Symbolic Summation Assists Combinatorics

Carsten Schneider

RISC-Linz, Austria

Part 1: Symbolic Summation Methods and Applications

Part 2: Summation in Difference Fields

Part 3: Multi-Summation and Applications

Warmup example (bonus problem 6.69 in "Concrete Mathematics")

FIND a closed form for

$$\sum_{k=1}^{n} k^2 \mathsf{H}_{n+k} = \boxed{?},$$

where $\mathsf{H}_n := \sum_{k=1}^{n} \frac{1}{k}$.

Telescoping

GIVEN $f(k) = k^2 H_{n+k}$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

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for all $1 \leq k \leq n$ and $n \geq 0$.

Sigma computes

$$g(k) = \frac{1}{36}(k(-4k^2 + (6n+3)k - 12n(n+1) + 1) \\ + 6(2k^3 - 3k^2 + k + n(n+1)(2n+1))H_{k+n}).$$

Telescoping

GIVEN $f(k)=k^2H_{n+k}$.

FIND $g(k)$:

$$\boxed{f(k)=g(k+1)-g(k)}$$

for all $1\leq k\leq n$ and $n\geq 0$.

Summing this equation over k from 1 to n gives

$$\boxed{\sum_{k=1}^{n}k^2\mathrm{H}_{n+k}}=\boxed{g(n+1)-g(1)}$$

$$=-\frac{1}{36}n(n+1)\left(10n+6(2n+1)H_n-12(2n+1)H_{2n}-1\right).$$

Example 1: Padé approximation

Quadratic Padé approximation to $\log(x)$ at $x = 1$

FIND $r_m(x) = \sum_{k=0}^{m} a_k x^k, \quad s_m(x) = \sum_{k=0}^{m} b_k x^k, \quad t_m(x) = \sum_{k=0}^{m} c_k x^k$:

$$r_m(x)\,(\log x)^2 + s_m(x)\,\log(x) + t_m(x) = O((x-1)^{3m+2}).$$

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$$r_m(x)\,(\log x)^2 + s_m(x)\,\log(x) + t_m(x) = O((x-1)^{3m+2}).$$

A. Weideman finds

$$\begin{aligned} r_m(x) &= c_3 A(m,x) \\ s_m(x) &= c_2 A(m,x) + 2c_3 B(m,x) \\ t_m(x) &= c_1 A(m,x) + c_2 B(m,x) + c_3 C(m,x) \end{aligned}$$

where

$$A(m,x) = \sum_{k=0}^{m} \binom{m}{k}^3 (-x)^k \qquad B(m,x) = \sum_{k=0}^{m} \Big[\frac{d}{dk}\binom{m}{k}^3\Big](-x)^k$$

$$C(m,x) = \sum_{k=0}^{m} \Big[\frac{d^2}{dk^2}\binom{m}{k}^3\Big](-x)^k.$$

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A. Weideman finds

$$\begin{aligned} r_m(x)&=c_3A(m,x)\\ s_m(x)&=c_2A(m,x)+2c_3B(m,x)\\ t_m(x)&=c_1A(m,x)+c_2B(m,x)+c_3C(m,x) \end{aligned}$$

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$$C(m,x)=\sum_{k=0}^{m}\left[\frac{d^2}{dk^2}\binom{m}{k}^3\right](-x)^k.$$

Tests at $x=1$:
$c_1=\pi^2, \quad c_2=0, \quad c_3=1$

For all $n \geq 0$*,*

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}^3\Big[3(H_{m-k}-H_k)^2+H_{m-k}^{(2)}+H_k^{(2)}\Big]=0$$

where

$$H_k=\sum_{i=1}^{k}\frac{1}{i},\quad H_k^{(2)}=\sum_{i=1}^{k}\frac{1}{i^2}.$$

Theorem (Sigma; 2002)

For all $n \geq 0$*,*

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}^3\Big[3(H_{m-k}-H_k)^2+H_{m-k}^{(2)}+H_k^{(2)}\Big]=0$$

where

$$H_k=\sum_{i=1}^{k}\frac{1}{i},\quad H_k^{(2)}=\sum_{i=1}^{k}\frac{1}{i^2}.$$

Proof.

Sigma

□

Z's creative telescoping paradigm

GIVEN

$$\text{SUM}(m) := \sum_{k=0}^{m} \underbrace{(-1)^k \binom{m}{k}^3 \left[3(H_{m-k} - H_k)^2 + H_{m-k}^{(2)} + H_k^{(2)}\right]}_{=:\, f(m,k)}.$$

FIND $c_0(m)$, $c_1(m)$, $c_2(m)$, and $g(m,k)$:

$$\boxed{g(m,k+1) - g(m,k)} = \boxed{c_0(m)\, f(m,k) + c_1(m)\, f(m+1,k) + c_2(m)\, f(m+2,k)}$$

for all $0 \leq k \leq m$ and all $m \geq 0$.

