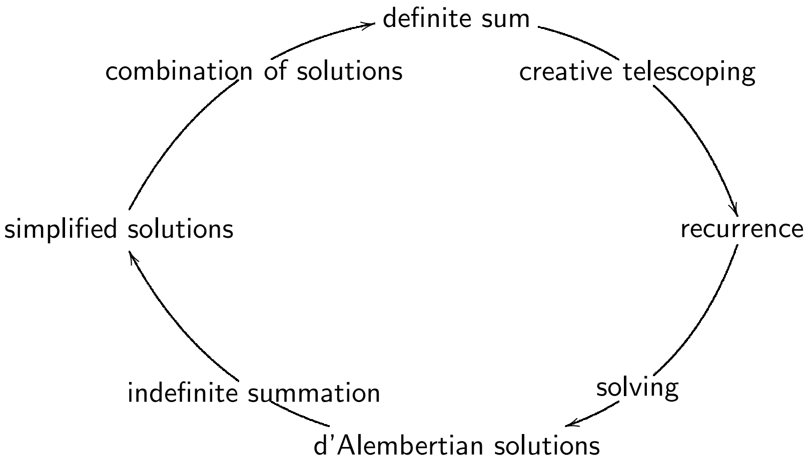
Paule, CS. Computer proofs of a new family of harmonic number identities. 2003.

Pemantle, CS. When is 0.999... equal to 1?

CS. The Summation Package Sigma: Underlying Principles and a Rhombus Tiling Application. 2004.

CS. An Implementation of Karr's Summation Algorithm in Mathematica. 2000.

The Sigma-summation spiral:



Telescoping

- GIVEN $\sum_{k=0}^{n} f(k)$.
- FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k).}$$

- THEN summation over k from 0 to n gives

$$\sum_{k=0}^{n} f(k) = g(n+1) - g(0).$$

Telescoping

FIND a closed form for

$$\sum_{k=1}^{n} H_k.$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{Q}, & & \\ \sigma(k) &= k+1, & \mathcal{S}\, k &= k+1, \\ \sigma(h) &= h + \frac{1}{k+1}, & \mathcal{S}\, H_k &= H_k + \frac{1}{k+1}. \end{aligned}$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

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$$\boxed{g(k+1) - g(k) = H_k}$$

with

$$g(k) = (H_k - 1)k.$$

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Hence,

$$(H_{n+1} - 1)(n+1) = \sum_{k=1}^{n} H_k.$$

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Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]$:

$d = 1$

$$\forall g \in \mathbb{Q}(k)(h): \quad \sigma(g) - g = h \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[h].$$

FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

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$b = 2$

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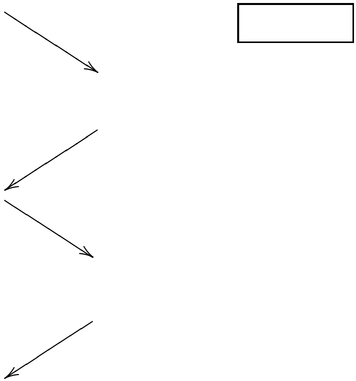
Polynomial Solution: FIND

$$g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h].$$

g=hk-k

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\sigma(g) - g = h$$



ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\left[\sigma(g_2)\left(h+\tfrac{1}{k+1}\right)^2+\sigma(g_1 h+g_0)\right] \\ -\left[g_2\,h^2+g_1 h+g_0\right]=h$$

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coeff. comp.

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$$g_2 = c \in \mathbb{Q}$$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = h - c\Big[\tfrac{2h(k+1)+1}{(k+1)^2}\Big]$$

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$c = 0,\quad g_1 = k + d,\ d \in \mathbb{Q}$

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$c = 0$, $g_1 = k + d$, $d \in \mathbb{Q}$

$$\sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}$$

$\leftarrow$ $g_0 = -k$, $d = 0$

Telescoping in difference fields

Let $(\mathbb{F}, \sigma)$ be a difference field and

$$\mathbb{K} = \{k \in \mathbb{F} | \sigma(k) = k\} \supseteq \mathbb{Q} \quad \text{(constant field)}$$

