# On a Conjectured Inequality for a Sum of Legendre Polynomials

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## 1 Introduction

When working on a new convergence proof for a certain higher order finite element scheme, J. Schöberl was led to conjecture that the inequality

$$\sum_{j=0}^{n}(4j+1)(2n-2j+1)P_{2j}(0)P_{2j}(x) \geq 0 \tag{1}$$

holds for $-1 \leq x \leq 1$ and $n \geq 0$, where $P_k(x)$ denotes the $k$-th Legendre polynomial. In this note we show that the inequality holds in certain neighborhoods (whose size decreases with $n$) of $-1$, $0$, and $1$, and give some asymptotic/numerical evidence concerning its validity on the whole interval.

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Let us mention two standard techniques for proving inequalities of this kind that do *not* work here (at least not directly). First, there is the Dirichlet-Mehler formula [1]

$$P_n(\cos\theta) = \frac{2}{\pi}\int_\theta^\pi \frac{\sin(n+1/2)\phi}{(2\cos\theta - 2\cos\phi)^{1/2}}\mathrm{d}\phi,$$

which can be used to prove Fejér's inequality $\sum_{j=0}^n P_j(x) \geq 0$. Namely, after exchanging summation and integration the resulting sum can be done in closed form and is obviously positive. This approach does not work for our sum, since it does not lead to a positive integrand.

Second, Gasper [4] has shown that there is a non-negative function $\kappa(x,y,t)$ such that

$$P_n(x)P_n(y) = \int_{-1}^1 \kappa(x,y,t)P_n(t)\mathrm{d}t.$$

Therefore, if the sum in (1) with the factor $P_{2j}(0)$ removed was non-negative (it is not), then it would follow that the sum itself is non-negative (even with $P_{2j}(y)$ instead of $P_{2j}(0)$).

In the following section we outline the background where the conjectured inequality (1) popped up. In Section 3 we show that the inequality (1) holds for $x = 0$ and for $x = \pm 1$. Then, by continuity, the inequality has to hold if $x$ is sufficiently close to these points. Precise statements of this kind follow from results about the behaviour of the Legendre polynomials near $x = 0$ and $x = \pm 1$. In Section 4 we derive a linear recurrence equation for the sum in (1) by symbolic summation. This recurrence, while interesting in its own right, is the basis for the investigations in Section 5. From the recurrence we compute a differential equation that is satisfied by the generating function of our sum. Unfortunately, it does not seem to admit of a closed form solution. The Fuchs-Frobenius theory about singularities of differential equations allows to extract some information on the asymptotics of the generating function and the sum itself. We show that there is a function of $x$ that is a lower envelope for our sum as $n$ tends to infinity. However, we do not know yet whether this envelope is positive.

## 2 Motivation

We want to define a family of functions $f_p$ which act as point evaluation functionals on polynomials, i.e.,

$$\int_{-1}^1 f_p(x)v(x)\,dx = v(0) \qquad \forall \text{ polynomials } v \text{ up to order } p. \tag{2}$$

As $p$ increases, the functions $f_p$ converge to the $\delta$ point evaluation distribution. We want to find functions $f_p$ whose Lebesgue norms satisfy

$$\|f_p\|_{L_1} \leq C$$

and

$$\|f_p\|_{L_\infty} \leq Cp,$$

where $C$ is a constant independent of $p$. In addition, we want the functions $f_p$ to be polynomials themself. If we restrict the order of $f_p$ to $p$, then the function is uniquely defined by (2). We call this candidate $f_p^p$. If we expand it in Legendre-polynomials, i.e.,

$$f_p^p(x) = \sum_{i=0}^{p} \alpha_i P_i(x),$$

the coefficients evaluate to

$$\alpha_i = \frac{P_i(0)}{\|P_i\|^2_{L_2(-1,1)}} = \frac{2i+1}{2} P_i(0).$$

But, unfortunately, this simple function does not seem to fulfill uniform bounds in the $L_1$-norm. This is indicated by numerical computations.

