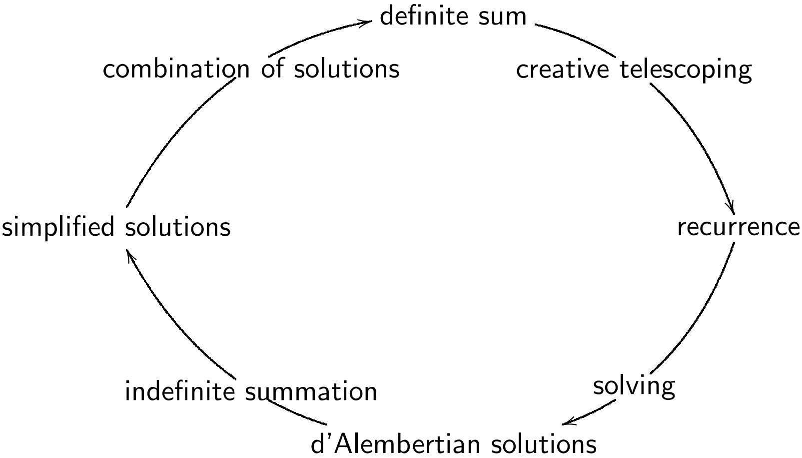
Symbolic Summation Assists Combinatorics

Carsten Schneider

RISC-Linz, Austria

The Sigma-summation spiral:



Typical example:

$$\sum_{k=0}^{2m} \binom{2m}{k}^3 (-1)^k H_k H_{2m-k} = \frac{(3m)!(-1)^m}{m!m!m!} \frac{1}{12} \big(3H_m^2 - 6H_m H_{3m} + 3H_{3m}^2 + H_m^{(2)} + 12H_{2m}(H_{2m} + H_m - H_{3m}) + 4H_{2m}^{(2)} - 3H_{3m}^{(2)}\big)$$

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Wenchang Chu asked:

$$\sum_{k=0}^{2m+1} (-1)^k \binom{2m+1}{k}^3 H_k^{(2)} = \text{Sigma}$$

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Wenchang Chu asked:

$$\sum_{k=0}^{2m+1}(-1)^k\binom{2m+1}{k}^3 H_k^{(2)} = \frac{(-1)^m(6m+1)(m!)^3(6m)!}{2(2m+1)^2((2m)!)^3(3m)!}\times \times\Big(-2+\sum_{i=1}^{m}\frac{(72i^2+36i+5)((2i)!)^3((3i)!)^2}{2i(2i+1)(6i+1)(i!)^6(6i)!}\Big)$$

Part 1: Symbolic Summation Methods and Applications

Part 2: Summation in Difference Fields

Part 3: Multi-Summation and Applications

Summation tools in difference fields

Telescoping

- GIVEN
$$S(n)=\sum_{k=0}^{n} f(k).$$
- FIND $g(k)$:
$$\boxed{f(k)=g(k+1)-g(k).}$$
- THEN (with some mild extra conditions)
$$S(n)=\sum_{k=0}^{n} f(k)=g(n+1)-g(0).$$

Telescoping

FIND a closed form for

$$\sum_{k=1}^{n} H_k.$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{Q}, & & \\ \sigma(k) &= k+1, & \mathcal{S}\,k &= k+1, \\ \sigma(h) &= h + \frac{1}{k+1}, & \mathcal{S}\,H_k &= H_k + \frac{1}{k+1}. \end{aligned}$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

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with

$$g(k) = (H_k - 1)k.$$

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Hence,

$$(H_{n+1} - 1)(n+1) = \sum_{k=1}^{n} H_k.$$

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Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]$:

$\boxed{d = 1}$

$$\forall g \in \mathbb{Q}(k)(h): \quad \sigma(g) - g = h \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[h].$$

FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

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$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

Degree bound: COMPUTE $b \geq 0$:

$$\boxed{b = 2}$$

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq b.$$

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Polynomial Solution: FIND

g=hk-k

$$g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h].$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\, h^2 + g_1 h + g_0\big] = h$$

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coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

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$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = h - c\left[\tfrac{2h(k+1)+1}{(k+1)^2}\right]$$