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for all $0 \le k \le m$ and all $m \ge 0$.

Sigma: $c_0(m) := 3(3m+2)(3m+4)(3m+8), \quad c_1(m) := 0,$
$c_2(m) := (m+2)^2(3m+8),$

$$g(m,k) := (-1)^k \binom{m}{k}^3 \frac{p_1(k, m, H_k, H_k^{(2)}, H_{m-k}, H_{m-k}^{(2)})}{(m-k+1)^5(m-k+2)^5},$$

$$g(m,k+1) := (-1)^k \binom{m}{k}^3 \frac{p_2(k, m, H_k, H_k^{(2)}, H_{m-k}, H_{m-k}^{(2)})}{(m-k+1)^5}.$$

Z's creative telescoping paradigm

GIVEN

$$\mathsf{SUM}(m) := \sum_{k=0}^{m} \underbrace{(-1)^k \binom{m}{k}^3 \left[3(H_{m-k} - H_k)^2 + H^{(2)}_{m-k} + H^{(2)}_k\right]}_{=:\, f(m,k)}.$$

GIVEN $c_0(m)$, $c_1(m)$, $c_2(m)$, and $g(m,k)$:

$$\boxed{g(m,k+1) - g(m,k)} = \boxed{c_0(m)\, f(m,k) + c_1(m)\, f(m+1,k) + c_2(m)\, f(m+2,k)}$$

for all $0 \le k \le m$ and all $m \ge 0$.

Summing this equation over k from 0 to m gives:

$$\boxed{g(m,m+1) - g(m,0)}$$

$$= \boxed{\begin{array}{l} c_0(m)\,\mathsf{SUM}(m) + \\ c_1(m)\,\left[\mathsf{SUM}(m+1) - f(m+1,m+1)\right] \\ c_2(m)\,\left[\mathsf{SUM}(m+2) - f(m+2,m+1) - f(m+2,m+2)\right]. \end{array}}$$

Proving $\xrightarrow{\text{Sigma}}$ Finding

$$\sum_{k=0}^{2m}\binom{2m}{k}^3(-1)^k H_k H_{2m-k} = \frac{(3m)!(-1)^m}{m!m!m!}\frac{1}{12}\big(3H_m^2-6H_mH_{3m}+3H_{3m}^2+H_m^{(2)}$$
$$+12H_{2m}(H_{2m}+H_m-H_{3m})+4H_{2m}^{(2)}-3H_{3m}^{(2)}\big)$$

$$\sum_{k=0}^{2m}\binom{2m}{k}^3(-1)^k H_k^2 = \frac{(3m)!(-1)^m}{m!m!m!}\frac{1}{12}\big(3H_m^2-6H_mH_{3m}+3H_{3m}^2-H_m^{(2)}$$
$$+12H_{2m}(H_{2m}+H_m-H_{3m})+2H_{2m}^{(2)}-3H_{3m}^{(2)}\big)$$

$$\sum_{k=0}^{2m}\binom{2m}{k}^3(-1)^k H_k^{(2)} = \frac{1}{2}\frac{(3m)!(-1)^m}{m!m!m!}(H_n^{(2)}+H_{2n}^{(2)})$$

Proving $\xrightarrow{\text{Sigma}}$ Finding

$$\sum_{k=0}^{2m}\binom{2m}{k}^3(-1)^k H_k H_{2m-k} = \frac{(3m)!(-1)^m}{m!m!m!}\frac{1}{12}\big(3H_m^2 - 6H_m H_{3m} + 3H_{3m}^2 + H_m^{(2)} + 12H_{2m}(H_{2m}+H_m-H_{3m}) + 4H_{2m}^{(2)} - 3H_{3m}^{(2)}\big)$$

$$\sum_{k=0}^{2m}\binom{2m}{k}^3(-1)^k H_k^2 = \frac{(3m)!(-1)^m}{m!m!m!}\frac{1}{12}\big(3H_m^2 - 6H_m H_{3m} + 3H_{3m}^2 - H_m^{(2)} + 12H_{2m}(H_{2m}+H_m-H_{3m}) + 2H_{2m}^{(2)} - 3H_{3m}^{(2)}\big)$$

$$\sum_{k=0}^{2m}\binom{2m}{k}^3(-1)^k H_k^{(2)} = \frac{1}{2}\frac{(3m)!(-1)^m}{m!m!m!}(H_n^{(2)} + H_{2n}^{(2)})$$