An approach similar to the de la Vallée-Poussin sums for trigonometric functions, see [2], page 273 is to take the sliding averages

$$f_p := \frac{1}{p} \sum_{k=p}^{2p-1} f_p^p \tag{3}$$

This is a polynomial of order $2p-1$, and it satisfies (2) for polynomials up to order $p$. An alternative notation is

$$f_p = \frac{1}{p} \sum_{k=0}^{2p-1} f_p^p - \frac{1}{p} \sum_{k=0}^{p-1} f_p^p. \tag{4}$$

By defining

$$S_p := \frac{1}{p} \sum_{k=0}^{p-1} f_p^p,$$

we can write

$$f_p := 2S_{2p} - S_p.$$

We will prove that the $L_1$-norms of the polynomials $S_p$ are uniformely bounded by a constant $C$. Then the triangle inequality allows to bound the $L_1$ norms of $f_p$ by $3C$.

First, observe that choosing $v = 1$ in (2) implies that

$$\int_{-1}^{1} S_p(x)\, dx = 1$$

If we can show that $S_p \geq 0$ on the whole interval $[-1, 1]$, then we can conclude that

$$\int_{-1}^{1} |S_p(x)|\, dx = \int_{-1}^{1} S_p(x)\, dx = 1.$$

Reordering the sums in $S_p$ leads to

$$\begin{aligned}
S_p &= \frac{1}{p} \sum_{k=0}^{p-1} \sum_{i=0}^{k} \frac{2i+1}{2} P_i(0) P_i(x) \\
&= \frac{1}{2p} \sum_{i=0}^{p-1} \sum_{k=i}^{p-1} \frac{2i+1}{2} P_i(0) P_i(x) \\
&= \frac{1}{2p} \sum_{i=0}^{p-1} (p-1-i)(2i+1) P_i(0) P_i(x).
\end{aligned}$$

By noting that $P_i(0) = 0$ for $i$ odd, it is now easy to see that $S_p \geq 0$ is equivalent to (1).

# 3 The Special Cases $x = 0, -1, 1$

Let $f_n = f_n(x)$ denote the sum in (1). Figure 1 illustrates the oscillatory behaviour of $f_n(x)$. It is clear that $f_n(x) = f_n(-x)$, since $P_{2j}(x) = P_{2j}(-x)$.

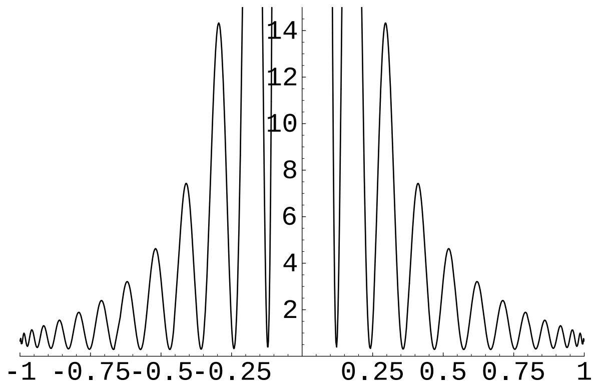


Figure 1: The function $f_{25}(x)$

We will frequently make use of the value

$$P_{2j}(0) = \frac{(-1)^j}{4^j}\binom{2j}{j} \sim \frac{(-1)^j}{\sqrt{\pi j}}$$

of the Legendre polynomials at $x = 0$. At $x = 0$, the inequality holds trivially, since we are summing positive terms. For $x$ close to zero, the summands are still positive. Indeed, if $x = \cos\theta_j$ denotes the smallest positive root of $P_{2j}(x)$, then $\theta_j \leq j\pi/(2j+1)$ [9, Theorem 6.21.3]. Hence

$$P_{2j}(\cos\theta) \neq 0 \qquad \text{for} \qquad \frac{j\pi}{2j+1} \leq \theta \leq \frac{\pi}{2}.$$

This shows that $(-1)^j P_{2j}(\cos\theta) > 0$ in this range.