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$c = 0,\quad g_1 = k + d,\; d \in \mathbb{Q}$

$$\boxed{\sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}}$$

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g=hk-k

$$\boxed{\sigma(g_2) - g_2 = 0}$$

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$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = h - c\left[\tfrac{2h(k+1)+1}{(k+1)^2}\right]$$

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$c = 0,\quad g_1 = k + d,\ d \in \mathbb{Q}$

$$g_0 = -k,\ d = 0 \quad \leftarrow \quad \boxed{\sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}}$$

Creative telescoping

- GIVEN

$$S(n) = \sum_{k=0}^{n} f(n,k), \qquad d \in \mathbb{N}.$$

- FIND $c_0(n), \dots, c_d(n)$ and $g(n,k)$:

$$\boxed{c_0(n) f(n,k) + \dots + c_d(n) f(n+d,k) = g(n,k+1) - g(n,k).}$$

- THEN (with some mild extra conditions)

$$c_0(n) S(n) + \dots + c_d(n) S(n+d) = h(n)$$

for some $h(n)$.

FIND a recurrence for

$$S(n) := \sum_{k=1}^{n} \binom{n}{k} H_k.$$

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Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(n)(k)(h)(b)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),$$

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$$\sigma(b) = \frac{n-k}{k+1}\, b, \qquad \mathcal{S} \binom{n}{k} = \frac{n-k}{k+1} \binom{n}{k}.$$

REPRESENT $f(n+i,k)$ in $\mathbb{F}$:

$$f(n,k) = \mathsf{H}_k \binom{n}{k} \quad \longleftrightarrow \quad h\,b := f_0$$

$$f(n+1,k) = \frac{(n+1)\,\mathsf{H}_k \binom{n}{k}}{n+1-k} \quad \longleftrightarrow \quad \frac{(n+1)\,h\,b}{n+1-k} := f_1$$

$$f(n+2,k) = \frac{(n+1)\,(n+2)\,\mathsf{H}_k \binom{n}{k}}{(n+1-k)\,(n+2-k)} \quad \longleftrightarrow \quad \frac{(n+1)\,(n+2)\,h\,b}{(n+1-k)\,(n+2-k)} =: f_2.$$

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FIND $c_0, c_1, c_2 \in \mathbb{Q}(n)$ and $g \in \mathbb{F}$:

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Sigma computes

$$c_0 := 4\,(1+n), \quad c_1 := -2\,(3+2\,n), \quad c_2 := 2+n,$$

$$g := \frac{(1+n)\,(-2+k-n+(2\,k-2\,k^2+k\,n)h)\,b)}{(1-k+n)\,(2-k+n)}.$$

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This gives

$$\boxed{c_0(n)f(n,k) + \cdots + c_2(n)f(n+2,k) = g(n,k+1) - g(n,k)}$$

with

$$c_0(n) := 4\,(1+n), \quad c_1(n) := -2\,(3+2\,n), \quad c_2(n) := 2+n,$$

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This gives

$$c_0(n)f(n,k)+\cdots+c_2(n)f(n+2,k)=g(n,k+1)-g(n,k)$$

Summing over k from 0 to n gives

$$4(1+n)S(n)-2(3+2n)S(n+1)+(2+n)S(n+3)=1$$

for

$$S(n)=\sum_{k=0}^{n}\binom{n}{k}H_k.$$

(Creative) Telescoping in difference fields

Let $(\mathbb{F}, \sigma)$ be a difference field and

$$\mathbb{K} = \{k \in \mathbb{F} | \sigma(k) = k\} \supseteq \mathbb{Q} \quad \text{(constant field)}$$

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Telescoping

- ▶GIVEN $f\in\mathbb{F}$.
- ▶FIND $g\in\mathbb{F}$:

$$\boxed{\sigma(g)-g=f.}$$

$$\downarrow\qquad\qquad\uparrow$$

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Parameterized telescoping

- GIVEN $f_0, \dots, f_d \in \mathbb{F}$, $a_0, a_1 \in \mathbb{F}$.
- FIND ALL $c_0, \dots, c_d \in \mathbb{K}$, $h \in \mathbb{F}$:

$$\boxed{a_1\,\sigma(h) - a_0\,h = c_0\,f_0 + \cdots + c_d\,f_d.}$$

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Remark: creative telescoping

- GIVEN $f_i=\text{summand}(n+i,k)\in\mathbb{F}$.
- FIND ALL $c_0,\dots,c_d\in\mathbb{K}$, $g\in\mathbb{F}$:

$$\boxed{\sigma(g)-g=c_0\,f_0+\dots+c_d\,f_d.}$$

Algorithms for (creative) telescoping

Hypergeometric case:

Gosper. Decision procedures for indefinite hypergeometric summation. 1978.