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Parameterized telescoping

- ▶GIVEN $f_0,\dots,f_d\in\mathbb{F}$, $a_0,a_1\in\mathbb{F}$.
- ▶FIND ALL $c_0,\dots,c_d\in\mathbb{K}$, $h\in\mathbb{F}$:

$$\boxed{a_1\,\sigma(h)-a_0\,h=c_0\,f_0+\dots+c_d\,f_d.}$$

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Remark: creative telescoping

- GIVEN $f_i=\text{summand}(n+i,k)\in\mathbb{F}$.
- FIND ALL $c_0,\dots,c_d\in\mathbb{K}$, $g\in\mathbb{F}$:

$$\boxed{\sigma(g)-g=c_0\,f_0+\dots+c_d\,f_d.}$$

Creative telescoping (for order 2)

GIVEN

$$S(n) = \sum_{k=0}^{n} f(n,k).$$

FIND $c_0(n), c_1(n), c_2(n)$ and $g(n,k)$:

$$\boxed{g(n,k+1) - g(n,k)}$$
$$= \boxed{c_0(n)f(n,k) + c_1(n)f(n+1,k) + c_2(n)f(n+2,k).}$$

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THEN summation over k from 0 to n gives

$$\boxed{g(n,n+1)-g(n,0)}$$
$$=\boxed{\begin{array}{l}c_0(n)\,\mathsf{S}(n)+\\ c_1(n)\,\big[\mathsf{S}(n+1)-f(n+1,n+1)\big]\\ c_2(n)\,\big[\mathsf{S}(n+2)-f(n+2,n+1)-f(n+2,n+2)\big].\end{array}}$$

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$$= \boxed{c_0(n) f(n,k) + c_1(n) f(n+1,k) + c_2(n) f(n+2,k).}$$

HENCE

$$h(n) = c_0(n) S(n) + c_1(n) S(n+1) + c_2(n) S(n+2)$$

for some $h(n)$.

FIND a recurrence for

$$S(n) := \sum_{k=1}^{n} \binom{n}{k} H_k.$$

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$$\sigma(b) = \frac{n-k}{k+1}\, b, \qquad \mathcal{S} \binom{n}{k} = \frac{n-k}{k+1} \binom{n}{k}.$$

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$$f(n+2,k) = \frac{(n+1)\,(n+2)\,\mathsf{H}_k \binom{n}{k}}{(n+1-k)\,(n+2-k)} \quad \longleftrightarrow \quad \frac{(n+1)\,(n+2)\,h\,b}{(n+1-k)\,(n+2-k)} =: f_2.$$

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FIND $c_0, c_1, c_2 \in \mathbb{Q}(n)$ and $g \in \mathbb{F}$:

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Sigma computes

$$c_0 := 4\,(1+n), \quad c_1 := -2\,(3+2\,n), \quad c_2 := 2+n,$$

$$g := \frac{(1+n)\,(-2+k-n+(2\,k-2\,k^2+k\,n)h)\,b)}{(1-k+n)\,(2-k+n)}.$$

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This gives

$$\boxed{g(n,k+1) - g(n,k) = c_0(n) f(n,k) + c_1(n) f(n+1,k) + c_2(n) f(n+2,k)}$$

with

$$c_0(n) := 4\,(1+n), \quad c_1(n) := -2\,(3+2\,n), \quad c_2(n) := 2+n,$$

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Summing over k from 0 to n gives

$$\boxed{1 = 4(1+n)S(n) - 2(3+2n)S(n+1) + (2+n)S(n+2)}$$

for

$$S(n) = \sum_{k=0}^{n} \binom{n}{k} H_k.$$

Algorithms for (creative) telescoping

Hypergeometric case:

Gosper. Decision procedures for indefinite hypergeometric summation. 1978.

Zeilberger. The method of creative telescoping. 1991.

Paule. Greatest factorial factorization and symbolic summation. 1995.

q-Hypergeometric case:

Paule, Riese. A Mathematica q-analogue of Zeilberger's algorithm... 1997.

Mixed case:

Bauer, Petkovšek. Multibasic and mixed hypergeometric Gosper-type algorithms. 1999.

$\Pi\Sigma^*$-field case:

Karr. Summation in finite terms. 1981.

Schneider. An Implementation of Karr's Summation Algorithm in Mathematica. 2000.

