

Symbolic Summation Assists Combinatorics

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Part 1: Symbolic Summation Methods and Applications

Part 2: Summation in Difference Fields

Part 3: Multi-Summation and Applications

Publications

Karr. Summation in finite terms. 1981.

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Zeilberger. A holonomic systems approach to special functions identities. 1990.

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1. Sister Celine/WZ-Method

Sister Celine Fasenmyer. Some generalized hypergeometric polynomials. PhD-thesis. 1945.

Verbaeten. The automatic construction of pure recurrence equations. 1974.

Wilf, Zeilberger. An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities. 1992.

Hornegger. Hypergeometrische Summation und polynomiale Rekursion. Diploma-thesis. 1992.

Wegschaider. Computer generated proofs of binomial multi-sum identities. Diploma thesis. 1997.

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2. Holonomic setting

Chyzak, Salvy. Non-commutative elimination in ore algebras proves multivariate identities. 1998.

Chyzak. An extension of Zeilberger's fast algorithm to general holonomic functions. 2000.

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Chyzak, Salvy. Non-commutative elimination in ore algebras proves multivariate identities. 1998.

Chyzak. An extension of Zeilberger's fast algorithm to general holonomic functions. 2000.

3. Karr + Specialized, but simplified Chyzak =

CS. A new Sigma approach to multi-summation. 2005.

Example 1: Indefinite summation

Telescoping

GIVEN

$$S(n) := \sum_{k=1}^n \frac{2k+1}{k+1} \overbrace{\sum_{j=0}^k \frac{(-k)_j (k+1)_j (2)_{k-j}}{j! k!}}{=: P(k) (= P^{(1,-1)}(x))} \left(\frac{1-x}{2}\right)^j.$$

FIND a closed form for $S(n)$.

Arose in joint cooperation with the JKU-Finite Element group.

Telescoping

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$$S(n) := \sum_{k=1}^n \frac{2k+1}{k+1} \overbrace{\sum_{j=0}^k \frac{(-k)_j (k+1)_j (2)_{k-j}}{j! k!}}{=: P(k) (= P^{(1,-1)}(x))} \left(\frac{1-x}{2}\right)^j.$$

Intermediate step: **FIND** a recurrence for $P(k)$.

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GIVEN

$$P(k+2) = \frac{(2k+3)x}{k+2} P(k+1) - \frac{k}{k+1} P(k).$$

FIND a closed form for $S(n)$.

Telescoping

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$$S(n) := \sum_{k=1}^n \frac{2k+1}{k+1} \overbrace{\sum_{j=0}^k \frac{(-k)_j (k+1)_j (2)_{k-j}}{j! k!} \left(\frac{1-x}{2}\right)^j}^{=: P(k) (= P^{(1,-1)}(x))}.$$

GIVEN

$$P(k+2) = \frac{(2k+3)x}{k+2} P(k+1) - \frac{k}{k+1} P(k).$$

FIND $g(k) := g_0(k)P(k) + g_1(k)P(k+1)$:

$$g(k+1) - g(k) = \frac{2k+1}{k+1} P(k)$$

for all $1 \leq k \leq n$ and all $n \geq 1$.

Telescoping

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$$S(n) := \sum_{k=1}^n \frac{2k+1}{k+1} \overbrace{\sum_{j=0}^k \frac{(-k)_j (k+1)_j (2)_{k-j}}{j! k!}}{=: P(k) (= P^{(1,-1)}(x))} \left(\frac{1-x}{2}\right)^j.$$

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$$P(k+2) = \frac{(2k+3)x}{k+2} P(k+1) - \frac{k}{k+1} P(k).$$

$$\boxed{\text{Sigma computes}} \quad g(k) = \frac{1+k-x-2kx}{(x-1)(k+1)} P(k) + \frac{1}{x-1} P(k+1):$$

$$\boxed{g(k+1) - g(k) = \frac{2k+1}{k+1} P(k)}$$

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for all $1 \leq k \leq n$ and all $n \geq 1$.

VERIFICATION:

$$g(k) \xrightarrow{\text{depends on}} P(k), P(k+1)$$

$$g(k+1) \xrightarrow{\text{depends on}} P(k+1), P(k+2).$$

Telescoping

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$$\boxed{g(k+1) - g(k) = \frac{2k+1}{k+1} P(k)}$$

for all $1 \leq k \leq n$ and all $n \geq 1$.

Summing this equation over k from 1 to n gives:

$$\sum_{k=1}^n \frac{2k+1}{k+1} P(k) = -\frac{x+1}{x-1} - \frac{nP(n)}{(n+1)(x-1)} + \frac{P(n+1)}{x-1}.$$

GIVEN

$$P(k+2) = \frac{(2k+3)x}{k+2}P(k+1) - \frac{k}{k+1}P(k).$$

ANSATZ: $g(k) = g_0(k)P(k) + g_1(k)P(k+1)$ such that

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 \Updownarrow

$$\left[g_0(k+1)P(k+1) + g_1(k+1)P(k+2) \right] - \left[g_0(k)P(k) + g_1(k)P(k+1) \right] = \frac{2k+1}{k+1}P(k)$$