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^3\Big[3(H_{2m-k}-H_k)^2 + H_{2m-k}^{(2)} + H_k^{(2)}\Big] = 0$$

Proving $\xrightarrow{\text{Sigma}}$ Finding

$$\boxed{\sum_{k=0}^{2m}\binom{2m}{k}^3(-1)^k H_k H_{2m-k}} = \text{FIND}$$

$$\sum_{k=0}^{2m}\binom{2m}{k}^3(-1)^k H_k^2 = \frac{(3m)!(-1)^m}{m!m!m!}\frac{1}{12}\big(3H_m^2 - 6H_m H_{3m} + 3H_{3m}^2 - H_m^{(2)} + 12H_{2m}(H_{2m}+H_m-H_{3m}) + 2H_{2m}^{(2)} - 3H_{3m}^{(2)}\big)$$

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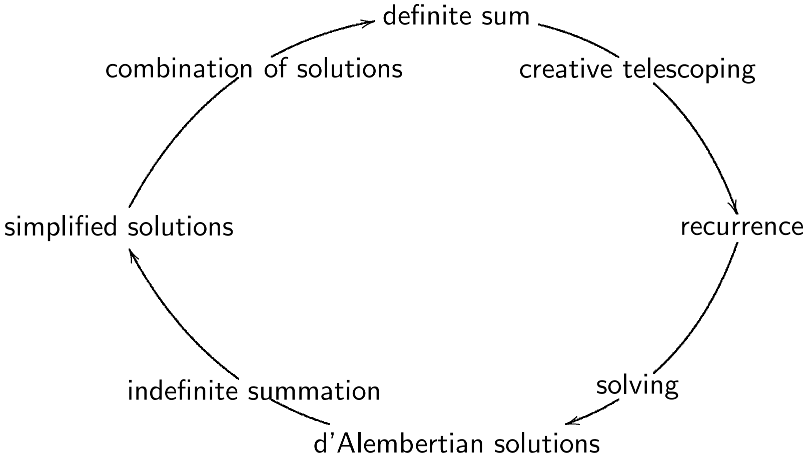
The Sigma-summation spiral:



Weideman. Padé Approximations to the Logarithm I: Derivation via Differential Equations. 2005.

Driver, Prodinger, CS, Weideman. Padé approximations to the logarithm II: Identities, recurrences, and symbolic computation. 2006.

Driver, Prodinger, CS, Weideman. Padé approximations to the logarithm III: Alternative methods and additional results.

Krattenthaler. Private communication (differentiation, hypergeometric transformations).

Wenchang Chu. Harmonic number identities and Hermite-Padé approximation to the logarithm function. 2005.

Example 2: Evaluation of a quadruple sum

A challenging email

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From: Doron Zeilberger
To: Robin Pemantle, Herbert Wilf
CC:Carsten Schneider



Robin and Herb,

I am willing to bet that Carsten Schneider's SIGMA package
for handling sums with harmonic numbers (among others)
can do it in a jiffy. I am Cc-ing this to Carsten.

Carsten: please do it, and Cc- the  answer to me.
             -Doron
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The problem

From: Robin Pemantle [University of Pennsylvania]
To: herb wilf; doron zeilberger

Herb, Doron,

I have a sum that, when I evaluate numerically, looks suspiciously like it comes out to exactly 1.
Is there a way I can automatically decide this?
The sum may be written in many ways, but one is:

$$\sum_{j,k=1}^{\infty} \frac{H_j(H_{k+1}-1)}{jk(k+1)(j+k)}; \qquad H_j := \sum_{i=1}^{j} \frac{1}{i}.$$

Of course you can expand out the H's and get a quadruple sum.
There are zillions of ways to play with it, summing by parts, but I have never managed to get rid of all the summations.

Robin

Take the truncated version:

$$S(a,b)=\sum_{k=1}^{b}\frac{H_{k+1}-1}{k(k+1)}\sum_{j=1}^{a}\frac{H_j}{j(j+k)},$$

i.e.,

$$\lim_{a,b\to\infty}S(a,b)=S.$$

Take the truncated version:

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$$\sum_{j=1}^{a}\frac{H_j}{j(j+k)}=$$ FIND