Petkovšek, Wilf, Zeilberger. $A = B$. 1996.

Paule. Greatest factorial factorization and symbolic summation. 1995.

q-**Hypergeometric case:**

Paule, Riese. A Mathematica q-analogue of Zeilberger's algorithm... 1997.

Mixed case:

Bauer, Petkovšek. Multibasic and mixed hypergeometric Gosper-type algorithms. 1999.

$\Pi\Sigma^*$-**field case:**

M. Karr. Summation in finite terms. 1981.

Schneider. An Implementation of Karr's Summation Algorithm in Mathematica. SLC '00.

Holonomic case:

Chyzak. An extension of Zeilberger's fast algorithm to general holonomic functions. 2000.

A constructive $\Pi\Sigma^*$-field theory

Σ^*-extensions

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$$\sigma(t) = t + f, \quad f \in \mathbb{F},$$

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- the constants remain unchanged:

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Theorem (Karr, 1981). Let $(\mathbb{F}(t), \sigma)$ be a difference field with

$$\sigma(t) = t + f \in \mathbb{F}.$$

This is a Σ^*-extension iff

$$\nexists g \in \mathbb{F}: \quad \sigma(g) - g = f.$$

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Example: Construct a difference field for

$$\sum_{k=0}^{n} H_k.$$

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Example: Construct a difference field for

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There is no $g \in \mathbb{Q}$ with

$$\sigma(g) - g = 1.$$

Hence, we can construct the Σ^*-extension $(\mathbb{Q}(k), \sigma)$ with

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There is no $g \in \mathbb{Q}(k)$ with

$$\sigma(g) - g = \frac{1}{k+1}.$$

Hence, we can construct the Σ^*-extension $(\mathbb{Q}(k)(h), \sigma)$ with

$$\sigma(h) = h + \frac{1}{k+1}.$$

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There is no need! Take

$$t := kh - k.$$

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Π-extensions

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Let $(\mathbb{F},\sigma)$ be a difference field with the constant field

$$\mathbb{K} := \{k \in \mathbb{F} | \sigma(k) = k\}.$$

We call a difference field $(\mathbb{F}(t),\sigma)$ a Π-extension of $(\mathbb{F},\sigma)$ if

- t is transcendental,
- the automorphism σ is extended from $\mathbb{F}$ to $\mathbb{F}(t)$ with

$$\sigma(t) = a t, \quad a \in \mathbb{F},$$

- the constants remain unchanged:

$$\mathbb{K} = \{k \in \mathbb{F}(t) | \sigma(k) = k\}.$$

Theorem (Karr 1981). Let $(\mathbb{F}(t),\sigma)$ be a difference field with

$$\sigma(t) = a t, \quad a \in \mathbb{F}.$$

This is a Π-extension iff $t \neq 0$ and

$$\nexists r > 0, g \in \mathbb{F} : \quad \frac{\sigma(g)}{g} = a^r.$$

Π-extensions

Example: Construct a difference field for

$$\binom{n}{k} H_k$$

There is no $g \in \mathbb{Q}(k)(h)$ and no $r > 0$ with

$$\frac{\sigma(g)}{g} = \left(\frac{n-k}{k+1}\right)^r.$$

Hence, we can construct the Π-extension $(\mathbb{Q}(k)(h)(b), \sigma)$ with

$$\sigma(b) = \frac{n-k}{k+1} b. \qquad \left[\binom{n}{k+1} = \binom{n}{k}\frac{n-k}{k+1}\right]$$

Theorem (Karr 1981). Let $(\mathbb{F}(t), \sigma)$ be a difference field with

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$\Pi\Sigma^*$-fields

A $\Pi\Sigma^*$-field $(\mathbb{K}(t_1)\dots(t_e),\sigma)$ is a tower of Π- or Σ^*-extensions.