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$$\left[g_0(k+1)P(k+1) + g_1(k+1)P(k+2) \right] - \left[g_0(k)P(k) + g_1(k)P(k+1) \right] = \frac{2k+1}{k+1}P(k)$$

$$\Leftrightarrow$$

$$P(k) \left[-\frac{k}{k+1}g_1(k+1) - g_0(k) - \frac{2k+1}{k+1} \right] \\ + P(k+1) \left[g_0(k+1) + \frac{(2k+3)x}{k+2}g_1(k+1) - g_1(k) \right] = 0$$

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 \uparrow

$$g_0(k) = -\frac{k}{k+1}g_1(k+1) - \frac{2k+1}{k+1}$$

$$g_0(k+1) + \frac{(2k+3)x}{k+2}g_1(k+1) - g_1(k) = 0$$

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↑

$$g_0(k) = -\frac{k}{k+1}g_1(k+1) - \frac{2k+1}{k+1}$$

$$\downarrow \mathcal{S}_k g_0(k)$$

$$g_0(k+1) + \frac{(2k+3)x}{k+2}g_1(k+1) - g_1(k) = 0$$

GIVEN

$$P(k+2) = \frac{(2k+3)x}{k+2}P(k+1) - \frac{k}{k+1}P(k).$$

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↑

$$g_0(k) = -\frac{k}{k+1}g_1(k+1) - \frac{2k+1}{k+1}$$

$$- \frac{k+1}{k+2}g_1(k+2) + \frac{(2k+3)x}{k+2}g_1(k+1) - g_1(k) = \frac{2k+3}{k+2}$$

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Sigma computes:

$$g_1(k) = \frac{1}{x-1},$$

GIVEN

$$P(k+2) = \frac{(2k+3)x}{k+2}P(k+1) - \frac{k}{k+1}P(k).$$

ANSATZ: $g(k) = g_0(k)P(k) + g_1(k)P(k+1)$ such that

$$g(k+1) - g(k) = \frac{2k+1}{k+1}P(k).$$

↑

$$g_0(k) = -\frac{k}{k+1}g_1(k+1) - \frac{2k+1}{k+1}$$

$$-\frac{k+1}{k+2}g_1(k+2) + \frac{(2k+3)x}{k+2}g_1(k+1) - g_1(k) = \frac{2k+3}{k+2}$$

Sigma computes:

$$g_1(k) = \frac{1}{x-1}, \quad g_0(k) = \frac{1+k-x-2kx}{(x-1)(k+1)}.$$

General case:

GIVEN $f(k) := h(k)P(k)$

and

$$P(k + s + 1) = a_0(k)P(k) + \cdots + a_s(k)P(k + s).$$

FIND $g(k) = g_0(k)P(k) + \cdots + g_s(k)P(k + s) :$

$$g(k + 1) - g(k) = f(k).$$

1. FIND a solution $g_s(k)$ for

$$\sum_{j=0}^s a_{s-j}(k + j)g_s(k + j + 1) - g_s(k) = h(k + 1).$$

2. COMPUTE the remaining g_0, \dots, g_{s-1} :

$$g_0(k) = a_0(k)g_s(k + 1) - h(k),$$

$$g_r(k) = a_r(k)g_s(k + 1) + g_{r-1}(k + 1), \quad 0 < r < s.$$

General case:

$$\text{GIVEN } f(k) := h_0(k)P(k) + \cdots + h_s(k)P(k+s)$$

and

$$P(k+s+1) = a_0(k)P(k) + \cdots + a_s(k)P(k+s).$$

$$\text{FIND } g(k) = g_0(k)P(k) + \cdots + g_s(k)P(k+s) :$$

$$g(k+1) - g(k) = f(k).$$

1. FIND a solution $g_s(k)$ for

$$\sum_{j=0}^s a_{s-j}(k+j)g_s(k+j+1) - g_s(k) = \sum_{j=0}^s h_{s-j}(k+j).$$

2. COMPUTE the remaining g_0, \dots, g_{s-1} :

$$g_0(k) = a_0(k)g_s(k+1) - h_0(k),$$

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General case:

$$\text{GIVEN } f_0(k) := h_0^{(0)}(k)P(k) + \cdots + h_s^{(0)}(k)P(k+s), \dots,$$

$$f_d(k) := h_0^{(d)}(k)P(k) + \cdots + h_s^{(d)}(k)P(k+s)$$

and

$$P(k+s+1) = a_0(k)P(k) + \cdots + a_s(k)P(k+s).$$

$$\text{FIND } g(k) = g_0(k)P(k) + \cdots + g_s(k)P(k+s) \text{ and } c_0, \dots, c_d:$$

$$g(k+1) - g(k) = c_0 f_0(k) + \cdots + c_d f_d(k).$$

1. FIND a solution $g_s(k)$ and c_0, \dots, c_d for

$$\sum_{j=0}^s a_{s-j}(k+j)g_s(k+j+1) - g_s(k) = \sum_{i=0}^d c_i \sum_{j=0}^s h_{s-j}^{(i)}(k+j).$$