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$$S(a,b)=\sum_{k=1}^{b}\frac{H_{k+1}-1}{k(k+1)}\sum_{j=1}^{a}\frac{H_j}{j(j+k)},$$

i.e.,

$$\lim_{a,b\to\infty}S(a,b)=S.$$

$$\sum_{j=1}^{a}\frac{H_j}{j(j+k)}=\frac{kH_k^2-2H_k+kH_k^{(2)}+2kH_a^{(2)}}{2k^2}$$

$$\underbrace{-\frac{(kH_a-1)}{k^2}\sum_{i=1}^{k}\frac{1}{a+i}-\frac{1}{k}\sum_{i=1}^{k}\frac{1}{i}\sum_{j=1}^{i}\frac{1}{a+j}}_{\text{Limits}}$$

Take the truncated version:

$$S(a,b)=\sum_{k=1}^{b}\frac{H_{k+1}-1}{k(k+1)}\sum_{j=1}^{a}\frac{H_j}{j(j+k)},$$

i.e.,

$$\lim_{a,b\to\infty}S(a,b)=S.$$

Hence, for

$$\boxed{S'(a,b):=\sum_{k=1}^{b}\frac{H_{k+1}-1}{k(k+1)}\frac{kH_k^2-2H_k+kH_k^{(2)}+2kH_a^{(2)}}{2k^2},}$$

we have

$$\lim_{a,b\to\infty}S'(a,b)=S.$$

Further simplification

$$S'(a,b) = \text{Simplify}$$

Further simplification

$$S'(a,b) = A(a,b) + B(a,b) + C(a,b)$$

where

$$\begin{aligned} A(a,b) :=& \frac{1}{2(b+1)^2}\Big(6H_b + 4bH_b + 4H_b^2 + 3bH_b^2 + H_b^3 + bH_b^3 - 6bH_a^{(2)} \\ &+ 2H_bH_a^{(2)} + 2bH_bH_a^{(2)} - 2H_b^{(2)} - 7bH_b^{(2)} + H_bH_b^{(2)} + bH_bH_b^{(2)}\Big), \\ B(a,b) :=& -\frac{2b^2}{(b+1)^2}\Big(H_a^{(2)} + H_b^{(2)}\Big), \\ C(a,b) :=& (H_a^{(2)} - 1)\sum_{i=1}^{b}\frac{H_i}{i^2} - \sum_{i=1}^{b}\frac{H_i^2}{i^3} + \frac{1}{2}\sum_{i=1}^{b}\frac{H_i^3}{i^2} + \frac{1}{2}\sum_{i=1}^{b}\frac{H_iH_i^{(2)}}{i^2}. \end{aligned}$$

Further simplification

$$S'(a,b) = A(a,b) + B(a,b) + C(a,b)$$

where

$$A(a,b) := \frac{1}{2(b+1)^2}\Big(6H_b + 4bH_b + 4H_b^2 + 3bH_b^2 + H_b^3 + bH_b^3 - 6bH_a^{(2)} + 2H_bH_a^{(2)} + 2bH_bH_a^{(2)} - 2H_b^{(2)} - 7bH_b^{(2)} + H_bH_b^{(2)} + bH_bH_b^{(2)}\Big),$$

$$B(a,b) := -\frac{2b^2}{(b+1)^2}\Big(H_a^{(2)} + H_b^{(2)}\Big),$$

$$C(a,b) := (H_a^{(2)} - 1)\sum_{i=1}^{b}\frac{H_i}{i^2} - \sum_{i=1}^{b}\frac{H_i^2}{i^3} + \frac{1}{2}\sum_{i=1}^{b}\frac{H_i^3}{i^2} + \frac{1}{2}\sum_{i=1}^{b}\frac{H_iH_i^{(2)}}{i^2}.$$

By

$$\lim_{a,b\to\infty} A(a,b) = 0 \qquad \text{and} \qquad \lim_{a,b\to\infty} B(a,b) = -4\zeta(2)$$

we get

$$S = \lim_{a,b\to\infty} S'(a,b) = -4\zeta(2) + \lim_{a,b\to\infty} C(a,b).$$

ζ-relations

This gives

$$S = \sum_{j,k=1}^{\infty} \frac{H_j(H_{k+1}-1)}{jk(k+1)(j+k)}$$
$$= -4\zeta(2) + (\zeta(2)-1)\sum_{i=1}^{\infty}\frac{H_i}{i^2} - \sum_{i=1}^{\infty}\frac{H_i^2}{i^3} + \frac{1}{2}\sum_{i=1}^{\infty}\frac{H_i^3}{i^2} + \frac{1}{2}\sum_{i=1}^{\infty}\frac{H_i H_i^{(2)}}{i^2}.$$

ζ-relations

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E.g., in J.M. Borwein, Girgensohn. Evaluation of triple Euler sums, 1996.