Holonomic case:

Chyzak. An extension of Zeilberger's fast algorithm to general holonomic functions. 2000.

CS. A new Sigma approach to multi-summation. 2005.

A constructive $\Pi\Sigma^*$-field theory

Σ^*-extensions

GIVEN a difference field $(\mathbb{F}, \sigma)$ with constant field

$$\mathbb{K} := \{k \in \mathbb{F} | \sigma(k) = k\}.$$

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2. the automorphism σ is extended from $\mathbb{F}$ to $\mathbb{F}(t)$ with

$$\sigma(t) = t + f, \quad f \in \mathbb{F},$$

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3. the constants remain unchanged:

$$\{k \in \mathbb{F}(t) | \sigma(k) = k\} = \mathbb{K}.$$

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Theorem (Karr'81). Given $(\mathbb{F}(t), \sigma)$ where σ is extended from $\mathbb{F}$ to $\mathbb{F}(t)$ with

$$\sigma(t) = t + f, \quad f \in \mathbb{F}.$$

This is a Σ^*-extension iff

$$\nexists g \in \mathbb{F} : \quad \sigma(g) = g + f.$$

Σ^*-extensions

Example: Construct a difference field for

$$\sum_{k=0}^{n-1} H_k.$$

We start with the difference field $(\mathbb{Q}, \sigma)$ with

$$\forall q \in \mathbb{Q}: \ \sigma(q) = q.$$

Theorem (Karr'81). Given $(\mathbb{F}(t), \sigma)$ where σ is extended from $\mathbb{F}$ to $\mathbb{F}(t)$ with

$$\sigma(t) = t + f, \quad f \in \mathbb{F}.$$

This is a Σ^*-extension iff

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Σ^*-extensions

Example: Construct a difference field for

$$\sum_{k=0}^{n-1} H_k \qquad\qquad \mathcal{S}k = k+1.$$

There is no $g \in \mathbb{Q}$ with

$$\sigma(g) = g + 1.$$

Theorem (Karr'81). Given $(\mathbb{F}(t), \sigma)$ where σ is extended from $\mathbb{F}$ to $\mathbb{F}(t)$ with

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There is no $g \in \mathbb{Q}$ with

$$\sigma(g) = g + 1.$$

Hence, we can construct the Σ^*-extension $(\mathbb{Q}(k), \sigma)$ with

$$\sigma(k) = k + 1.$$

Theorem (Karr'81). Given $(\mathbb{F}(t), \sigma)$ where σ is extended from $\mathbb{F}$ to $\mathbb{F}(t)$ with

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This is a Σ^*-extension iff

$$\nexists g \in \mathbb{F}: \quad \sigma(g) = g + f.$$

Σ^*-extensions

Example: Construct a difference field for

$$\sum_{k=0}^{n-1} H_k \qquad\qquad \mathcal{S}H_k = H_k + \frac{1}{k+1}.$$

There is no $g \in \mathbb{Q}(k)$ with

$$\sigma(g) = g + \frac{1}{k+1}.$$

Theorem (Karr'81). Given $(\mathbb{F}(t), \sigma)$ where σ is extended from $\mathbb{F}$ to $\mathbb{F}(t)$ with

$$\sigma(t) = t + f, \quad f \in \mathbb{F}.$$

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Σ^*-extensions

Example: Construct a difference field for

$$\sum_{k=0}^{n-1} H_k \qquad \mathcal{S}H_k = H_k + \frac{1}{k+1}.$$

There is no $g \in \mathbb{Q}(k)$ with

$$\sigma(g) = g + \frac{1}{k+1}.$$

Hence, we can construct the Σ^*-extension $(\mathbb{Q}(k)(h), \sigma)$ with

$$\sigma(h) = h + \frac{1}{k+1}.$$

Theorem (Karr'81). Given $(\mathbb{F}(t), \sigma)$ where σ is extended from $\mathbb{F}$ to $\mathbb{F}(t)$ with

$$\sigma(t) = t + f, \quad f \in \mathbb{F}.$$

This is a Σ^*-extension iff

$$\nexists g \in \mathbb{F}: \quad \sigma(g) = g + f.$$

Σ^*-extensions

Example: Construct a difference field for

$$\sum_{k=0}^{n-1} H_k \qquad \mathcal{S}\sum_{k=1}^{n-1} H_k = \sum_{k=1}^{n-1} H_k + H_n.$$