2. COMPUTE the remaining g_0, \dots, g_{s-1} :

$$g_0(k) = a_0(k)g_s(k+1) - \sum_{i=0}^d c_i h^{(i)_0}(k),$$

$$g_r(k) = a_r(k)g_s(k+1) + g_{r-1}(k+1) - \sum_{i=0}^d c_i h^{(r)_0}(k), \quad 0 < r < s.$$

Example 2: Definite summation

Definite summation

Show that

$$\sum_{k=0}^n \sum_{s=0}^n \underbrace{(-1)^{n+k+s} \binom{n}{k} \binom{n}{s} \binom{n+k}{k} \binom{n+s}{s} \binom{2n-s-k}{n}}_{= P(n, k)} = \sum_{k=0}^n \binom{n}{k}^4.$$

($A = B$, M. Petkovšek, H.S. Wilf, and D. Zeilberger)

Definite summation

Show that

$$\sum_{k=0}^n \sum_{s=0}^n \underbrace{(-1)^{n+k+s} \binom{n}{k} \binom{n}{s} \binom{n+k}{k} \binom{n+s}{s} \binom{2n-s-k}{n}}_{= P(n,k)} = \sum_{k=0}^n \binom{n}{k}^4.$$

Proof. We **COMPUTE** for both sides the same recurrence

$$\begin{aligned} & -4(1+n)(3+4n)(5+4n)S(n) \\ & -2(3+2n)(7+9n+3n^2)S(1+n) + (2+n)^3 S(2+n) = 0. \end{aligned}$$

Both sides agree at $n = 0, 1$. □

Definite summation

Show that

$$\sum_{k=0}^n \sum_{s=0}^n \underbrace{(-1)^{n+k+s} \binom{n}{k} \binom{n}{s} \binom{n+k}{k} \binom{n+s}{s} \binom{2n-s-k}{n}}_{= P(n,k)} = \sum_{k=0}^n \binom{n}{k}^4.$$

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Both sides agree at $n = 0, 1$. □

Chyzak's approach (2500s) and Wegschaider's approach (500s) are much slower on that.

Creative telescoping

GIVEN

$$P(n, k + 2) = a_0(n, k)P(n, k) + a_1(n, k)P(n, k + 1)$$

$$P(n + 1, k) = b_0(n, k)P(n, k) + b_1(n, k)P(n, k + 1).$$

FIND $g(n, k) = g_0(n, k)P(n, k) + g_1(n, k)P(n, k + 1)$, $c_0(n)$:

$$g(n, k + 1) - g(n, k) = c_0(n)P(n, k)$$

Sigma:

Creative telescoping

GIVEN

$$P(n, k + 2) = a_0(n, k)P(n, k) + a_1(n, k)P(n, k + 1)$$

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Sigma:

No solution

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FIND $g(n, k) = g_0(n, k)P(n, k) + g_1(n, k)P(n, k + 1)$, $c_0(n)$, $c_1(n)$:

$$g(n, k + 1) - g(n, k) = c_0(n)P(n, k) + c_1(n)P(n + 1, k)$$

Sigma:

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FIND $g(n, k) = g_0(n, k)P(n, k) + g_1(n, k)P(n, k + 1)$, $c_0(n)$, $c_1(n)$:

$$g(n, k + 1) - g(n, k) = c_0(n)P(n, k) + c_1(n) \left[b_0(n, k)P(n, k) + b_1(n, k)P(n, k + 1) \right]$$

Sigma:

Creative telescoping

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Sigma:

No solution

Creative telescoping

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FIND $g(n, k) = g_0(n, k)P(n, k) + g_1(n, k)P(n, k + 1)$, $c_0(n)$, $c_1(n)$, $c_2(n)$:

$$\begin{aligned}g(n, k + 1) - g(n, k) &= c_0(n)P(n, k) + \\ &\quad c_1(n)P(n + 1, k) + \\ &\quad c_2(n)P(n + 2, k)\end{aligned}$$

Sigma:

Creative telescoping

GIVEN

$$P(n, k + 2) = a_0(n, k)P(n, k) + a_1(n, k)P(n, k + 1)$$

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FIND $g(n, k) = g_0(n, k)P(n, k) + g_1(n, k)P(n, k + 1)$, $c_0(n)$, $c_1(n)$, $c_2(n)$:

$$\begin{aligned}
 g(n, k + 1) - g(n, k) &= c_0(n)P(n, k) + \\
 &\quad c_1(n) \left[b_0(n, k)P(n, k) + b_1(n, k)P(n, k + 1) \right] + \\
 &\quad c_2(n) \left[\beta_0(n, k)P(n, k) + \beta_1(n, k)P(n, k + 1) \right]
 \end{aligned}$$

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$$P(n + 1, k) = b_0(n, k)P(n, k) + b_1(n, k)P(n, k + 1).$$

FIND $g(n, k) = g_0(n, k)P(n, k) + g_1(n, k)P(n, k + 1)$, $c_0(n)$, $c_1(n)$, $c_2(n)$:

$$g(n, k + 1) - g(n, k) = c_0(n)P(n, k) +$$

$$c_1(n) \left[b_0(n, k)P(n, k) + b_1(n, k)P(n, k + 1) \right] +$$

$$c_2(n) \left[\beta_0(n, k)P(n, k) + \beta_1(n, k)P(n, k + 1) \right]$$

Sigma:

Solution

Strategy:

GIVEN

$$S(n) = \sum_{k=0}^n f(n, k)P(n, k)$$

where $P(n, k)$ is a hypergeometric sum and $f(n, k)$ is a $\Pi\Sigma^*$ -term.