Flajolet, Salvy. Euler sums and contour integral representations, 1998.

we find

$$\sum_{i=1}^{\infty}\frac{H_i}{i^2} = 2\zeta(3), \qquad \sum_{i=1}^{\infty}\frac{H_i^2}{i^3} = -\zeta(2)\zeta(3) + \frac{7}{2}\zeta(5),$$
$$\sum_{i=1}^{\infty}\frac{H_i^3}{i^2} = \zeta(2)\zeta(3) + 10\zeta(5), \qquad \sum_{i=1}^{\infty}\frac{H_i H_i^{(2)}}{i^2} = \zeta(2)\zeta(3) + \zeta(5).$$

ζ-relations

Theorem.

$$S=\sum_{j,k=1}^{\infty}\frac{H_j(H_{k+1}-1)}{jk(k+1)(j+k)}$$
$$=-4\zeta(2)-2\zeta(3)+4\zeta(2)\zeta(3)+2\zeta(5)=0.999222...$$

Pemantle, CS. When is 0.999... equal to 1?

Panholzer, Prodinger. Computer-free evaluation of an infinite double sum. SLC'05.

Main contribution of Sigma

FIND a closed form for

$$\mathsf{SUM}(a,k)=\sum_{j=1}^{a}\frac{H_j}{j(j+k)}.$$

1. Compute a recurrence (creative telescoping)

$$k^2\mathsf{SUM}(a,k)-(k+1)(2k+1)\mathsf{SUM}(a,k+1)+(k+1)(k+2)\mathsf{SUM}(a,k+2)=\frac{a(a+k+2)-(a+1)(k+1)H_a}{(k+1)(a+k+1)(a+k+2)}.$$

2. Solve the recurrence (d'Alembertian solutions)

$$h_1=\frac{1}{k},\qquad h_2=\frac{1}{k}\sum_{j=2}^{k}\frac{1}{j-1},\qquad p=\frac{1}{k}\sum_{j=2}^{k}\frac{\sum_{i=2}^{j}-\left(\frac{-a^2-H_a-aH_a-ai+H_a i+aH_a i}{(-1+i)(-1+a+i)(a+i)}\right)}{j-1}.$$

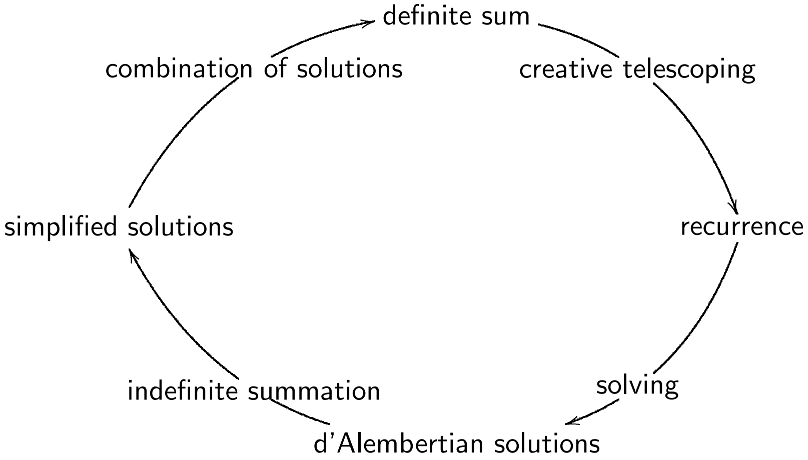
3. Simplification (indefinite summation)

$$h_1=\frac{1}{k},\ h_2=\frac{kH_k-1}{k^2},\ p=\frac{kH_a-H_k-aH_k+kH_k{}^2+akH_k{}^2}{(1+a)k^2}-\frac{kH_a-1}{k^2}\sum_{i=1}^{k}\frac{1}{i+a}-\frac{1}{k}\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{a+i}}{j}.$$

4. FIND the linear combination

$$\sum_{j=1}^{a}\frac{H_j}{j(j+k)}=\frac{-H_a+(1+a)H_a{}^2}{1+a}h_1+0\,h_2+p.$$

The Sigma-summation spiral:



A problem in rhombus tilings

Define

$$S_n := \sum_{k=0}^{n-1} \frac{(-1)^{n+k}(n+k+4)!H_{k+1}}{(k+2)!(k+3)!(n-k-1)!}, \quad T_n := \sum_{k=0}^{n-1} \frac{(-1)^k(n+k+4)!}{(k+1)(k+2)!^2(n-k-1)!}.$$

Lemma 26 (Fulmek, Krattenthaler; 2000) For $n \geq 0$,

$$S_n + \frac{(1-(-1)^n(n+2))n!}{(n+3)!} T_n = (-1)^n(n+2) - 2.$$

Proof: 4 pages involving highly non-trivial transformations.