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- creative telescoping
- a decision procedure for

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$\Rightarrow$ Completely constructive summation theory!

Generalizations, extensions and **examples**

Generalization I: Refined telescoping

- GIVEN

$$\sum_{k=0}^{n} f(k).$$

- FIND $g(k)$ and $f'(k)$:

$$\boxed{f(k) = g(k+1) - g(k) + f'(k)}$$

where $f'(k)$ is simpler than $f(k)$.

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- THEN (with some mild extra conditions)
$$\sum_{k=0}^{n} f(k) = g(n+1) - g(0) + \sum_{k=0}^{n} f'(k).$$

Degree optimal w.r.t the top extension

- GIVEN $f \in \mathbb{F}(t)$.
- FIND $(f', g) \in \mathbb{F}(t)^2$:

$$\sigma(g) - g + f' = f$$

where in f' the degree of the denominator (and numerator) is minimal.

$$\sum_{k=2}^{n} \frac{2 - kH_k + H_k^4 - kH_k^5}{H_k - kH_k^2} =$$

Sigma

$$\sum_{k=1}^{n} H_k^4 =$$

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$$\sum_{k=2}^{n} \frac{2 - kH_k + H_k^4 - kH_k^5}{H_k - kH_k^2} = \sum_{k=2}^{n} \frac{k^2 + H_k}{k^2 H_k} + (n+1)H_n^3 - (2n+1)\left(\tfrac{3}{2}H_n^2 - 3H_n + \tfrac{3}{2}\right) + \tfrac{1}{H_n}$$

$$\sum_{k=1}^{n} H_k^4 = -H_n^{(3)} - 2H_n^{(2)} + 2\sum_{k=1}^{n} \frac{H_k}{k^2} + (n+1)H_n^4 - 2(2n+1)H_n^3 + 6(2n+1)H_n^2 - 12(2n+1)H_n + 24n.$$

Further examples

$$\sum_{k=1}^{n}\frac{k+1}{k(k+2)}=-\frac{n(3n+5)}{4(n+1)(n+2)}+\sum_{k=1}^{n}\frac{1}{k}$$

$$\sum_{k=2}^{n}\frac{1}{k(k-1)2^k}=\frac{-1}{n2^{n+1}}+\frac{1}{4}-\frac{1}{2}\sum_{k=2}^{n}\frac{1}{k2^k}$$

$$\sum_{k=1}^{n}\frac{k!\left(k^2+k+k!\left(k(k+1)^2+k!\left(k(k+1)^2+\left(2k^2-1\right)k!-3\right)-2\right)+1\right)+1}{(k!)^3(k!+1)((k+1)k!+1)}$$
$$=\frac{3(n+1)(n!)^3+(3-2n)(n!)^2-2(n+2)n!-2}{2(n!)^2((n+1)n!+1)}+\sum_{k=1}^{n}\frac{k(k!)^3+k!+1}{(k!)^3(k!+1)}$$

$$\sum_{k=2}^{n}\frac{(k+1)\left(k(k+1)^2(k+2)H_k^3+k\left(3k^2+8k+5\right)H_k^2-(k+2)H_k-k-2\right)}{H_k\left(k(k+1)^2(k+2)H_k^3+2\left(k^3+2k^2-1\right)H_k^2-\left(k^2+5k+5\right)H_k-2k-3\right)}$$
$$=\frac{-6(n+1)(n+2)H_n^2-6(2n+3)H_n+11(n+1)(n+2)}{11H_n\left(2n+(n+1)(n+2)H_n+3\right)}+\sum_{k=2}^{n}\frac{k(k+1)}{kH_k-1}$$