1. Try to compute recurrences

$$P(n, k + s + 1) = a_0(n, k)P(n, k) + \cdots + a_s(n, k)P(n, k + s)$$

$$P(n + 1, k) = b_0(n, k)P(n, k) + \cdots + b_s(n, k)P(n, k + s)$$

for some $s \geq 0$.

2. Try to compute a recurrence for $S(n)$ (as showed above).

Strategy:

GIVEN

$$S(n) = \sum_{k=0}^n f(n, k)P(n, k)$$

where $P(n, k)$ is a **multi**-sum and $f(n, k)$ is a $\Pi\Sigma^*$ -term.

1. Try to compute (**recursively**) recurrences

$$P(n, k + s + 1) = a_0(n, k)P(n, k) + \cdots + a_s(n, k)P(n, k + s)$$

$$P(n + 1, k) = b_0(n, k)P(n, k) + \cdots + b_s(n, k)P(n, k + s)$$

for some $s \geq 0$.

2. Try to compute a recurrence for $S(n)$ (as showed above).

Example 3: Stembridge's TSPP Theorem

(G. Andrews, P. Paule, Sigma)

Plane partitions

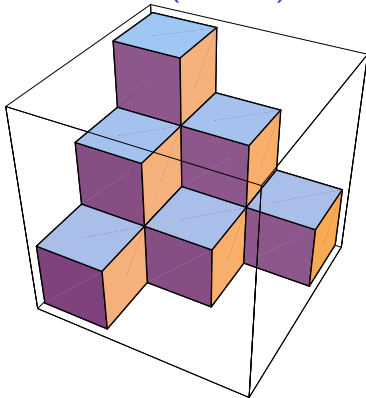
A plane partition with largest part $\leq n$ is

$$\begin{array}{ccccccc}
 n \geq & & & & & & \\
 a_{11} & \geq & a_{12} & \geq & a_{13} & \geq & \dots & a_{1r} \\
 \vee | & & \vee | & & & & & \vee | \\
 a_{21} & \geq & a_{22} & & & & \dots & a_{2r} \\
 \vee | & & & & & & & \vee | \\
 a_{31} & & & & & & & a_{3r} \\
 & & & & & & & \vdots \\
 & & & & & & & \vee | \\
 a_{s1} & \geq & a_{s2} & \geq & a_{s3} & \geq & \dots & \geq & a_{s,r} \\
 & & & & & & & & \geq 0.
 \end{array}$$

Totally Symmetric Plane Partition (TSPP)

Example:

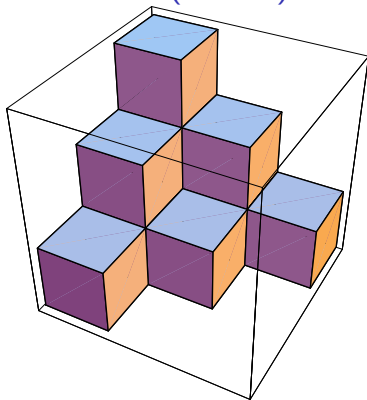
3	2	1	
2	1	0	↔
1	0	0	



Totally Symmetric Plane Partition (TSPP)

Example:

$$\begin{array}{ccc} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{array} \longleftrightarrow$$



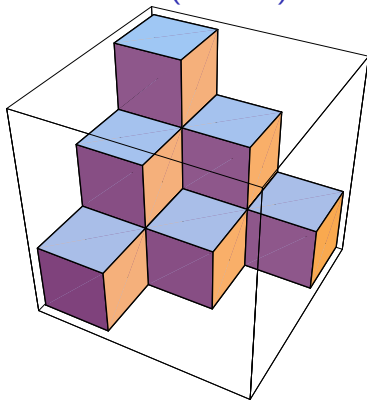
Conjecture (R. Stanley, 1986). Let T_n equal the number of totally symmetric plane partitions with largest part $\leq n$. Then for $n \geq 1$

$$T_n = \prod_{1 \leq i \leq j \leq k \leq n} \frac{i + j + k - 1}{i + j + k - 2}.$$

Totally Symmetric Plane Partition (TSPP)

Example:

$$\begin{array}{ccc} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{array} \longleftrightarrow$$



Theorem (Stembridge, 1995). Let T_n equal the number of totally symmetric plane partitions with largest part $\leq n$. Then for $n \geq 1$

$$T_n = \prod_{1 \leq i \leq j \leq k \leq n} \frac{i + j + k - 1}{i + j + k - 2}.$$

Okada's Theorem (1989). For $n \geq 3$,

$$T_{n-2}^2 = \begin{cases} \det(M(n))x^{-1} & \text{if } n \text{ is odd,} \\ \det(M(n)) & \text{if } n \text{ is even,} \end{cases}$$

with $M(n) = (\mu_1(i, j))_{0 \leq i, j \leq n-1}$

where

$$\mu(i, j) = \begin{cases} 0 & \text{if } j \leq i, \\ 2^{j-1} + (-1)^{j-1} & \text{if } i = 0, i < j, \\ (-1)^{j-i-1} + \sum_{s=i}^{j-1} \binom{i+j-2}{s} & \text{if } 0 < i < j, \end{cases}$$

$$\mu_1(i, j) = \begin{cases} x & \text{if } i = j = 0, \\ (-1)^{j-1} & \text{if } i = 0, j > 0, \\ (-1)^i & \text{if } j = 0, i > 0, \\ 0 & \text{if } i = j > 0, \\ \mu(i-1, j-1) & \text{if } j > i \geq 1, \\ -\mu(j-1, i-1) & \text{if } 1 \leq j < i. \end{cases}$$

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PROBLEM: Evaluate $\det(M)$.