Sigma computes $g = kh - k \in \mathbb{Q}(k)(h)$ with

$$\sigma(g) = g + h.$$

Hence, there is no Σ^*-extension $(\mathbb{Q}(k)(h)(t), \sigma)$ with

$$\sigma(t) = t + h.$$

Theorem (Karr'81). Given $(\mathbb{F}(t), \sigma)$ where σ is extended from $\mathbb{F}$ to $\mathbb{F}(t)$ with

$$\sigma(t) = t + f, \quad f \in \mathbb{F}.$$

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Σ^*-extensions

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Sigma computes $g = kh - k \in \mathbb{Q}(k)(h)$ with

$$\sigma(g) = g + h.$$

Hence, there is no Σ^*-extension $(\mathbb{Q}(k)(h)(t), \sigma)$ with

$$\sigma(t) = t + h.$$

There is no need! Take

$$t := kh - k.$$

Theorem (Karr'81). Given $(\mathbb{F}(t), \sigma)$ where σ is extended from $\mathbb{F}$ to $\mathbb{F}(t)$ with

$$\sigma(t) = t + f, \quad f \in \mathbb{F}.$$

This is a Σ^*-extension iff

$$\nexists g \in \mathbb{F}: \quad \sigma(g) = g + f.$$

Π-extensions

GIVEN a difference field $(\mathbb{F}, \sigma)$ with the constant field

$$\mathbb{K} := \{k \in \mathbb{F} | \sigma(k) = k\}.$$

A difference field $(\mathbb{F}(t), \sigma)$ is a Π-extension of $(\mathbb{F}, \sigma)$ if

1. t is a transcendental extension,
2. the automorphism σ is extended from $\mathbb{F}$ to $\mathbb{F}(t)$ with

$$\sigma(t) = at, \quad a \in \mathbb{F},$$

3. the constants remain unchanged:

$$\{k \in \mathbb{F}(t) | \sigma(k) = k\} = \mathbb{K}.$$

Π-extensions

GIVEN a difference field $(\mathbb{F},\sigma)$ with the constant field

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3. the constants remain unchanged:
$$\{k \in \mathbb{F}(t) | \sigma(k) = k\} = \mathbb{K}.$$

Theorem (Karr'81). Given $(\mathbb{F}(t),\sigma)$ where σ is extended from $\mathbb{F}$ to $\mathbb{F}(t)$ with

$$\sigma(t) = at, \quad a \in \mathbb{F}.$$

This is a Π-extension iff $t \neq 0$ and

$$\nexists r > 0, g \in \mathbb{F}^* : \quad \sigma(g) = a^r g.$$

Π-extensions

Example: Construct a difference field for

$$\binom{n}{k} H_k \qquad \mathcal{S}\binom{n}{k} = \frac{n-k}{k+1}\binom{n}{k}$$

There is no $g \in \mathbb{Q}(k)(h)^*$ and no $r > 0$ with

$$\sigma(g) = \left(\frac{n-k}{k+1}\right)^r g.$$

Hence, we can construct the Π-extension $(\mathbb{Q}(k)(h)(b), \sigma)$ with

$$\sigma(b) = \frac{n-k}{k+1} b.$$

Theorem (Karr'81). Given $(\mathbb{F}(t), \sigma)$ where σ is extended from $\mathbb{F}$ to $\mathbb{F}(t)$ with

$$\sigma(t) = at, \quad a \in \mathbb{F}.$$

This is a Π-extension iff $t \neq 0$ and

$$\nexists r > 0, g \in \mathbb{F}^* : \quad \sigma(g) = a^r g.$$

Π-extensions

Problem:

$$(-1)^k \qquad \mathcal{S}(-1)^k = -(-1)^k$$

For $g = 1$ and $r = 2$ we have

$$\sigma(g) = (-1)^r g.$$

Hence, there is no Π-extension $(\mathbb{Q}(P), \sigma)$ with

$$\sigma(P) = -P.$$

Theorem (Karr'81). Given $(\mathbb{F}(t), \sigma)$ where σ is extended from $\mathbb{F}$ to $\mathbb{F}(t)$ with