A problem in rhombus tilings

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$$S_n := \sum_{k=0}^{n-1} \frac{(-1)^{n+k}(n+k+4)!H_{k+1}}{(k+2)!(k+3)!(n-k-1)!}, \quad T_n := \sum_{k=0}^{n-1} \frac{(-1)^k(n+k+4)!}{(k+1)(k+2)!^2(n-k-1)!}.$$

Lemma 26 (Fulmek, Krattenthaler; 2000) For $n \geq 0$,

$$S_n + \frac{(1-(-1)^n(n+2))n!}{(n+3)!}T_n = (-1)^n(n+2) - 2.$$

Proof: Sigma finds (like above):

$$S_n = \frac{-5-3n}{(1+n)(2+n)} - 2\mathsf{H}_n + (-1)^n\Big(\frac{5+2n-2n^2-n^3}{(1+n)(2+n)} + 2(2+n)\mathsf{H}_n\Big),$$

$$T_n = 1 - 9n - 9n^2 - 2n^3 + 2(1+n)(2+n)(3+n)H_n - (-1)^n.$$

Generalized reciprocity laws for sums of harmonic numbers

(joined work with H. Prodinger and M. Kuba)

In the analysis of Hoare's Find algorithm (Kirschenhofer, Prodinger):

$$\sum_{k=1}^{j}\frac{H_{n-k}}{k}+\sum_{k=1}^{n+1-j}\frac{H_{n-k}}{k}=H_jH_{n+1-j}+H_n^2-H_n^{(2)}-\frac{1}{j(n+1-j)}.$$

Generalized reciprocity laws for sums of harmonic numbers

(joined work with H. Prodinger and M. Kuba)

Sigma could generalize and prove:

$$\sum_{k=1}^{j}\frac{H_{n-k}^{(a)}}{k^b}+\sum_{k=1}^{n+1-j}\frac{H_{n-k}^{(b)}}{k^a}=-\frac{1}{j^b(n+1-j)^a}+H_j^{(b)}H_{n+1-j}^{(a)}+\sum_{k=1}^{n}\frac{H_{n-k}^{(b)}}{k^a}.$$

Generalized reciprocity laws for sums of harmonic numbers

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$$\sum_{k=1}^{j}\frac{H_{n-k}^{(a)}}{k^b}+\sum_{k=1}^{n+1-j}\frac{H_{n-k}^{(b)}}{k^a}=-\frac{1}{j^b(n+1-j)^a}+H_j^{(b)}H_{n+1-j}^{(a)}+\sum_{k=1}^{n}\frac{H_{n-k}^{(b)}}{k^a}.$$

Sigma helped to discover:

$$\sum_{k=1}^{n}\frac{H_{n-k}^{(b)}}{k^a}=\sum_{i=1}^{a}\binom{i+b-2}{b-1}\zeta_n(i+b-1,a+1-i)$$
$$+\sum_{i=1}^{b}\binom{i+a-2}{a-1}\zeta_n(i+a-1,b+1-i)$$

where the finite multiple zeta functions are defined as follows:

$$\zeta_N(a_1,\dots,a_l):=\sum_{N\geq n_1>n_2>\dots>n_l\geq 1}\frac{1}{n_1^{a_1}n_2^{a_2}\dots n_l^{a_l}}.$$

Example 5: Summation with unspecified sequences

(joint work with M. Kauers)

Identity 1

$$\sum_{k=0}^{n} kk! = (n+1)! - 1$$

$$\sum_{k=0}^{n} \frac{1-q^k}{1-q} \prod_{i=1}^{k} \frac{1-q^i}{1-q} = \frac{1}{q}\Big(\prod_{i=1}^{n+1} \frac{1-q^i}{1-q} - 1\Big)$$

$$\sum_{k=0}^{n} \frac{kH_k + H_k - k}{k+1} \prod_{i=1}^{k} H_i = \prod_{i=1}^{n+1} H_i - 1$$

Identity 1

$$\sum_{k=0}^{n} k k! = (n+1)! - 1 \qquad X_k := k$$

$$\sum_{k=0}^{n} \frac{1-q^k}{1-q} \prod_{i=1}^{k} \frac{1-q^i}{1-q} = \frac{1}{q}\Big(\prod_{i=1}^{n+1} \frac{1-q^i}{1-q} - 1\Big) \qquad X_k := \frac{1-q^k}{1-q}$$

$$\sum_{k=0}^{n} \frac{kH_k + H_k - k}{k+1} \prod_{i=1}^{k} H_i = \prod_{i=1}^{n+1} H_i - 1 \qquad X_k := H_k$$

$$\boxed{\sum_{k=0}^{n} (X_{k+1} - 1) \prod_{i=1}^{k} X_i = \prod_{k=1}^{n+1} X_k - 1}$$