Okada's Theorem (1989). For $n \geq 3$,

$$T_{n-2}^2 = \begin{cases} \det(M(n))x^{-1} & \text{if } n \text{ is odd,} \\ \det(M(n)) & \text{if } n \text{ is even,} \end{cases}$$

with $M(n) = (\mu_1(i, j))_{0 \leq i, j \leq n-1}$

G. Andrews guessed (1990)

$$W = \begin{pmatrix} * & \dots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \quad \text{with} \quad \det(W) = 1 \quad MW = \begin{pmatrix} * & & 0 \\ \vdots & \ddots & \\ * & \dots & * \end{pmatrix} =: U.$$

($M = W^{-1}U$ is "LU-decomposition")

Okada's Theorem (1989). For $n \geq 3$,

$$T_{n-2}^2 = \begin{cases} \det(M(n))x^{-1} & \text{if } n \text{ is odd,} \\ \det(M(n)) & \text{if } n \text{ is even,} \end{cases}$$

with $M(n) = (\mu_1(i, j))_{0 \leq i, j \leq n-1}$

G. Andrews guessed (1990)

$$W = \begin{pmatrix} * & \dots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \quad \text{with} \quad \det(W) = 1 \quad MW = \begin{pmatrix} * & & 0 \\ \vdots & \ddots & \\ * & \dots & * \end{pmatrix}.$$

THEN

$$\begin{aligned} \det(M) &= \det(M)\det(W) = \det(MW) \\ &= \text{product of diagonal elements of } MW \end{aligned}$$

Define

$$\left\{ \begin{matrix} x \\ n \end{matrix} \right\} = \frac{1}{2} \left(\binom{x}{n} + \binom{x-1}{n} \right);$$

$$t_1(n) = \begin{cases} 1 & \text{if } n = 0, \\ \frac{(n+1)(n+3)\dots(3n-1)}{\binom{n}{n}} & \text{if } n > 0; \end{cases}$$

$$t(n) = \begin{cases} 1 & \text{if } n = 0, \\ \frac{t_1(n)}{t_1(n-1)} & \text{if } n > 0. \end{cases}$$

$$r_3(s, j) = 4^{-s} \sum_{k=0}^s \frac{(j-k)(j)_k (-3j-1)_k}{jk!(-2j+\frac{1}{2})_k},$$

$$f_1(c, j) =$$

$$(-1)^c \sum_{s=0}^{\lfloor \frac{c}{2} \rfloor} \frac{(-1)^s \binom{j-1-s}{c-2s} (j)_s (-3j+1)_s (3j-3s-1)}{4^s s! (-2j+\frac{3}{2})_s (3j-1)},$$

$$f_2(c, j) = (-1)^c \sum_{s=0}^{\lfloor \frac{c}{2} \rfloor} (-1)^s \left\{ \begin{matrix} j-s \\ c-2s \end{matrix} \right\} r_3(s, j),$$

Conjecture (G. Andrews, 1990) For each $n \geq 1$,

$$M(n)W(n) = \begin{pmatrix} x & 0 & 0 & 0 & \dots & 0 \\ A_{10} & \frac{t_1(0)^2}{x} & 0 & 0 & \dots & 0 \\ A_{20} & A_{21} & t_1(1)^2 x & 0 & \dots & 0 \\ \vdots & & & & \ddots & \vdots \\ A_{n-1,0} & A_{n-1,1} & \dots & \dots & \dots & t_1(n-2)^2 x^{\pm 1} \end{pmatrix}.$$

$$r_2(j) = \begin{cases} \frac{t_1(j-1)}{2} & \text{if } j \text{ even,} \\ \frac{t_1(j-1)}{2} + \frac{f_2(j-2, \frac{j-1}{2})}{2} & \text{if } j \text{ odd,} \end{cases}$$

$$r_1(j) = \begin{cases} t_1(j-1) & \text{if } j \text{ even,} \\ 0 & \text{if } j \text{ odd,} \end{cases}$$

$$e_1(i, j) = \begin{cases} 0 & \text{if } i > j, \\ 1 & \text{if } i = j, \\ r_1(j) & \text{if } i = 0, i < j, \\ r_2(j) & \text{if } i = 1, i < j, \\ f_1(j-i, \frac{j}{2}) & \text{if } 2 \leq i < j, j \text{ even,} \\ f_2(j-i, \frac{j-1}{2}) & \text{if } 2 \leq i < j, j \text{ odd,} \end{cases}$$

$$e(i, j) = \begin{cases} 0 & \text{if } i > j, \\ 1 & \text{if } i = j, \\ e_1(i, j) - t(j-1) \times \\ \times x^{-1} e_1(i, j-1) & \text{if } i < j, \end{cases}$$

$$W(n) = (e(i, j))_{0 \leq i, j \leq n-1}$$

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$$W(n) = (e(i, j))_{0 \leq i, j \leq n-1}$$