$$\sigma(t) = at, \quad a \in \mathbb{F}.$$

This is a Π-extension iff $t \neq 0$ and

$$\nexists r > 0, g \in \mathbb{F}^* : \quad \sigma(g) = a^r g.$$

Π-extensions

Problem:

$$(-1)^k \qquad \mathcal{S}(-1)^k = -(-1)^k$$

There are zero-divisors $(-1)^k$:

$$(1-(-1)^k)(1+(-1)^k) = (1-(-1)^{2k}) = 0.$$

Theorem (Karr'81). Given $(\mathbb{F}(t), \sigma)$ where σ is extended from $\mathbb{F}$ to $\mathbb{F}(t)$ with

$$\sigma(t) = at, \quad a \in \mathbb{F}.$$

This is a Π-extension iff $t \neq 0$ and

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$\Pi\Sigma^*$-fields

A $\Pi\Sigma^*$-field $(\mathbb{K}(t_1)\dots(t_e),\sigma)$ is a tower of Π- or Σ^*-extensions.

$\Pi\Sigma^*$-fields

A $\Pi\Sigma^*$-field $(\mathbb{K}(t_1)\dots(t_e),\sigma)$ is a tower of Π- or Σ^*-extensions.

There are algorithms for

- telescoping
- creative telescoping
- a decision procedure for

$$\nexists r>0, g\in\mathbb{F}^*: \quad \sigma(g)=a^r g.$$

$\Rightarrow$ Completely constructive summation theory!

$\Pi\Sigma^*$-fields

A $\Pi\Sigma^*$-field $(\mathbb{K}(t_1)\dots(t_e),\sigma)$ is a tower of Π- or Σ^*-extensions.

ASSUMPTION on $\mathbb{K}$:

- for any $k\in\mathbb{K}$ it can be decided if $k\in\mathbb{Z}$;
- multivariate polynomials over $\mathbb{K}$ can be factored;
- for any vector $(c_1,\dots,c_k)\in(K^*)^k$, a basis of the module

$$\{(n_1,\dots,n_k)\in\mathbb{Z}^k|c_1^{n_1}\cdots c_k^{n_k}=1\}\subseteq\mathbb{Z}^k$$

  can be computed.

$\Pi\Sigma^*$-fields

A $\Pi\Sigma^*$-field $(\mathbb{K}(t_1)\dots(t_e),\sigma)$ is a tower of Π- or Σ^*-extensions.

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  can be computed.

E.g., $\mathbb{K}$ can be any rational function field over an algebraic number field.

CS. Product Representations in $\Pi\Sigma$-Fields. 2005.

Kauers. Algorithms for Nonlinear Higher Order Difference Equations. PhD Thesis. 2005

Generalizations, extensions and **examples**

Generalization I: Refined telescoping

- GIVEN

$$\sum_{k=0}^{n} f(k).$$

- FIND $g(k)$ and $f'(k)$:

$$\boxed{f(k) = g(k+1) - g(k) + f'(k)}$$

where $f'(k)$ is simpler than $f(k)$.

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- FIND $g(k)$ and $f'(k)$:

$$\boxed{f(k) = g(k+1) - g(k) + f'(k)}$$

where $f'(k)$ is simpler than $f(k)$.

- THEN (with some mild extra conditions)

$$\sum_{k=0}^{n} f(k) = g(n+1) - g(0) + \sum_{k=0}^{n} f'(k).$$

Degree optimal w.r.t the top extension

- GIVEN $f \in \mathbb{F}(t)$.
- FIND $(f', g) \in \mathbb{F}(t)^2$:

$$\sigma(g) - g + f' = f$$

where in f' the degree of the denominator (and numerator) is minimal.