Sigma

Identity 1

Graham, Knuth, Patashnik. *Concrete Mathematics*, Bonus Problem 5.93:

$$\sum_{k=1}^{n}\frac{1}{X_k}\prod_{i=1}^{k}\frac{X_i}{a+X_i}=\frac{1}{a}\Big(1-\prod_{k=1}^{n}\frac{X_k}{a+X_k}\Big)$$

van der Poorten. A proof that Euler missed...Apéry's proof of the irrationality of $\zeta(3)$, 1979:

$$\sum_{k=0}^{n}\frac{\prod_{j=1}^{k}\frac{(X_j+\alpha)}{X_j}}{X_j+\alpha}=\frac{1}{\alpha}\Big(\prod_{j=1}^{k}\frac{\prod_{j=1}^{n}(X_j+\alpha)}{X_j}-\frac{X_0}{X_0+\alpha}\Big)$$

$$\boxed{\sum_{k=0}^{n}(X_{k+1}-1)\prod_{i=1}^{k}X_i=\prod_{k=1}^{n+1}X_k-1}$$

Sigma

Identity 2

$$\sum_{k=1}^{n} k^2 \sum_{i=1}^{k} X_i =$$ Sigma

Identity 2

$$\sum_{k=1}^{n} k^2 \sum_{i=1}^{k} X_i = \frac{1}{6}\Big(n(n+1)(2n+1)\sum_{k=1}^{n} X_k - \sum_{k=1}^{n} k X_k + 3\sum_{k=1}^{n} k^2 X_k - 2\sum_{k=1}^{n} k^3 X_k\Big)$$

Identity 2

$$\sum_{k=1}^{n}k^2\sum_{i=1}^{k}X_i=\tfrac{1}{6}\Big(n(n{+}1)(2n{+}1)\sum_{k=1}^{n}X_k-\sum_{k=1}^{n}kX_k+3\sum_{k=1}^{n}k^2X_k-2\sum_{k=1}^{n}k^3X_k\Big)$$

$$\downarrow X_k=\frac{1}{n+k}$$

$$\sum_{k=1}^{n}k^2\sum_{i=1}^{k}\frac{1}{n+k}=\frac{1}{36}n(n+1)(1-10n+12(2n+1)\sum_{k=1}^{n}\frac{1}{n+k})$$

Identity 2

$$\sum_{k=1}^{n} k^2 \sum_{i=1}^{k} X_i = \tfrac{1}{6}\Big(n(n{+}1)(2n{+}1)\sum_{k=1}^{n} X_k - \sum_{k=1}^{n} kX_k + 3\sum_{k=1}^{n} k^2 X_k - 2\sum_{k=1}^{n} k^3 X_k\Big)$$

$$\downarrow X_k = \frac{1}{n+k}$$

$$\sum_{k=1}^{n} k^2 \sum_{i=1}^{k} \frac{1}{n+k} = \frac{1}{36} n(n+1)(1-10n+12(2n+1)\sum_{k=1}^{n} \frac{1}{n+k})$$

$$\downarrow H_{n+k} = H_n + \sum_{i=1}^{k} \frac{1}{n+i}$$

$$\sum_{k=1}^{n} k^2 H_{n+k} = \tfrac{1}{3} n(n+\tfrac{1}{2})(n+1)(2H_{2n} - H_n) - \tfrac{1}{36}(10n^2+9n-1)$$

(The warmup example)

Identity 3

$$\sum_{k=1}^{n} a^k \sum_{j=1}^{k} X_j = \text{Sigma}$$

Identity 3

$$\sum_{k=1}^{n} a^k \sum_{j=1}^{k} X_j = \frac{a^{n+1} \sum_{k=1}^{n} X_k - \sum_{k=1}^{n} a^k X_k}{a-1}, \quad a \neq 1$$

Identity 3

$$\sum_{k=1}^{n} a^k \sum_{j=1}^{k} X_j = \frac{a^{n+1} \sum_{k=1}^{n} X_k - \sum_{k=1}^{n} a^k X_k}{a-1}, \quad a \neq 1$$

- $X_j := \frac{1}{j}$:

$$\sum_{k=1}^{n} a^k H_k = \frac{1}{a-1}\Big[a^{n+1} H_n - \sum_{k=1}^{n} \frac{a^k}{k}\Big].$$