The analogue problem for Π-extensions

$$\prod_{k=1}^{n}\frac{(-k-1)(k+7)}{(k+4)^2}=\frac{4}{35}\,\frac{(n+5)(n+6)(n+7)}{(n+2)(n+3)(n+4)}\,(-1)^n,$$

$$\prod_{k=1}^{n}\frac{(k+3)(H_k(k+1)+1)^2(H_k(k+2)(k+1)+2k+3)}{(k+1)^2H_k(H_k(k+3)(k+2)(k+1)+3(k+4)k+11)}$$
$$=\frac{11}{6}\,\frac{(n+3)(n+2)(H_n(n+1)+1)^2}{(n+1)(H_n(n+3)(n+2)(n+1)+3(n+4)n+11)}\prod_{k=1}^{n}H_k,$$

$$\prod_{k=1}^{n}\frac{k!(H_k(k+2)(k+1)+2k+3)(H_k(k+1)+1)}{H_k(k+3)(k+2)(k+1)+3(k+4)k+11}$$
$$=\frac{11(H_n(n+1)+1)}{H_n(n+3)(n+2)(n+1)+3(n+4)n+11}\prod_{k=1}^{n}k!H_k,$$

$$\prod_{k=1}^{n}\frac{(q^{k+2}+(k+1)!)(q^{k+1}+k!)(k+2)(k+1)}{(q^{k+3}+(k+2)!)(k+3)}$$
$$=\frac{3(q^3+2)}{q+1}\,\frac{(q^{n+1}(n+1)+(n+1)!)}{(q^{n+3}+(n+2)!)(n+3)}\prod_{k=1}^{n}(kq^k+k!)$$

CS. Product Representations in $\Pi\Sigma$-Fields. 2005.

Simpler w.r.t. the depth

- GIVEN $f \in \mathbb{F}$.
- FIND an extension $\mathbb{E} \supseteq \mathbb{F}$ and $(f', g) \in \mathbb{E}^2$:

$$\sigma(g) - g + f' = f$$

where f' has minimal depth.

$$\sum_{k=1}^{n} H_k^2 H_k^{(2)} =$$

$$\sum_{k=0}^{a} \left(\sum_{i=0}^{k} \binom{n}{i} \right)^2 =$$

Sigma

$$\sum_{k=1}^{n} \frac{1}{k^3} \sum_{j=1}^{k} \frac{H_j}{j^2} =$$

Simpler w.r.t. the depth

- GIVEN $f \in \mathbb{F}$.
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$$\sigma(g) - g + f' = f$$

where f' has minimal depth.

$$\sum_{k=1}^{n} H_k^2 H_k^{(2)} = \frac{1}{3} H_n^{(3)} - \frac{1}{3} H_n^3 + \left((n+1) H_n^{(2)} + 1\right) H_n^2 + (2n+1)\left(1 - H_n\right) H_n^{(2)} - 2H_n$$

$$\sum_{k=0}^{a} \left(\sum_{i=0}^{k} \binom{n}{i}\right)^2 = (n-a)\binom{n}{a} \sum_{i=0}^{a} \binom{n}{i} - \frac{n-2a-2}{2} \left(\sum_{i=0}^{a} \binom{n}{i}\right)^2 - \frac{n}{2} \sum_{i=1}^{n} \binom{n}{i}^2$$

$$\sum_{k=1}^{n} \frac{1}{k^3} \sum_{j=1}^{k} \frac{H_j}{j^2} = H_n^{(3)} \sum_{j=1}^{n} \frac{H_j}{j^2} - \sum_{j=1}^{n} \frac{H_j (j^3 H_j^{(3)} - 1)}{j^5}$$

CS. Symbolic summation with single-nested sum extensions. 2004.

CS. Finding telescopers with minimal depth for indefinite nested sum and product expressions. 05.

Further examples

$$\sum_{k=1}^{n} H_k^3 = \frac{1}{2}\left(2(n+1)H_n^3 - 3(2n+1)H_n^2 + 6(2n+1)H_n - 12n - 1 + H_n^{(2)}\right)$$

$$\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j}\sum_{i=1}^{j}\frac{1}{i} = \frac{1}{6}\left[H_n^3 + 3H_n\, H_n^{(2)} + 2\, H_n^{(3)}\right]$$

$$\sum_{k=1}^{n}\Big(\sum_{j=1}^{k}\binom{n}{j}\Big)\Big(\sum_{j=1}^{k}\binom{n}{j}^2\Big) = \frac{n+2}{2}\sum_{j=1}^{n}\binom{n}{j}\sum_{j=1}^{n}\binom{n}{j}^2 - \frac{1}{n}\sum_{j=1}^{n}(n^2-nj+j^2)\binom{n}{j}^3$$