$$\sum_{k=2}^{n} \frac{2 - kH_k + H_k^4 - kH_k^5}{H_k - kH_k^2} =$$

Sigma

$$\sum_{k=1}^{n} H_k^4 =$$

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$$\sum_{k=2}^{n} \frac{2 - kH_k + H_k^4 - kH_k^5}{H_k - kH_k^2} = \sum_{k=2}^{n} \frac{k^2 + H_k}{k^2 H_k} + (n+1)H_n^3 - (2n+1)\left(\tfrac{3}{2}H_n^2 - 3H_n + \tfrac{3}{2}\right) + \frac{1}{H_n}$$

$$\sum_{k=1}^{n} H_k^4 = -H_n^{(3)} - 2H_n^{(2)} + 2\sum_{k=1}^{n} \frac{H_k}{k^2} + (n+1)H_n^4 - 2(2n+1)H_n^3 + 6(2n+1)H_n^2 - 12(2n+1)H_n + 24n.$$

Further examples

$$\sum_{k=1}^{n}\frac{k+1}{k(k+2)}=-\frac{n(3n+5)}{4(n+1)(n+2)}+\sum_{k=1}^{n}\frac{1}{k}$$

$$\sum_{k=2}^{n}\frac{1}{k(k-1)2^k}=\frac{-1}{n2^{n+1}}+\frac{1}{4}-\frac{1}{2}\sum_{k=2}^{n}\frac{1}{k2^k}$$

$$\sum_{k=1}^{n}\frac{k!\left(k^2+k+k!\left(k(k+1)^2+k!\left(k(k+1)^2+\left(2k^2-1\right)k!-3\right)-2\right)+1\right)+1}{(k!)^3(k!+1)((k+1)k!+1)}$$
$$=\frac{3(n+1)(n!)^3+(3-2n)(n!)^2-2(n+2)n!-2}{2(n!)^2((n+1)n!+1)}+\sum_{k=1}^{n}\frac{k(k!)^3+k!+1}{(k!)^3(k!+1)}$$

$$\sum_{k=2}^{n}\frac{(k+1)\left(k(k+1)^2(k+2)H_k^3+k\left(3k^2+8k+5\right)H_k^2-(k+2)H_k-k-2\right)}{H_k\left(k(k+1)^2(k+2)H_k^3+2\left(k^3+2k^2-1\right)H_k^2-\left(k^2+5k+5\right)H_k-2k-3\right)}$$
$$=\frac{-6(n+1)(n+2)H_n^2-6(2n+3)H_n+11(n+1)(n+2)}{11H_n\left(2n+(n+1)(n+2)H_n+3\right)}+\sum_{k=2}^{n}\frac{k(k+1)}{kH_k-1}$$

The analogue problem for Π-extensions

$$\prod_{k=1}^{n}\frac{(-k-1)(k+7)}{(k+4)^2}=\frac{4}{35}\,\frac{(n+5)(n+6)(n+7)}{(n+2)(n+3)(n+4)}\,(-1)^n,$$

$$\prod_{k=1}^{n}\frac{(k+3)(H_k(k+1)+1)^2(H_k(k+2)(k+1)+2k+3)}{(k+1)^2H_k(H_k(k+3)(k+2)(k+1)+3(k+4)k+11)}$$
$$=\frac{11}{6}\,\frac{(n+3)(n+2)(H_n(n+1)+1)^2}{(n+1)(H_n(n+3)(n+2)(n+1)+3(n+4)n+11)}\prod_{k=1}^{n}H_k,$$

$$\prod_{k=1}^{n}\frac{k!(H_k(k+2)(k+1)+2k+3)(H_k(k+1)+1)}{H_k(k+3)(k+2)(k+1)+3(k+4)k+11}$$
$$=\frac{11(H_n(n+1)+1)}{H_n(n+3)(n+2)(n+1)+3(n+4)n+11}\prod_{k=1}^{n}k!H_k,$$

$$\prod_{k=1}^{n}\frac{(q^{k+2}+(k+1)!)(q^{k+1}+k!)(k+2)(k+1)}{(q^{k+3}+(k+2)!)(k+3)}$$
$$=\frac{3(q^3+2)}{q+1}\,\frac{(q^{n+1}(n+1)+(n+1)!)}{(q^{n+3}+(n+2)!)(n+3)}\prod_{k=1}^{n}(kq^k+k!)$$

CS. Product Representations in $\Pi\Sigma$-Fields. 2005.