**Proposition 1.** *The inequality* (1) *holds for* $|\theta - \pi/2| < \pi/(4n+2)$, *where* $x = \cos\theta$.

We proceed to investigate the behaviour of $f_n(x)$ near $x = 1$. Write $p(n,j) := (4j+1)(2n-2j+1)$. A straightforward application of Zeilberger's algorithm [8] shows that

$$\sum_{j=0}^{n}(p(n,j)-1)P_{2j}(0) = 0, \qquad n \geq 0.$$

Thus, since $P_{2j}(1) = 1$,

$$\lim_{n\to\infty} f_n(1) = \sum_{j=0}^{\infty}\left(-\frac{1}{4}\right)^j\binom{2j}{j} = \frac{1}{\sqrt{1-4z}}\bigg|_{z=-1/4} = \frac{1}{\sqrt{2}},$$

and $f_n(1) \geq \frac{1}{2}$ for all $n \geq 0$. The latter observation will be used in the proof of the following result, where we show that positivity of $f_n(x)$ at $x = 1$ can be extended to the left by an asymptotic property of the Legendre polynomials.

**Proposition 2.** *The inequality* (1) *holds for* $0 \leq \theta \leq n^{-5/4}$, *where* $x = \cos\theta$.

*Proof.* Let us assume, more generally, that $0 \leq \theta \leq cn^{-\alpha}$ for some positive constants $c$ and $\alpha$. By an estimate due to Gatteschi [9], we have, provided that $0 \leq \theta \leq \pi/(4n)$,

$$\begin{aligned} P_{2n}(\cos\theta) &= (\theta/\sin\theta)^{1/2}\mathrm{J}_0((2n+\tfrac{1}{2})\theta) + e_3 \\ &=: (1+e_1)(1-e_2) + e_3, \end{aligned}$$

where

$$|e_3| < \tfrac{9}{100}\theta^2.$$

From the alternating series for sin and the Bessel function $\mathrm{J}_0$ it is easy to derive the bounds

$$0 \leq e_1 \leq \tfrac{1}{10}\theta^2 \qquad \text{and} \qquad 0 \leq e_2 \leq \tfrac{1}{4}(2n+\tfrac{1}{2})^2\theta^2.$$

Upon putting

$$M := \max\Big(\frac{c^2}{10n^{2\alpha}} + \frac{9c^2}{100n^{2\alpha}}, \frac{c^2}{4n^{2\alpha}}(2n+\tfrac{1}{2})^2 + \frac{9c^2}{100n^{2\alpha}} + \frac{c^4}{40n^{4\alpha}}(2n+\tfrac{1}{2})^2\Big),$$

it is therefore clear that

$$f_n(\cos\theta) = f_n(1)\cdot(1 + e_1 - e_2 + e_3 - e_1e_2)$$

is positive for $n \geq n_0$ if $n_0$ is such that both $cn^{-\alpha} \leq \pi/(4n)$ and $M < 1$ hold for $n \geq n_0$. The assertion of the proposition follows by setting $c = 1$ and $\alpha = \frac{5}{4}$. □

# 4 Linear Recurrence Equation

The sum $f_n(x)$ satisfies a linear recurrence relation (w.r.t. $n$), which can be computed by the `SumCracker` package [6]. Applying the `LinearRecurrence` command of that package to the representation

$$f_n(x) = 2n\sum_{j=0}^{n}(4j+1)P_{2j}(0)P_{2j}(x) + \sum_{j=0}^{n}(4j+1)(1-2j)P_{2j}(0)P_{2j}(x),$$

the package delivers the recurrence equation

$$\begin{aligned} &-(2n+3)^2(4n+11)\left(16x^2n^2 - 16n^2 + 88x^2n - 88n + 117x^2 - 118\right)f_n(x) \\ &+ p(n,x)f_{n+1}(x) \\ &+ \left(48x^2n^2 - 16n^2 + 216x^2n - 72n + 243x^2 - 90 - p(n,x)\right)f_{n+2}(x) \\ &4(4n+7)(n+3)^2\left(16x^2n^2 - 16n^2 + 56x^2n - 56n + 45x^2 - 46\right)f_{n+3}(x) = 0, \end{aligned} \tag{5}$$

where

$$p(n,x) = \left(-1024x^4 + 1792x^2 - 768\right) n^5 + \left(-11520x^4 + 20160x^2 - 8640\right) n^4 - \left(50560x^4 - 88592x^2 + 38016\right) n^3 - \left(108000x^4 - 189780x^2 + 81656\right) n^2 + \left(-112036x^4 + 197880x^2 - 85536\right) n - 45045x^4 + 80244x^2 - 34956.$$