Identity 3

$$\sum_{k=1}^{n} a^k \sum_{j=1}^{k} X_j = \frac{a^{n+1}\sum_{k=1}^{n} X_k - \sum_{k=1}^{n} a^k X_k}{a-1}, \quad a \neq 1$$

- $X_j := \frac{1}{j}$:

$$\sum_{k=1}^{n} a^k H_k = \frac{1}{a-1}\Big[a^{n+1} H_n - \sum_{k=1}^{n} \frac{a^k}{k}\Big].$$

- $X_j = \binom{m}{j-1}$, $a=-1$, and $n := m+1$:

$$\sum_{k=0}^{m} (-1)^{k+1} \sum_{j=0}^{k} \binom{m}{j} = \frac{1}{2}(-1)^{m+1} 2^m.$$

Z. Zhang. A kind of binomial identity. *Discrete Math.*, 1999.

a=1

$$\sum_{k=1}^{n}\sum_{j=1}^{k}X_j = $$ Sigma

a=1

$$\sum_{k=1}^{n}\sum_{j=1}^{k}X_j=(n{+}1)\sum_{k=1}^{n}X_k-\sum_{k=1}^{n}kX_k$$

- $X_j:=\frac{1}{j^2}$:

$$\sum_{k=1}^{n}H_k^{(2)}=(n+1)H_n^{(2)}-H_k.$$

a=1

$$\sum_{k=1}^{n}\sum_{j=1}^{k} X_j = (n+1)\sum_{k=1}^{n} X_k - \sum_{k=1}^{n} k X_k$$

- $X_j = \binom{m}{j}$:

$$\sum_{k=0}^{n}\sum_{j=0}^{k} \binom{m}{j} = (n+1)\sum_{k=0}^{n}\binom{m}{k} - \sum_{k=0}^{n} k\binom{m}{k}$$

a=1

$$\sum_{k=1}^{n}\sum_{j=1}^{k}X_j=(n{+}1)\sum_{k=1}^{n}X_k-\sum_{k=1}^{n}kX_k$$

- $X_j=\binom{m}{j}$:

$$\sum_{k=0}^{n}\sum_{j=0}^{k}\tbinom{m}{j}=(n+1)\sum_{k=0}^{n}\binom{m}{k}-\sum_{k=0}^{n}k\binom{m}{k}$$
$$=\frac{1}{2}(m-n)\binom{m}{n}+(2n-m+2)\sum_{i=0}^{n}\binom{m}{i}$$

a=1

$$\sum_{k=1}^{n}\sum_{j=1}^{k}X_j=(n{+}1)\sum_{k=1}^{n}X_k-\sum_{k=1}^{n}kX_k$$

- $X_j=\binom{m}{j}$:

$$\sum_{k=0}^{n}\sum_{j=0}^{k}\tbinom{m}{j}=(n+1)\sum_{k=0}^{n}\binom{m}{k}-\sum_{k=0}^{n}k\binom{m}{k}$$

$$=\frac{1}{2}(m-n)\binom{m}{n}+(2n-m+2)\sum_{i=0}^{n}\binom{m}{i}$$

$$\overset{(m=n)}{=}\frac{m+2}{2}2^m$$

Andrews, Paule. MacMahon's Partition Analysis IV: Hypergeometric Multisums. SLC'99

X-Collection

$$\sum_{k=1}^{a}(-1)^k\Big(\sum_{j=1}^{k}X_j-\frac{X_k}{2}\Big)^2=\frac{1}{2}(-1)^a\Big(\sum_{k=1}^{a}X_k\Big)^2-\frac{1}{4}\sum_{k=1}^{a}(-1)^kX_k^2,$$

$$\sum_{k=1}^{a}\Big(\sum_{j=1}^{k}X_j+X_k(k-1)\Big)^2=n\Big(\sum_{k=1}^{a}X_k\Big)^2-\sum_{k=1}^{a}kX_k^2+\sum_{k=1}^{a}k^2X_k^2\Big),$$

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}X_{n-i}\right)^2=2\sum_{k=0}^{n}X_k\sum_{j=0}^{k}jX_{j-1}+\sum_{k=0}^{n}X_k^2+\sum_{k=0}^{n}kX_k^2,$$

$$\sum_{k=1}^{a}(-1)^k\binom{n}{k}\sum_{j=1}^{k}X_j=\frac{1}{n}\Big[(n-a)\binom{n}{a}(-1)^a\sum_{k=1}^{a}X_k+\sum_{k=1}^{a}(-1)^kk\binom{n}{k}X_k\Big].$$

Part 2: Summation in Difference Fields