Simpler w.r.t. the depth

- GIVEN $f \in \mathbb{F}$.
- FIND an extension $\mathbb{E} \supseteq \mathbb{F}$ and $(f', g) \in \mathbb{E}^2$:

$$\sigma(g) - g + f' = f$$

where f' has minimal depth.

$$\sum_{k=1}^{n} H_k^2 H_k^{(2)} =$$

$$\sum_{k=0}^{a} \left(\sum_{i=0}^{k} \binom{n}{i} \right)^2 =$$

Sigma

$$\sum_{k=1}^{n} \frac{1}{k^3} \sum_{j=1}^{k} \frac{H_j}{j^2} =$$

Simpler w.r.t. the depth

- GIVEN $f \in \mathbb{F}$.
- FIND an extension $\mathbb{E} \supseteq \mathbb{F}$ and $(f', g) \in \mathbb{E}^2$:

$$\sigma(g) - g + f' = f$$

where f' has minimal depth.

$$\sum_{k=1}^{n} H_k^2 H_k^{(2)} = \frac{1}{3} H_n^{(3)} - \frac{1}{3} H_n^3 + \Big((n+1) H_n^{(2)} + 1\Big) H_n^2 + (2n+1)(1 - H_n) H_n^{(2)} - 2H_n$$

$$\sum_{k=0}^{a} \left(\sum_{i=0}^{k} \binom{n}{i} \right)^2 = (n-a) \binom{n}{a} \sum_{i=0}^{a} \binom{n}{i} - \frac{n-2a-2}{2} \left(\sum_{i=0}^{a} \binom{n}{i} \right)^2 - \frac{n}{2} \sum_{i=1}^{a} \binom{n}{i}^2$$

$$\sum_{k=1}^{n} \frac{1}{k^3} \sum_{j=1}^{k} \frac{H_j}{j^2} = H_n^{(3)} \sum_{j=1}^{n} \frac{H_j}{j^2} - \sum_{j=1}^{n} \frac{H_j (j^3 H_j^{(3)} - 1)}{j^5}$$

CS. Symbolic summation with single-nested sum extensions. 2004.

CS. Finding telescopers with minimal depth for indefinite nested sum and product expressions. 05.

Further examples

$$\sum_{k=1}^{n} H_k^3 = \frac{1}{2}\left(2(n+1)H_n^3 - 3(2n+1)H_n^2 + 6(2n+1)H_n - 12n - 1 + H_n^{(2)}\right)$$

$$\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j}\sum_{i=1}^{j}\frac{1}{i} = \frac{1}{6}\left[H_n^3 + 3H_n\,H_n^{(2)} + 2\,H_n^{(3)}\right]$$

$$\sum_{k=1}^{n}\Big(\sum_{j=1}^{k}\binom{n}{j}\Big)\Big(\sum_{j=1}^{k}\binom{n}{j}^2\Big) = \frac{n+2}{2}\sum_{j=1}^{n}\binom{n}{j}\sum_{j=1}^{n}\binom{n}{j}^2 - \frac{1}{n}\sum_{j=1}^{n}(n^2 - nj + j^2)\binom{n}{j}^3$$

$$\sum_{k=1}^{n}\frac{1}{k}\sum_{i=1}^{k}\frac{1}{H_i} = -n + H_n\sum_{i=1}^{n}\frac{1}{H_i} + \sum_{i=1}^{n}\frac{1}{iH_i},$$

$$\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\frac{H_j^{(2)}}{j^3}\right)^2 = -({H_n^{(2)}}^2 + H_n^{(4)})\sum_{j=1}^{n}\frac{H_j^{(2)}}{j^3} + (n+1)\times$$

$$\times\left(\sum_{j=1}^{n}\frac{H_j^{(2)}}{j^3}\right)^2 + \sum_{j=1}^{n}\frac{H_j^{(2)}\left((jH_j^{(2)})^2 - H_j^{(2)} + j^2 H_j^{(4)}\right)}{j^5}.$$