$$\sum_{k=1}^{n}\frac{1}{k}\sum_{i=1}^{k}\frac{1}{H_i} = -n + H_n\sum_{i=1}^{n}\frac{1}{H_i} + \sum_{i=1}^{n}\frac{1}{iH_i},$$

$$\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\frac{H_j^{(2)}}{j^3}\right)^2 = -\big({H_n^{(2)}}^2 + H_n^{(4)}\big)\sum_{j=1}^{n}\frac{H_j^{(2)}}{j^3} + (n+1)\times$$

$$\times\left(\sum_{j=1}^{n}\frac{H_j^{(2)}}{j^3}\right)^2 + \sum_{j=1}^{n}\frac{H_j^{(2)}\big((jH_j^{(2)})^2 - H_j^{(2)} + j^2 H_j^{(4)}\big)}{j^5}.$$

Generalization II: Difference equations of higher order

Linear difference equations

▶GIVEN $f, a_0, \dots, a_m \in \mathbb{F}$.

▶FIND ALL $g \in \mathbb{F}$:

$$\boxed{a_m\,\sigma^{m}(g) + \dots + a_0\,g = f.}$$

$\downarrow$ $\uparrow$

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- ▶GIVEN $f, a_0, \dots, a_m \in \mathbb{F}$.
- ▶FIND ALL $g \in \mathbb{F}$:

$$\boxed{a_m\,\sigma^m(g) + \dots + a_0\,g = f.}$$

$$\downarrow \qquad\qquad \uparrow$$

Parameterized linear difference equations

- ▶GIVEN $a_0, \dots, a_m \in \mathbb{F}$,$f_0, \dots, f_d \in \mathbb{F}$.
- ▶FIND ALL $g \in \mathbb{F}$, $c_0, \dots, c_d \in \mathbb{K}$:

$$\boxed{a_m\,\sigma^m(g) + \dots + a_0\,g = c_0\,f_0 + \dots + c_d\,f_d.}$$

Algorithms for parameterized linear difference equations

Rational case:

Abramov. Rational solutions of linear differential and difference equations with polynomial coefficients. 1989.

Hoeij. Rational solutions of linear difference equations. 1998.

q-Rational case:

Abramov. Rational solutions of linear difference and q-difference equations with polynomial coefficients. 1995.

$\Pi\Sigma^*$-field case: (open subproblems)

M. Karr. Summation in finite terms. 1981.

Bronstein. On solutions of linear ordinary difference equations in their coefficient field. 2000.

CS. Solving parameterized linear difference equations in terms of indefinite nested sums and products. 2005.

CS. A collection of denominator bounds to solve parameterized linear difference equations in $\Pi\Sigma$-extensions. 2004.

CS. Degree bounds to find polynomial solutions of parameterized linear difference equations in $\Pi\Sigma$-fields.

Example: Sigma

Generalization III: Solving recurrences with extensions

Rational case:

Petkovšek. Hypergeometric solutions of linear recurrences with polynomial coefficients. 1992.

Hoeij. Finite singularities and hypergeometric solutions of linear recurrence equations. 1999.

Abramov, Petkovšek. D'Alembertian solutions of linear differential and difference equations. 1994.

Hendriks, Singer. Solving difference equations in finite terms. 1999.

$\Pi\Sigma^*$**-field case: (open subproblems)**

CS. Symbolic summation in difference fields. PhD-thesis, 2001.

Communication with (M. Bronstein), Abramov, Petkovšek.

Example: Sigma

Generalization IV: Unspecified sequences

GIVEN

$$\mathbb{K}(\dots, x_{-1}, x_0, x_1, \dots), \quad \text{where} \quad \sigma(x_i) = x_{i+1}$$

plus a tower of $\Pi\Sigma^*$-extensions on top.

There are algorithms for

- telescoping
- creative telescoping
- solving recurrences (open subproblems)

Kauers, CS. Application of unspecified sequences in symbolic summation. 2006.
Kauers, CS. Indefinite summation with unspecified sequences.

Generalization V: Multi-Summation

Generalization V: Multi-Summation

$$||$$

Part 3