Theorem (Andrews, Paule, Sigma, 2005) For each $n \geq 1$,

$$M(n)W(n) = \begin{pmatrix} x & 0 & 0 & 0 & \dots & 0 \\ A_{10} & \frac{t_1(0)^2}{x} & 0 & 0 & \dots & 0 \\ A_{20} & A_{21} & t_1(1)^2 x & 0 & \dots & 0 \\ \vdots & & & & \ddots & \vdots \\ A_{n-1,0} & A_{n-1,1} & \dots & \dots & \dots & t_1(n-2)^2 x^{\pm 1} \end{pmatrix}.$$

Reduction to summation problems (P. Paule):

E.g., define

$$h(k, m) := \sum_{s=0}^{\lfloor \frac{2m-k}{2} \rfloor - 1} \frac{k}{m-s} \binom{m-s}{2m-2s-k} \frac{(-1)^{s+k}}{2m 4^s} \sum_{r=0}^s \frac{(m-r)(m)_r (-3m-1)_r}{r! (\frac{1}{2} - 2m)_r}$$

and

FIND Rec

$$A_0(i, m) := \sum_{k=0}^{2m} \binom{i+k-3}{i-2} h(k, m),$$

$$A_2(i, m) := \sum_{k=i}^{2m} (-1)^k h(k, m).$$

Theorem. For all $m \geq 1$ and $3 \leq i \leq 2m + 1$,

$$2h(i-2, m) - 5h(i-1, m)$$

$$- A_0(i, m) + 6(-1)^i A_2(i, m) - 3(-1)^i \prod_{s=1}^{2m-1} \frac{2(m+s-1)}{2m+s-2} = 0.$$

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Theorem. For all $m \geq 1$ and $3 \leq i \leq 2m + 1$,

ADD

$$2h(i-2, m) - 5h(i-1, m)$$

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Zeilberger's Opinion 65:

Seeing all the details, (that nowadays can (and should!) be easily relegated to the computer), even if they are extremely hairy, is a hang-up that traditional mathematicians should learn to wean themselves from. A case in point is the **excellent** but unnecessarily long-winded recent article

Andrew, Paule, Schneider. Plane Partitions VI: Stembridge's TSPP Theorem. 2005.

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You don't need 30 pages, and frankly all this EXPLICIT LANGUAGE of hairy computer output is almost pornographic.

Example 4: Handbook of Mathematical Functions (Abramowitz and Stegun)

From: Frank W. J. Olver [University of Maryland]

To: Peter Paule

"...The writing of DLMF Chapter Bessel Functions by Leonard Maximon and myself is now largely complete;...

However, a problem has arisen in connection with about a dozen formulas from Chapter 10 of Abramowitz and Stegun for which we have not yet tracked down proofs, and the author of this chapter, Henry Antosiewicz, died about a year ago.

Since it is the editorial policy for the DLMF not to state formulas without indications of proofs, I am hoping that you will be willing to step into the breach and supply verifications by computer algebra methods..."

My favorite of the dozen problems

Verify with computer algebra the identity

$$J_0(z \sin \theta) = \sum_{n=0}^{\infty} (4n + 1) \frac{(2n)!}{2^{2n} n!^2} j_{2n}(z) P_{2n}(\cos \theta) \quad (10.1.48)$$

where

$P_n(z)$: Legendre polynomials,

$J_n(z)$: Bessel functions of the first kind,

$j_n(z)$: spherical Bessel functions of the first kind.

My favorite of the dozen problems

Verify with computer algebra the identity

$$J_0(z \sin \theta) = \sum_{n=0}^{\infty} (4n + 1) \frac{(2n)!}{2^{2n} n!^2} j_{2n}(z) P_{2n}(\cos \theta) \quad (10.1.48)$$

$$t := \cos \theta \quad \sin(\theta) = \sqrt{1 - t^2}$$

where

$P_n(z)$: Legendre polynomials,

$J_n(z)$: Bessel functions of the first kind,

$j_n(z)$: spherical Bessel functions of the first kind.