Generalization II: Difference equations of higher order

Linear difference equations

- ▶GIVEN $f, a_0, \dots, a_m \in \mathbb{F}$.
- ▶FIND ALL $g \in \mathbb{F}$:

$$\boxed{a_m\, \sigma^m(g) + \dots + a_0\, g = f.}$$

$\downarrow$ $\uparrow$

Generalization II: Difference equations of higher order

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- ▶GIVEN $f, a_0, \dots, a_m \in \mathbb{F}$.
- ▶FIND ALL $g \in \mathbb{F}$:

$$\boxed{a_m\,\sigma^{m}(g) + \dots + a_0\,g = f.}$$

$$\downarrow \qquad\qquad \uparrow$$

Parameterized linear difference equations

- ▶GIVEN $a_0, \dots, a_m \in \mathbb{F}$,$f_0, \dots, f_d \in \mathbb{F}$.
- ▶FIND ALL $g \in \mathbb{F}$, $c_0, \dots, c_d \in \mathbb{K}$:

$$\boxed{a_m\,\sigma^{m}(g) + \dots + a_0\,g = c_0\,f_0 + \dots + c_d\,f_d.}$$

Algorithms for parameterized linear difference equations

Rational case:

Abramov. Rational solutions of linear differential and difference equations with polynomial coefficients. 1989.

Hoeij. Rational solutions of linear difference equations. 1998.

q-Rational case:

Abramov. Rational solutions of linear difference and q-difference equations with polynomial coefficients. 1995.

$\Pi\Sigma^*$-field case: (open subproblems)

M. Karr. Summation in finite terms. 1981.

Bronstein. On solutions of linear ordinary difference equations in their coefficient field. 2000.

CS. Solving parameterized linear difference equations in terms of indefinite nested sums and products. 2005.

CS. A collection of denominator bounds to solve parameterized linear difference equations in $\Pi\Sigma$-extensions. 2004.

CS. Degree bounds to find polynomial solutions of parameterized linear difference equations in $\Pi\Sigma$-fields.

Example: Sigma

Generalization III: Solving recurrences with extensions

Rational case:

Petkovšek. Hypergeometric solutions of linear recurrences with polynomial coefficients. 1992.

Hoeij. Finite singularities and hypergeometric solutions of linear recurrence equations. 1999.

Abramov, Petkovšek. D'Alembertian solutions of linear differential and difference equations. 1994.

Hendriks, Singer. Solving difference equations in finite terms. 1999.

$\Pi\Sigma^*$**-field case: (open subproblems)**

CS. Symbolic summation in difference fields. PhD-thesis, 2001.

Communication with (M. Bronstein), Abramov, Petkovšek.

Example: Sigma

Generalization IV: Unspecified sequences

GIVEN

$$\mathbb{K}(\dots, x_{-1}, x_0, x_1, \dots), \quad \text{where} \quad \sigma(x_i) = x_{i+1}$$

plus a tower of $\Pi\Sigma^*$-extensions on top.

There are algorithms for

- telescoping
- creative telescoping
- solving recurrences (open subproblems)

Kauers, CS. Application of unspecified sequences in symbolic summation. 2006.
Kauers, CS. Indefinite summation with unspecified sequences.

X-Examples:

$$\sum_{k=1}^{a}(-1)^k\Big(\sum_{j=1}^{k}X_j-\frac{X_k}{2}\Big)^2=\frac{1}{2}(-1)^a\Big(\sum_{k=1}^{a}X_k\Big)^2-\frac{1}{4}\sum_{k=1}^{a}(-1)^kX_k^2,$$

$$\sum_{k=1}^{a}\Big(\sum_{j=1}^{k}X_j+X_k(k-1)\Big)^2=n\Big(\sum_{k=1}^{a}X_k\Big)^2-\sum_{k=1}^{a}kX_k^2+\sum_{k=1}^{a}k^2X_k^2\Big),$$

$$\sum_{k=0}^{n}\Big(\sum_{i=0}^{k}X_{n-i}\Big)^2=2\sum_{k=0}^{n}X_k\sum_{j=0}^{k}jX_{j-1}+\sum_{k=0}^{n}X_k^2+\sum_{k=0}^{n}kX_k^2,$$

$$\sum_{k=1}^{a}(-1)^k\binom{n}{k}\sum_{j=1}^{k}X_j=\frac{1}{n}\Big[(n-a)\binom{n}{a}(-1)^a\sum_{k=1}^{a}X_k+\sum_{k=1}^{a}(-1)^kk\binom{n}{k}X_k\Big].$$