Alternatively, a recurrence equation for $f_n$ can be obtained by the `MultiSum` package [10], but this leads in the present case to a recurrence of order 4 with even uglier polynomials as coefficients.

# 5 Asymptotics via the Generating Function

Let $F(z) := \sum_{n\geq 0} f_n z^n$ denote the generating function of $f_n$ (we frequently suppress the parameter $x$). Since the absolute value of the Legendre polynomials is at most one in the unit interval, the function $F(z)$ is analytic in $|z| < 1$ (for all $x \in [-1, 1]$). We will determine the singularities of $F(z)$ and its asymptotic behaviour near them, from which we can deduce asymptotic information on $f_n$. From the recurrence (5) we can compute a fifth order linear ordinary differential equation with polynomial coefficients for $F(z)$, by means of C. Mallinger's `GeneratingFunctions` package [7]. The singularities of the equation are the roots of its leading coefficient. From now on we assume that $x \notin \{-1, 0, 1\}$. Then the singularities of the equation are $z = 1, \rho, \bar{\rho}$, where

$$\rho = \rho(x) := 1 - 2x^2 + 2\,\mathrm{i}|x|\sqrt{1 - x^2}.$$

These three numbers, which lie on the unit circle, are candidates for singularities of $F(z)$. We continue our investigations by appealing to the Fuchs-Frobenius theory [5]. Consider first $z = 1$. There the indicial equation has the solutions $-1, 0, 1, 2, 3$, hence $F(z)$ is of the form

$$F(z) = \frac{K}{1-z} \log \frac{1}{1-z} + \frac{A}{1-z} + L \log \frac{1}{1-z} + \mathrm{O}((1-z) \log \frac{1}{1-z}) \qquad (6)$$

as $z \to 1$, for some complex coefficients $K, A, L$. As we will see below, the explicit representation of $F(z)$ allows to show that $\lim_{z\to 1}(1 - z)F(z)$ exists, which implies that the coefficient of $(1 - z)^{-1} \log(1 - z)^{-1}$ in (6) must vanish.

The other two singularities $\rho, \bar{\rho}$ can be analyzed completely analogously (the roots of the indicial polynomials are the same). Summing up, there are complex numbers $A = A(x)$, $B = B(x)$, $C = C(x)$ s.t.

$$F(z) = \frac{A}{1-z} + \frac{B}{\rho - z} + \frac{C}{\bar{\rho} - z} + g(z), \qquad |z| < 1,$$

where the function $g(z)$ is analytic inside the unit circle, has no singularities on the circle except $1, \rho, \bar{\rho}$, and is of at most logarithmic growth at these points:

$$g(z) = \mathrm{O}(\log \frac{1}{z_0 - z}), \quad z \to z_0, \quad z_0 \in \{1, \rho, \bar{\rho}\}.$$

Letting $z$ tend to one in the reals, we find that $A$ is real. The binomial theorem and Flajolet and Odlyzko's singularity analysis [3] yield

$$f_n = A + B\rho^{-n-1} + \bar{B}\bar{\rho}^{-n-1} + \mathrm{O}(\frac{\log n}{n}). \qquad (7)$$

Here we have already used the fact that $C = \bar{B}$ (otherwise $f_n$ could not be real). Our sequence $f_n$ thus equals

$$f_n = A + 2|B| \sin(2n\pi\theta + \varphi) + \mathrm{O}(\frac{\log n}{n})$$

for some real numbers $\theta = \theta(x)$, $\varphi = \varphi(x)$. Thus, the functions $A(x) - 2|B(x)|$ and $A(x)+2|B(x)|$ are, asymptotically, a lower and an upper envelope for $f_n(x)$, respectively (cf. Figure 2). They can be determined by Abel's limit theorem:

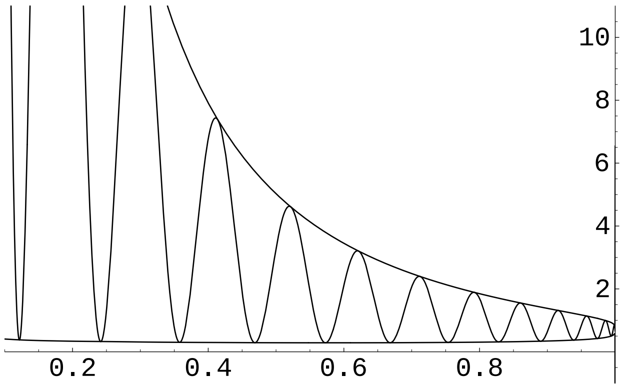


Figure 2: The function $f_{25}(x)$ and approximations of the two envelopes $A(x) \pm 2|B(x)|$

$$\begin{aligned}
(\rho - 1)(\bar{\rho} - 1)A &= \lim_{z\to 1} (1 - z)(\rho - z)(\bar{\rho} - z)F(z) && (8)\\
&= \lim_{z\to 1} (1 - z)(1 + (4x^2 - 2)z + z^2)F(z)\\
&= \lim_{z\to 1} \big(\sum_{n\geq 0} a_n z^n + q(z, x)\big)\\
&= \sum_{n\geq 0} a_n + q(1, x), && (9)
\end{aligned}$$

where $q(z, x)$ is the polynomial

$$q(z, x) := 1 + \tfrac{1}{4}(5 + x^2)z + \tfrac{1}{64}(17 + 22x^2 - 15x^4)z^2,$$

and the sums inside the coefficient $a_n$ cancel:

$$\begin{aligned}
a_n = a_n(x) :=& f_{n+3} + (4x^2 - 3)f_{n+2} + (3 - 4x^2)f_{n+1} - f_n\\
=& U_n P_{2n}(0)\Big(V_n(x)P_{2n}(x) + W_n(x)P_{2n+1}(x)\Big),
\end{aligned}$$

where

$$\begin{aligned}
U_n &:= \frac{(2n + 1)}{64(n + 1)^2(n + 2)^2(n + 3)^2},\\
V_n(x) &:= (2n + 1)\Big((8n^2 + 36n + 45) + 2(24n^3 + 134n^2 + 252n + 153)x^2\\
&\qquad - (4n + 5)(4n + 7)(4n + 9)x^4\Big),\\
W_n(x) &:= (2n + 1)^2(8n^2 + 36n + 45)x - 2(4n + 5)(32n^3 + 160n^2 + 256n + 117)x^3\\
&\qquad + (4n + 3)(4n + 5)(4n + 7)(4n + 9)x^5.
\end{aligned}$$

Since $a_n = \mathrm{O}(n^{-3/2})$ for all $x$, the sum in (9) is convergent. Note that Abel's limit theorem would not be applicable without the factor $(\rho - z)(\bar{\rho} - z)$ in (8), as the resulting sum would not converge at $z = 1$. Since $(\rho - 1)(\bar{\rho} - 1) = 4x^2$, we arrive at

$$A(x) = \frac{1}{4x^2} \left( \sum_{n \geq 0} a_n(x) + \tfrac{1}{64}(161 + 38x^2 - 15x^4) \right). \tag{10}$$

Similarly, we obtain

$$|B(x)| = \frac{1}{8x^2\sqrt{1-x^2}} \left| \sum_{n \geq 0} a_n(x)\rho(x)^n + q(\rho(x), x) \right|. \tag{11}$$

We have thus found two sequences of explicitly given functions that converge to $A$ and $|B|$, respectively. The series $\sum_{n \geq 3} a_n(x)$ converges absolutely and uniformly in $[-1, 1]$, hence $A(x)$ and $|B(x)|$ are continuous in $]0, 1[$. The limit of $A(x)$ at one can be evaluated by Mathematica, after exchanging limit and summation:

$$\lim_{x \to 1^-} A(x) = \frac{1}{4} \left( \sum_{n \geq 0} a_n(1) + q(1, 1) \right) = \frac{1}{\sqrt{2}}.$$

Observe that this equals $\lim_{n \to \infty} f_n(1)$. According to numerical evidence, $B(x)$ seems to tend to 0 as $x \to 1$.