My favorite of the dozen problems

Verify with computer algebra the identity

$$J_0(z\sqrt{1-t^2}) = \sum_{n=0}^{\infty} (4n+1) \frac{(2n)!}{2^{2n} n!^2} j_{2n}(z) P_{2n}(t)$$

where

$P_n(z)$: Legendre polynomials,

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My favorite of the dozen problems

Verify with computer algebra the identity

$$J_0(z\sqrt{1-t^2}) = \sum_{n=0}^{\infty} (4n+1) \frac{(2n)!}{2^{2n} n!^2} j_{2n}(z) P_{2n}(t)$$

By hypergeometric series representation we get

$$J_0(z\sqrt{1-t^2}) = \sum_{k=0}^{\infty} \frac{(-\frac{1}{2}z^2)^k}{k!^2} \left(\frac{1-t^2}{2}\right)^k,$$

$$P_{2n}(t) = \sum_{k=0}^{\infty} \frac{(-2n)_k (2n+1)_k}{k!^2} \left(\frac{1-t}{2}\right)^k,$$

$$j_{2n}(z) = \sum_{i=0}^{\infty} \frac{\sqrt{\pi} (-z^2/4)^i (z/2)^{2n}}{2 i! (2n+i+1/2)!}.$$

Show that

$$\sum_{k=0}^{\infty} a_k \left(\frac{1-t^2}{2} \right)^k = \sum_{k=0}^{\infty} b_k \left(\frac{1-t}{2} \right)^k$$

where

$$a_k = \frac{\left(-\frac{1}{2}z^2\right)^k}{k!^2},$$

$$b_k = \sum_{n=0}^{\infty} (4n+1) \frac{(2n)!}{2^{2n}n!} \frac{(-2n)_k (2n+1)_k}{k!^2} \overbrace{\sum_{i=0}^{\infty} \frac{\sqrt{\pi}}{2} \frac{(-z^2/4)^i (z/2)^{2n}}{i!(2n+i+1/2)!}}{=: P(k)}.$$

Show that

$$\sum_{k=0}^{\infty} a_k \left(\frac{1-t^2}{2} \right)^k = \sum_{k=0}^{\infty} b_k \left(\frac{1-t}{2} \right)^k$$

where

$$a_k = \frac{\left(-\frac{1}{2}z^2\right)^k}{k!^2},$$

$$b_k = \sum_{n=0}^{\infty} (4n+1) \frac{(2n)!}{2^{2n}n!} \frac{(-2n)_k (2n+1)_k}{k!^2} \overbrace{\sum_{i=0}^{\infty} \frac{\sqrt{\pi}}{2} \frac{(-z^2/4)^i (z/2)^{2n}}{i!(2n+i+1/2)!}}{=: P(k)}.$$

FIND recurrences for a_k and b_k .

Show that

$$\sum_{k=0}^{\infty} a_k \left(\frac{1-t^2}{2} \right)^k = \sum_{k=0}^{\infty} b_k \left(\frac{1-t}{2} \right)^k$$

where

$$z^2 a_k + 2(k+1)^3 a_{k+1} = 0$$

and

$$\begin{aligned} &4(k+3)z^2 b_k - 2(2k+5)z^2 b_{k+1} \\ &- (k+2)(k^2 + 5k + 6 - z^2) b_{k+2} + (k+2)(k+3)^2 b_{k+3} = 0. \end{aligned}$$

Show that

$$\sum_{k=0}^{\infty} a_k \left(\frac{1-t^2}{2} \right)^k = \sum_{k=0}^{\infty} b_k \left(\frac{1-t}{2} \right)^k$$

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FIND ODEs for

$$A(t) = \sum_{k=0}^{\infty} a_k t^k \quad \text{and} \quad B(t) = \sum_{k=0}^{\infty} b_k t^k.$$

Show that

$$\sum_{k=0}^{\infty} a_k \left(\frac{1-t^2}{2} \right)^k = \sum_{k=0}^{\infty} b_k \left(\frac{1-t}{2} \right)^k$$

We know that

$$A(t) = \sum_{k=0}^{\infty} a_k t^k \quad \text{and} \quad B(t) = \sum_{k=0}^{\infty} b_k t^k :$$

are solutions of

$$z^2 A(t) + 2A'(t) + 2tA''(t) = 0$$

and

$$\begin{aligned} & 12z^2 B(t) + 10(2tz^2 - z^2)B'(t) \\ & + (4t^2 z^2 - 4tz^2 + z^2 - 6)B''(t) - 3(2t-1)B^{(3)}(t) - t(t-1)B^{(4)}(t) = 0. \end{aligned}$$

Show that

$$\overbrace{\sum_{k=0}^{\infty} a_k \left(\frac{1-t^2}{2}\right)^k}^{=A\left(\frac{1-t^2}{2}\right)} = \overbrace{\sum_{k=0}^{\infty} b_k \left(\frac{1-t}{2}\right)^k}^{=B\left(\frac{1-t}{2}\right)}$$

We know that

SUBSTITUTE

$$A(t) = \sum_{k=0}^{\infty} a_k t^k \quad \text{and} \quad B(t) = \sum_{k=0}^{\infty} b_k t^k :$$

are solutions of

$$z^2 A(t) + 2A'(t) + 2tA''(t) = 0$$

and

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Show that

$$\overbrace{\sum_{k=0}^{\infty} a_k \left(\frac{1-t^2}{2}\right)^k} =: A(t) = \overbrace{\sum_{k=0}^{\infty} b_k \left(\frac{1-t}{2}\right)^k} =: B(t)$$

We know that $A(t)$ and $B(t)$ are solutions of

$$-t^3 z^2 A(t) + (t^2 + 1)A'(t) + t(t+1)(t-1)A''(t) = 0$$

and

$$-3z^2 B(t) - 5tz^2 B'(t) - (t^2 z^2 - 6)B''(t) + 6tB^{(3)}(t) + (t+1)(t-1)B^{(4)}(t) = 0.$$