The behaviour of $A(x)$ at $x = 0$ is

$$A(x) \sim \frac{2}{\pi x^2} \qquad \text{as} \quad x \to 0,$$

which follows from

$$\sum_{n \geq 0} a_n(0) + q(1, 0) = \frac{8}{\pi}.$$

This was again obtained with Mathematica. Using $\lim_{x \to 0} \rho(x) = 1$, the analogous calculation for $B$ yields

$$|B(x)| \sim \frac{1}{\pi x^2} \qquad \text{as} \quad x \to 0.$$

Hence the upper envelope $A + 2|B|$ is unbounded near zero. As for the lower envelope $A - 2|B|$, we so far only know that it is $\mathrm{o}(1/x^2)$. Numerical evidence suggests that it is in fact continuous at zero. One difficulty in obtaining the lower order terms is that we do not know whether the termwise differentiated series $\sum \mathrm{d}a_n(x)/\mathrm{d}x$ converges. We summarize our results in the following theorem. By symmetry, it suffices to formulate it for positive $x$.

**Theorem 3.** *There are real functions $A(x)$, $\theta(x)$, and $\varphi(x)$, and a complex function $B(x)$, all defined on $]0, 1[$, such that for all $x \in ]0, 1[$*

$$f_n = A + 2|B| \sin(2n\pi\theta + \varphi) + \mathrm{O}(\frac{\log n}{n}), \qquad n \to \infty.$$

*In particular,*

$$A - 2|B| \leq f_n + \mathrm{O}(\frac{\log n}{n}) \leq A + 2|B|.$$

*Formulas* (10) *and* (11) *give series of functions that converge to* $A$ *and* $B$*, respectively. The upper envelope* $A + 2|B|$ *is asymptotically equal to* $4/(\pi x^2)$ *as* $x \to 0$.

# 6 Outlook

If we could show that the lower envelope $A(x) - 2|B(x)|$ is larger than some positive constant for all $x \in \,]0, 1[$, then it would follow that for all $x \in \,]0, 1[$ we have $f_n(x) > 0$ for large $n$. This could serve as an achievable first goal. A natural way to proceed is to truncate the sums in (10) and (11) at some $N$, say $N = 20$. Without taking the cancellation of the dominant terms of $A$ and $-2|B|$ near zero into account, bounds for the tails of $A$ and $|B|$ are easily derived. (However, these bounds get bad as $x$ approaches zero or one.) Then it remains to show that the truncated version of $A - 2|B|$ is large enough so that the tail cannot make $A - 2|B|$ negative. This will probably work for $x \in [0.1, 0.9]$, say, but in order to establish the positivity of $A - 2|B|$ for all $x \in \,]0, 1[$ we need more information on the asymptotics of $A$ and $|B|$ at zero and one, including effective error estimates.

But we should actually try to prove the stronger assertion that there is $n_0$ such that $f_n(x) > 0$ for $n \geq n_0$ and $-1 \leq x \leq 1$. With our approach, this would hinge on an effective estimate for the error term in (7). If we can find such $n_0$ and it is not too large, the values $n \leq n_0$ can be easily checked by computer algebra, of course.

Finally we mention a completely different way to tackle such problems. A method that has been used to prove non-trivial inequalities of this kind before [1] is to apply transformations to the sum that reveal its positivity, for instance by expressing it as a sum of squares of special functions. This should certainly be tried on this example, too, both on $f_n(x)$ directly and on the lower envelope $A(x) - 2|B(x)|$.

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