ADD

Show that

$$\overbrace{\sum_{k=0}^{\infty} a_k \left(\frac{1-t^2}{2}\right)^k} =: A(t) = \overbrace{\sum_{k=0}^{\infty} b_k \left(\frac{1-t}{2}\right)^k} =: B(t)$$

We know that

$$C(t) := A(t) - B(t)$$

is a solution of

$$-3z^2 C(t) - 5tz^2 C'(t) - (t^2 z^2 - 6) C''(t) + 6t C^{(3)}(t) + (t+1)(t-1) C^{(4)}(t) = 0.$$

Show that

$$\overbrace{\sum_{k=0}^{\infty} a_k \left(\frac{1-t^2}{2}\right)^k} =: A(t) = \overbrace{\sum_{k=0}^{\infty} b_k \left(\frac{1-t}{2}\right)^k} =: B(t)$$

We know that

$$C(t) := A(t) - B(t)$$

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By inspection we get the initial conditions

$$a_0 = b_0$$

$$b_1 = 2a_1,$$

$$b_2 = -2a_1 + 4a_2.$$

Proving them shows that

$$C(t) = 0.$$

Show that

$$\underbrace{\sum_{n=0}^{\infty} -(4n+1)n \frac{(2n+1)!}{n!^2 2^{2n-1}} \overbrace{\sum_{i=0}^{\infty} \frac{\sqrt{\pi} (-z^2/4)^i (z/2)^{2n}}{2 i! (2n+i+1/2)!}}^{P(n)}}_{=: S(z)} = -z^2.$$

► **FIND** an ODE.

Show that

$$\underbrace{\sum_{n=0}^{\infty} -(4n+1)n \frac{(2n+1)!}{n!^2 2^{2n-1}} \overbrace{\sum_{i=0}^{\infty} \frac{\sqrt{\pi} (-z^2/4)^i (z/2)^{2n}}{2 i! (2n+i+1/2)!}}^{P(n)}}_{=: S(z)} = -z^2.$$

► We get

$$2S(z) - zS'(z) = 0.$$

Show that

$$\underbrace{\sum_{n=0}^{\infty} -(4n+1)n \frac{(2n+1)!}{n!^2 2^{2n-1}} \overbrace{\sum_{i=0}^{\infty} \frac{\sqrt{\pi} (-z^2/4)^i (z/2)^{2n}}{2 i! (2n+i+1/2)!}}^{P(n)}}_{=: S(z)} = -z^2.$$

► We get

$$2S(z) - zS'(z) = 0.$$

► Hence $S(z) = cz^2$ for some constant c .

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$$\underbrace{\sum_{n=0}^{\infty} -(4n+1)n \frac{(2n+1)!}{n!^2 2^{2n-1}} \sum_{i=0}^{\infty} \frac{\sqrt{\pi} (-z^2/4)^i (z/2)^{2n}}{2 i! (2n+i+1/2)!}}_{=: S(z)} \stackrel{P(n)}{=} -z^2.$$

► We get

$$2S(z) - zS'(z) = 0.$$

► Hence $S(z) = cz^2$ for some constant c .

► By inspection we get $S(z) = -z^2$. □

Creative telescoping

GIVEN

$$P(z, n+2) = -\frac{4n+7}{4n+3}P(z, n) + \frac{(4n+5)(16n^2+40n-2z^2+21)}{(4n+3)z^2}P(z, n+1)$$

$$\frac{d}{dz}P(z, n) = \frac{6n+8n^2-z^2}{4n+3}P(z, n) - \frac{z^2}{4n+3}P(z, n+1).$$

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FIND $g(z, n) = g_0(z, n)P(z, n) + g_1(z, n)P(z, n+1)$:

$$g(z, n+1) - g(z, n) = c_0(z)P(z, n) + c_1(z)\frac{d}{dz}P(z, n)$$

Creative telescoping

GIVEN

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FIND $g(z, n) = g_0(z, n)P(z, n) + g_1(z, n)P(z, n+1)$:

$$g(z, n+1) - g(z, n) = c_0(z)P(z, n) + c_1(z) \left[\frac{6n+8n^2-z^2}{4n+3}P(z, n) - \frac{z^2}{4n+3}P(z, n+1) \right]$$

Creative telescoping

GIVEN

$$P(z, n+2) = -\frac{4n+7}{4n+3}P(z, n) + \frac{(4n+5)(16n^2+40n-2z^2+21)}{(4n+3)z^2}P(z, n+1)$$

$$\frac{d}{dz}P(z, n) = \frac{6n+8n^2-z^2}{4n+3}P(z, n) - \frac{z^2}{4n+3}P(z, n+1).$$

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$$g(z, n+1) - g(z, n) = c_0(z)P(z, n) + c_1(z) \left[\frac{6n+8n^2-z^2}{4n+3}P(z, n) - \frac{z^2}{4n+3}P(z, n+1) \right]$$

Sigma: $g(z, n) =$

$$\frac{(2n)!}{n!2^{2n}} \frac{4(n-1)n(2n+1)}{4n+3} \left[(z^2 - 16n^2 - 16n - 3)P(z, n) + z^2P(z, n+1) \right].$$